POSITIVE p-HARMONIC FUNCTIONS WITH ZERO BOUNDARY DATA ON CONE DOMAINS

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1. Introduction

Let $1 < p < \infty$ and let D be a domain in $\mathbb{C}$. The Euler-Lagrange equation for the problem of minimizing the p-Dirichlet integral $\int_D |\nabla u|^p dx$ over a suitable function class is written in weak form as

$$\int_D |\nabla u|^{p-2} \nabla u \cdot \nabla \eta = 0, \tag{1.1}$$

which must hold at least for all $\eta \in C_0^\infty(D)$. If $u \in C^2(D)$, this implies that

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0 \tag{1.2}$$

in D. This equation is equivalent to

$$(p-2) \sum_{i,j=1}^{2} u_{x_i} u_{x_j} u_{x_i x_j} + |\nabla u|^2 \Delta u = 0. \tag{1.3}$$

Either of the three equations is called the p-harmonic equation and the solutions are called p-harmonic functions.

Let $0 < \phi < \pi$. We denote a cone of aperture ϕ by

$$D_\phi = \{z \in \mathbb{C} : |\arg z| < \phi\}.$$

In this paper we find positive p-harmonic functions $u(z)$ on D_ϕ with the boundary condition,

$$u(z) = \begin{cases} 0 & \text{for } |\arg z| = \phi \text{ and } z = 0, \\ \infty & \text{for } z = \infty, \end{cases} \tag{1.4}$$

or

$$u(z) = \begin{cases} 0 & \text{for } |\arg z| = \phi \text{ and } z = \infty, \\ \infty & \text{for } z = 0. \end{cases} \tag{1.5}$$

We consider the form $u(z) = r^k f(\theta)$ for $z = re^{i\theta}$, $k \neq 0$. Aronsson [1] determined all p-harmonic functions in $\mathbb{C}$ of the form $u(z) = r^k f(\theta)$, assuming $p > 2$. Here, for $p > 1$, we determine all positive p-harmonic functions

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in D_ϕ of the form $u(z) = r^k f(\theta)$ satisfying the boundary condition (1.4) or (1.5).

If $u(z)$ satisfy the boundary condition (1.4), then $k > 0$. This k is denoted by k_+^p. If $u(z)$ satisfy the boundary condition (1.5), then $k < 0$. This k is denoted by k_-^p. Let $\beta = \pi/(2\phi)$. For $p = 2$, it is easy to calculate k_+^2, k_-^2, and $f(\theta)$. We see that

$$\begin{cases} k_+^2 = \beta, \\ k_-^2 = -\beta. \end{cases}$$

and

$$f(\theta) = C \cos \beta\theta,$$

where C is a arbitrary positive constant. For general $p > 1$, we obtain the following theorems.

Theorem 1.1. *Let $\alpha = (p-2)/(p-1)$ and $\beta = \pi/(2\phi)$. If*

$$k_+^p = \frac{2\beta^2 - \alpha(\beta-1)^2 + (\beta-1)\sqrt{4\beta^2 - 4\alpha\beta^2 + \alpha^2(\beta-1)^2}}{2(2\beta-1)},$$

then there exists $f(\theta)$ such that $u(z) = r^{k_+^p} f(\theta)$ is p-harmonic in D_ϕ and satisfy the boundary condition (1.4).

Theorem 1.2. *Let $\alpha = (p-2)/(p-1)$ and $\beta = \pi/(2\phi)$. If*

$$k_-^p = \frac{-2\beta^2 + \alpha(\beta+1)^2 - (\beta+1)\sqrt{4\beta^2 - 4\alpha\beta^2 + \alpha^2(\beta+1)^2}}{2(2\beta+1)},$$

then there exists $f(\theta)$ such that $u(z) = r^{k_-^p} f(\theta)$ is p-harmonic in D_ϕ and satisfy the boundary condition (1.5).

These theorems are main results of this paper.

2. Separation equation

In this section we give the representation formula for $f(\theta)$. See [1] for these accounts.

