ASYMPTOTIC EXPANSION OF SOLUTIONS TO THE DISSIPATIVE EQUATION WITH ANOMALOUS DIFFUSION

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1. Introduction

The following Cauchy problem for the linear dissipative equation is studied by many authors:

$$\begin{cases} \partial_t u - \Delta u + a(t,x)u = 0, & t>0,\ x\in\mathbb{R}^n, \\ u(0,x) = u_0(x), & x\in\mathbb{R}^n, \end{cases} \tag{1.1}$$

where $n\in\mathbb{N}$ and the coefficient $a:(0,\infty)\times\mathbb{R}^n\to\mathbb{R}$ and the initial data $u_0:\mathbb{R}^n\to\mathbb{R}$ are given functions. Upon the suitable condition for $a(t,x)$ and $u_0(x)$, the well-posedness, the global in time existence and the decay of solutions to (1.1) are shown. Moreover the asymptotic expansion of the solution to (1.1) as $t\to\infty$ is derived (cf [6]). Here we consider those problems when the dissipative effect on the equation is provided by "the anomalous diffusion". In this manuscript, we define the Fourier transform and the Fourier inverse transform by

$$\mathcal{F}[\varphi](\xi) := (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}\varphi(x)dx \quad \text{and} \quad \mathcal{F}^{-1}[\varphi](x) := (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\varphi(\xi)d\xi.$$

Then, for $\theta>0$, the fractional Laplacian is given by

$$(-\Delta)^{\theta/2}\varphi(x) = \mathcal{F}^{-1}\left[|\xi|^\theta\mathcal{F}[\varphi]\right](x).$$

The fractional Laplacian with $\theta=2$ is the positive Laplacian. On the other hand, when $1<\theta<2$, this operator provides the anomalous diffusion on dissipative equations (see [2, 8]). Namely, for the fundamental solution of $\partial_t u + (-\Delta)^{\theta/2}u = 0$, we see the following property.

Lemma 1.1 ([1])**.** *Let $n\in\mathbb{Z}$, $\theta>0$, $C_\theta := \theta 2^{\theta-1}\pi^{-\frac{n}{2}-1}\sin\frac{\theta\pi}{2}\Gamma\left(\frac{n+\theta}{2}\right)\Gamma\left(\frac{\theta}{2}\right)$ and*

$$G_\theta(t,x) := \mathcal{F}^{-1}[e^{-t|\xi|^\theta}](x). \tag{1.2}$$

Then the following property holds:

$$|x|^{n+\theta}G_\theta(t,x)\to C_\theta t \quad \textit{as} \quad |x|\to\infty$$

for any $t>0$.

Here we remark that $G_\theta(t,x)$ is the fundamental solution of $\partial_t u+(-\Delta)^{\theta/2}u=0$. When $\theta=2$, the fundamental solution of $\partial_t u-\Delta u=0$ is given by the Gaussian $G(t,x)=(4\pi t)^{-n/2}e^{-|x|^2/(4t)}$. This satisfies

$$|x|^M G(t,x)\to 0 \quad \text{as} \quad |x|\to\infty$$

for any $t>0$ and $M>0$. This property and Lemma 1.1 are not contradictory. Indeed, when $\theta=2$, we see that $C_\theta=0$ in Lemma 1.1. When $1<\theta<2$, Lemma 1.1 immediately gives that

$$\|x^\alpha G_\theta(t)\|_1 = +\infty$$

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for any $t > 0$ and $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \geq 2$. We study the following linear dissipative equation with anomalous diffusion:

$$(1.3) \qquad \begin{cases} \partial_t u + (-\Delta)^{\theta/2} u + a(t,x)u = 0, & t > 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases}$$

where $n \in \mathbb{N},\ 1 < \theta \leq 2$. When $\theta = 2$, this and (1.1) are equivalent. We assume the following assumption for the coefficient:

$$(1.4) \qquad \|x^\alpha a(t)\|_p \leq C(1+t)^{-\mu + \frac{|\alpha|}{\theta} + \frac{n}{\theta p}}$$

for any $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| < \nu,\ n/(\nu - |\alpha|) < p \leq \infty$ and $t > 0$, where C is a positive constant, and $\mu > 1 + 1/\theta$ and $\nu > 0$ are some parameters. For example, if $a(t,x)$ satisfies the inequality

$$|a(t,x)| \leq C(1+t)^{-\mu} \left\langle (1+t)^{-1/\theta} x \right\rangle^{-\nu}$$

for some $C > 0$ and any $(t,x) \in (0,\infty) \times \mathbb{R}^n$, then the condition (1.4) is satisfied. Here we have used the notation $\langle x \rangle := \sqrt{1+|x|^2}$. We assume that the solution to (1.3) is well-posed in $L^p(\mathbb{R}^n)$ with some $1 \leq p \leq \infty$ and satisfies

$$(1.5) \qquad \|x^\alpha u(t)\|_p \leq C(1+t)^{-\frac{n}{\theta}(1-\frac{1}{p}) + \frac{|\alpha|}{\theta}}$$

for any $t > 0,\ \alpha \in \mathbb{Z}_+^n$ with $|\alpha| < n + \theta$ and $n/(n+\theta-|\alpha|) < p \leq \infty$. Indeed, for some problems, we obtain this inequality. We should remark that the anomalous diffusion causes $\|x^\alpha u(t)\|_1 = +\infty$ when $|\alpha| \geq 2$. Namely a moment of the solution with high-order diverges to infinity. We consider the large-time behavior of the solution to (1.3). Especially we give the estimate on the difference between the solution and its asymptotic expansion as $t \to \infty$. For (1.1) (namely, for (1.3) with $\theta = 2$), Ishige, Ishiwata and Kawakami [6] derived the large-time behavior of the solution completely. In their asymptotic expansion, the coefficients contain the moments of the solution. Unfortunately we cannot extend this idea to our problem since the moments of the solution cannot be defined. We provide the other way to reach our goal. For some $(l,\beta) \in \mathbb{Z}_+ \times \mathbb{Z}_+^n$, we introduce the following notation:

$$(1.6) \qquad \begin{aligned} U_{l,\beta}(t,x) :=& \frac{\partial_t^l \nabla^\beta G_\theta(1+t,x)}{l!\beta!} \left(\int_{\mathbb{R}^n} (-1)^l (-y)^\beta u_0(y) dy \right. \\ & \left. - \int_0^t \int_{\mathbb{R}^n} (-1-s)^l (-y)^\beta (au)(s,y) dy ds \right), \end{aligned}$$

where $G_\theta(t)$ is defined by (1.2). Then we see the following proposition.

Proposition 1.2. *Let $n \in \mathbb{N},\ 1 < \theta \leq 2,\ \mu > 1 + 1/\theta$ and $\nu > 0$. Let (1.4) and (1.5) be satisfied. Assume that $u_0 \in L_N^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ for $N := \max\{m \in \mathbb{Z}_+ \mid m < (\mu-1)\theta, \nu + \theta\}$ and $u(t)$ is the solution to (1.3), where*

$$(1.7) \qquad L_N^1(\mathbb{R}^n) := \left\{ \varphi \in L^1(\mathbb{R}^n) \;\middle|\; |x|^N \varphi \in L^1(\mathbb{R}^n) \right\}.$$

Let

$$(1.8) \qquad \bar{\Lambda}_N := \left\{ (l,\beta) \in \mathbb{Z}_+ \times \mathbb{Z}_+^n \;\middle|\; \theta l + |\beta| \leq N \right\}$$

and the functions $\{U_{l,\beta}(t)\}_{(l,\beta)\in\bar{\Lambda}_N}$ *be given by* (1.6). *Then the following estimate holds:*

$$\left\| u(t) - \sum_{(l,\beta)\in\bar{\Lambda}_N} U_{l,\beta}(t) \right\|_p = o\left(t^{-\frac{n}{\theta}(1-\frac{1}{p})-\frac{N}{\theta}} \right) \quad as \quad t\to\infty$$

for $1 \le p \le \infty$.

When $\theta = 2$, the asymptotic expansion of solution of this type was firstly observed by Escobedo and Zuazua. In [4], they derived the asymptotic expansion of solutions to the heat-convection equation. For the solution of the Navier-Stokes equation, the asymptotic expansion was provided by Carpio [3], and Fujigaki and Miyakawa [5]. The large-time behavior of the solution to the Keller-Segel equation in $L^p(\mathbb{R}^n)$ was considered by Nagai, Syukuinn and Umesako [9], Kato [7], and Nagai and Yamada [10].

Since the conditions (1.4), (1.5) and $N < \min\{(\mu-1)\theta, \nu+\theta\}$ are assumed, the coefficient $\int_0^\infty \int_{\mathbb{R}^n} (-1-s)^l (-y)^\beta (au)(s,y)dyds$ is uniformly integrable when $(l,\beta)\in\bar{\Lambda}_N$ holds. Hence, under this assumption, the asymptotic expansion (1.6) is well-defined. However, when $N \ge (\mu-1)\theta$ or $N \ge \nu+\theta$ is satisfied, some coefficients in the asymptotic expansion diverge to infinity. Thus, in this case, we cannot define the higher-order asymptotic expansion by the form as (1.6). Before proceeding next step, we study the decay-rates of the solution as $|x|\to\infty$. Then we obtain the following proposition.

Proposition 1.3. *Let* $n\in\mathbb{N}$, $1<\theta\le 2$, $\mu > 1+1/\theta$ *and* $\nu > 0$. *Let* (1.4) *and* (1.5) *be satisfied. Assume that* $u_0 \in L^1_N(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ *for* $N := \max\{m\in\mathbb{Z}_+ \mid m < (\mu-1)\theta, \nu+\theta\}$ *and* $u(t)$ *is the solution to* (1.3), *where* $L^1_N(\mathbb{R}^n)$ *is defined by* (1.7). *Let*

$$\Lambda_N := \left\{ (l,\beta)\in\mathbb{Z}_+\times\mathbb{Z}_+^n \;\middle|\; \theta l + |\beta| < N \right\} \tag{1.9}$$

and

$$U_{\Lambda_N}(t,x) := \sum_{(l,\beta)\in\Lambda_N} U_{l,\beta}(t,x), \tag{1.10}$$

where the functions $\{U_{l,\beta}(t)\}_{(l,\beta)\in\Lambda_N}$ *be defined by* (1.6). *Then* $u(t) - U_{\Lambda_N}(t) \in L^1_N(\mathbb{R}^n)$ *holds for any* $t>0$. *Moreover there exists a positive constant* C *such that*

$$\|x^\alpha (u(t) - U_{\Lambda_N}(t))\|_1 \le C(1+t)^{-\frac{N-|\alpha|}{\theta}}$$

for any $t>0$ *and* $\alpha\in\mathbb{Z}_+^n$ *with* $|\alpha|\le N$.

We should remark that, for any $\alpha\in\mathbb{Z}_+^n$ with $|\alpha|\ge 2$, the function $U_{\Lambda_N}(t)$ in Proposition 1.3 satisfies $\|x^\alpha U_{\Lambda_N}(t)\|_1 = \infty$. Thus this function gives the approximation of the solution as $|x|\to\infty