We observe that $u(z) = r^k f(\theta)$ satisfies (1.3) if and only if $f(\theta)$ satisfies the separation equation

(2.1)
$$[(p-1)(f')^2 + k^2 f^2] f'' + (2kp - 3k - p + 2)kf(f')^2 + (kp - k - p + 2)k^3 f^3 = 0.$$

Hence we find $f(\theta)$ satisfying the separation equation (2.1) with the condition

$$\begin{cases} f(\theta) > 0 & \text{for } -\phi < \theta < \phi, \\ f(\pm\phi) = 0. \end{cases} \tag{2.2}$$

Lemma 2.1. *Let I be an open interval and $f(\theta) \in C^2(I)$. Assume that $f(\theta) > 0$ and $f'(\theta) \neq 0$ on I. Put $\alpha = (p-2)/(p-1)$ and $g(\theta) = f'(\theta)^2 + (k-\alpha)kf(\theta)^2$.*
(1) *If $f(\theta)$ satisfies the separation equation (2.1) on I, then either (i) or (ii) holds:*

(i) $g \neq 0$ on I, and there is a constant $C_1 > 0$ such that

$$[(f')^2 + k^2 f^2]^k = C_1^2 |g|^{k-1}. \tag{2.3}$$

(ii) $g \equiv 0$ on I. Further, $f(\theta) = Ce^{\pm\mu\theta}$ where $\mu = \sqrt{(\alpha - k)k}$.
(2) *Conversely, if either (i) or (ii) holds, then $f(\theta)$ satisfies the separation equation (2.1) on I.*

Proof. Put $s = k^2 f(\theta)^2 > 0$. Let $J = s(I)$. We consider the inverse mapping $F : J \ni s \mapsto \theta \in I$. Obviously, $F \in C^2(J)$. Define a function $w(s)$ for $s \in J$ by

$$w(s) = \frac{f'(F(s))^2}{s} + 1.$$

We observe that $w(s) \in C^1(J)$ and

$$(w-1) + s\frac{dw}{ds} = \frac{f''}{k^2 f}.$$

Hence, $f(\theta)$ satisfies the separation equation (2.1) if and only if $w(s)$ satisfies the ordinary differential equation

$$\Big(w - \frac{\alpha}{k}\Big)w = -s(w-\alpha)\frac{dw}{ds},$$

where $\alpha = (p-2)/(p-1)$. If $f(\theta)$ satisfies the separation equation (2.1), then $w - \frac{\alpha}{k}$ is $\neq 0$ or $\equiv 0$ on J. On the other hand, we have

$$w - \frac{\alpha}{k} = \frac{f'(\theta)^2 + k^2 f(\theta)^2}{k^2 f(\theta)^2} - \frac{\alpha}{k} = \frac{g(\theta)}{k^2 f(\theta)^2}.$$

Hence g is $\neq 0$ or $\equiv 0$ on I. Let us consider three cases.

Case 1: $g(\theta) > 0$. The separation equation (2.1) is equivalent to

$$\frac{dw}{ds}\Big(\frac{k}{w} - \frac{k-1}{w - \frac{\alpha}{k}}\Big) + \frac{1}{s} = 0, \qquad \text{or}$$

$$\frac{d}{ds}\Big[\log w^k - \log\Big(w - \frac{\alpha}{k}\Big)^{k-1} + \log s\Big] = 0.$$

This holds if and only if

$$w^k s = C_1^2\Big(w - \frac{\alpha}{k}\Big)^{k-1}$$

for all $s \in J$, for some $C_1 > 0$. Thus we obtain

$$[(f')^2 + k^2 f^2]^k = C_1^2 g^{k-1}.$$

Case 2: $g(\theta) < 0$. The separation equation (2.1) is equivalent to

$$\frac{dw}{ds}\left(\frac{k}{w} - \frac{1-k}{\frac{\alpha}{k} - w}\right) + \frac{1}{s} = 0, \quad \text{or}$$
$$\frac{d}{ds}\left[\log w^k + \log\left(\frac{\alpha}{k} - w\right)^{1-k} + \log s\right] = 0.$$

This holds if and only if

$$w^k s = C_1^2\left(\frac{\alpha}{k} - w\right)^{k-1}$$

for all $s \in J$, for some $C_1 > 0$. Thus we obtain

$$[(f')^2 + k^2 f^2]^k = C_1^2(-g)^{k-1}.$$

Case 3: $g(\theta) \equiv 0$. Then we have

$$f'(\theta)^2 + (k-\alpha)kf(\theta)^2 \equiv 0.$$

Since $f'(\theta) \neq 0$, we see $(k-\alpha)k < 0$. Put $\mu = \sqrt{(\alpha-k)k}$. Then we have $f'(\theta) = \pm\mu f(\theta)$. Thus, $f(\theta) = Ce^{\pm\mu\theta}$. Conversely, if $(k-\alpha)k < 0$ and $f(\theta) = Ce^{\pm\mu\theta}$ where $\mu = \sqrt{(\alpha-k)k}$, then $f(\theta)$,obviously, satisfies the separation equation (2.1). □

Lemma 2.2. *Let I be an open interval and $f(\theta) \in C^2(I)$. Put $\alpha = (p-2)/(p-1)$ and $g(\theta) = f'(\theta)^2 + (k-\alpha)kf(\theta)^2$. Assume that $f(\theta) > 0$, $f'(\theta) \neq 0$, and $g(\theta) \neq 0$ on I. If there is a constant $C_1 > 0$ satisfying (2.3), then $f(\theta)$ has a parametric representation, given by*

$$\begin{cases} f(t) = \frac{C_1}{k}\left|1 - \frac{\alpha}{k}\cos^2 t\right|^{\frac{k-1}{2}} \cdot \cos t, \\ \theta(t) = \theta^* + \int_{t^*}^{t} \frac{1-\alpha\cos^2 t'}{k-\alpha\cos^2 t'} dt'. \end{cases}$$

Proof. Assume that $g(\theta) > 0$. We introduce polar coordinates in the plane:

$$\begin{cases} kf = \rho\cos t, \\ -f' = \rho\sin t \quad (\neq 0). \end{cases} \tag{2.4}$$

We see that $\rho = \rho(\theta)$ and $t = t(\theta)$ are in $C^1(I)$. The equation (2.3) gives

$$\rho^{2k} = C_1^2\left[\rho^2\left(1 - \frac{\alpha}{k}\cos^2 t\right)\right]^{k-1}.$$

Then

$$\rho = C_1\left(1 - \frac{\alpha}{k}\cos^2 t\right)^{(k-1)/2}. \tag{2.5}$$

Thus we have

$$f = \frac{C_1}{k}\left(1 - \frac{\alpha}{k}\cos^2 t\right)^{\frac{k-1}{2}} \cdot \cos t$$

Next we give a representation of $\theta = \theta(t)$. Since $kf = \rho \cos t$ and $f'(\theta) \neq 0$, we see that $\theta = \theta(t) \in C^1$. By (2.4), we have

$$k = \frac{dt}{d\theta} - \frac{1}{\rho \tan t}\frac{d\rho}{d\theta}.$$

Then

$$\frac{dt}{d\theta}\left(1 - \frac{1}{\tan t}\frac{d(\log\rho)}{dt}\right) = k.$$

By (2.5), we get

$$\frac{d(\log\rho)}{dt} = (k-1)\frac{\alpha \sin t \cos t}{k - \alpha\cos^2 t}.$$

Then

$$\frac{d\theta}{dt} = \frac{1 - \alpha\cos^2 t}{k - \alpha\cos^2 t}.$$

This implies the representation formula in the case $g(\theta) > 0$.

In the case $g(\theta) < 0$, the representation formula follows by a similar argument. Thus the lemma is proved. □

The following lemma is proved by easy computations. See [1].

Lemma 2.3. *Let I be a maximal open interval such that $\alpha\cos^2 t \neq k$ for $t \in I$. We consider the mapping $t \mapsto (f, \theta)$ defined by*

$$\begin{cases} f(t) = \left|1 - \frac{\alpha}{k}\cos^2 t\right|^{\frac{k-1}{2}} \cdot \cos t \\ \theta(t) = \int_{t^*}^{t} \frac{1-\alpha\cos^2 t'}{k-\alpha\cos^2 t'} dt' \end{cases}$$

for $t \in I$. Then $f(\theta)$ satisfies the separation equation (2.1).

3. Proof of Theorem 1.1 and Theorem 1.2

In this section we prove Theorem 1.1 and Theorem 1.2. Assume that $p \neq 2$. Let us cinsider the following four cases:

(i) $p > 2$ and $k > 0$,
(ii) $1 < p < 2$ and $k > 0$,
(iii) $p > 2$ and $k < 0$,
(iv) $1 < p < 2$ and $k < 0$.

Put $\alpha = (p-2)/(p-1)$ and $\beta = \pi/(2\phi)$. For simplicity, we let

$$\lambda = \frac{\sqrt{|k-\alpha|}}{\sqrt{|k|} + \sqrt{|\alpha|}}$$

and

$$\mu = \frac{\sqrt{|k|}}{\sqrt{|\alpha| + |k|}}.$$

3.1. **The case $p > 2$ and $k > 0$.** We observe that if $k \le \alpha$, then there is no function $f(\theta)$ satisfying the separation equation (2.1) with the condition (2.2) (see [1]). Hence we assume that $k > \alpha$. Then $g(\theta) = f'(\theta)^2 + (k-\alpha)kf(\theta)^2 > 0$. Since Lemma 2.2, all solutions of the separation equation (2.1) are given by

$$\begin{cases} f(t) = \frac{C}{k}(1 - \frac{\alpha}{k}\cos^2 t)^{\frac{k-1}{2}} \cdot \cos t \\ \theta(t) = t - t^* + (1-k)\int_0^t \frac{dt'}{k - \alpha\cos^2 t'} \end{cases}$$

for $-\infty < t < \infty$. We see that $\theta(t)$ is strictly increasing. By the condition (2.2), we have $-\pi/2 \le t \le \pi/2$, $t^* = 0$, and $\theta(\pi/2) = \phi$. Easy computations gives

$$\theta(t) = t - \frac{k-1}{\sqrt{(k-\alpha)k}}[\arctan(\lambda \tan\frac{t}{2}) + \arctan(\lambda^{-1}\tan\frac{t}{2})]$$

for $-\pi/2 \le t \le \pi/2$. Since $\theta(\pi/2) = \phi$, we have

$$\frac{\pi}{2} - \frac{k-1}{\sqrt{(k-\alpha)k}} \cdot \frac{\pi}{2} = \phi. \tag{3.1}$$

If $\phi = \pi/2$, then $k = 1$. We assume that $\phi \ne \pi/2$. Squaring and rewriting gives

$$(2\beta - 1)k^2 - [2\beta^2 - \alpha(\beta-1)^2]k + \beta^2 = 0.$$

The roots of this equation are

$$k_1 = \frac{2\beta^2 - \alpha(\beta-1)^2 + |\beta - 1|\sqrt{4\beta^2 - 4\alpha\beta^2 + \alpha^2(\beta-1)^2}}{2(2\beta-1)}$$

and

$$k_2 = \frac{2\beta^2 - \alpha(\beta-1)^2 - |\beta - 1|\sqrt{4\beta^2 - 4\alpha\beta^2 + \alpha^2(\beta-1)^2}}{2(2\beta-1)}.$$

We observe that $\alpha < k_2 < 1 < k_1$ and (3.1) has only one root. If $0 < \phi < \pi/2$, then $\beta < 1$ and only k_2 satisfies (3.1). If $\pi/2 < \phi < \pi$, then $\beta > 1$ and only k_1 satisfies (3.1). Thus, the following theorem is obtained.

Theorem 3.1. *Let $p > 2$. Put*

$$k_+^p = \frac{2\beta^2 - \alpha(\beta-1)^2 + (\beta - 1)\sqrt{4\beta^2 - 4\alpha\beta^2 + \alpha^2(\beta-1)^2}}{2(2\beta-1)}.$$

Let $f(\theta)$ be a function given by

$$\begin{cases} f(t) = C(1 - \frac{\alpha}{k_+^p}\cos^2 t)^{\frac{k_+^p-1}{2}} \cos t \\ \theta(t) = t - \frac{k_+^p - 1}{\sqrt{(k_+^p-\alpha)k_+^p}}[\arctan(\lambda\tan\frac{t}{2}) + \arctan(\lambda^{-1}\tan\frac{t}{2})] \end{cases}$$

for $-\pi/2 < t < \pi/2$, where C is a arbitrary positive constant. Then $f(\theta)$ satisfies the separation equation (2.1) with the condition (2.2).

3.2. **The case $1 < p < 2$ and $k > 0$.** We obtain the following theorem by a similar argument of the case $p > 2$ and $k > 0$.

Theorem 3.2. *Let $1 < p < 2$. Put*

$$k_+^p = \frac{2\beta^2 - \alpha(\beta-1)^2 - (\beta-1)\sqrt{4\beta^2 - 4\alpha\beta^2 + \alpha^2(\beta-1)^2}}{2(2\beta-1)}.$$

Let $f(\theta)$ be a function given by

$$\begin{cases} f(t) = C(1 - \frac{\alpha}{k_+^p}\cos^2 t)^{\frac{k_+^p - 1}{2}}\cos t \\ \theta(t) = t - \frac{k_+^p - 1}{\sqrt{(k_+^p - \alpha)k_+^p}}\tan^{-1}(\mu \tan t) \end{cases}$$

for $-\pi/2 < t < \pi/2$, where C is a arbitrary positive constant. Then $f(\theta)$ satisfies the separation equation (2.1) with the condition (2.2).

Thus Theorem 3.1 and Theorem 3.2 imply Theorem 1.1.

Remark 3.3. If $\phi = \pi/2$, then $k_+^p = 1$ and $f(\theta) = C\cos\theta$ for all $1 < p < \infty$. In fact, $u(z) = x$ for $z = x + iy$ is a positive p-harmonic function in D_ϕ and satisfy the boundary condition (1.4).

3.3. **The case $p > 2$ and $k < 0$.** Then $g(\theta) = f'(\theta)^2 + (k-\alpha)kf(\theta)^2 > 0$. Since Lemma 2.2, all solutions of the separation equation (2.1) are given by

$$\begin{cases} f(t) = \frac{C}{k}(1 - \frac{\alpha}{k}\cos^2 t)^{\frac{k-1}{2}} \cdot \cos t \\ \theta(t) = t - t^* + (1-k)\int_0^t \frac{dt'}{k - \alpha\cos^2 t'} \end{cases}$$

for $-\infty < t < \infty$. We see that $\theta(t)$ is strictly decreasing. By the condition (2.2), we have $-\pi/2 \le t \le \pi/2$, $t^* = 0$, and $\theta(\pi/2) = -\phi$. Easy computations gives

$$\theta(t) = t - \frac{1-k}{\sqrt{(k-\alpha)k}}\arctan(\mu\tan t)$$

for $-\pi/2 \le t \le \pi/2$. Since $\theta(\pi/2) = -\phi$, we have

$$\frac{\pi}{2} - \frac{1-k}{\sqrt{(k-\alpha)k}} \cdot \frac{\pi}{2} = -\phi. \tag{3.2}$$

Squaring and rewriting gives

$$(2\beta+1)k^2 + [2\beta^2 - \alpha(\beta+1)^2]k - \beta^2 = 0.$$

The roots of this equation are

$$k_1 = \frac{-2\beta^2 + \alpha(\beta+1)^2 - (\beta+1)\sqrt{4\beta^2 - 4\alpha\beta^2 + \alpha^2(\beta+1)^2}}{2(2\beta+1)}$$

and

$$k_2 = \frac{-2\beta^2 + \alpha(\beta+1)^2 - (\beta+1)\sqrt{4\beta^2 - 4\alpha\beta^2 + \alpha^2(\beta+1)^2}}{2(2\beta+1)}$$

We see that $\alpha < k_2 \leq 1 \leq k_1$ and only k_2 satisfies (3.2). Thus, the following theorem is obtained.

Theorem 3.4. *Let $p > 2$. Put*

$$k_-^p = \frac{-2\beta^2 + \alpha(\beta+1)^2 - (\beta+1)\sqrt{4\beta^2 - 4\alpha\beta^2 + \alpha^2(\beta+1)^2}}{2(2\beta+1)}.$$

Let $f(\theta)$ be a function given by

$$\begin{cases} f(t) = C(1 - \frac{\alpha}{k_-^p}\cos^2 t)^{\frac{k_-^p - 1}{2}} \cos t \\ \theta(t) = t - \frac{1-k_-^p}{\sqrt{(k_-^p - \alpha)k_-^p}} \tan^{-1}(\mu \tan t) \end{cases}$$

for $-\pi/2 < t < \pi/2$, where C is a arbitrary positive constant. Then $f(\theta)$ satisfies the separation equation (2.1) *with the condition* (2.2)*.*

3.4. **The case $1 < p < 2$ and $k < 0$.** We obtain the following theorem by a similar argument of the case $p > 2$ and $k < 0$.

Theorem 3.5. *Let $1 < p < 2$. Put*

$$k_-^p = \frac{-2\beta^2 + \alpha(\beta+1)^2 - (\beta+1)\sqrt{4\beta^2 - 4\alpha\beta^2 + \alpha^2(\beta+1)^2}}{2(2\beta+1)}.$$

Let $f(\theta)$ be a function given by

$$\begin{cases} f(t) = C(1 - \frac{\alpha}{k_-^p}\cos^2 t)^{\frac{k_-^p - 1}{2}} \cos t \\ \theta(t) = t - \frac{1-k_-^p}{\sqrt{(k_-^p - \alpha)k_-^p}} [\arctan(\lambda \tan \frac{t}{2}) + \arctan(\lambda^{-1} \tan \frac{t}{2})] \end{cases}$$

for $-\pi/2 < t < \pi/2$, where C is a arbitrary positive constant. Then $f(\theta)$ satisfies the separation equation (2.1) *with the condition* (2.2)*.*

Thus Theorem 3.4 and Theorem 3.5 imply Theorem 1.2.

References

[1] G. Aronsson, *Construction of singular solutions to the p-harmonic equation and its limit equation for $p = \infty$*, Manuscripta Math. **56** (1986), no. 2, 135–158.

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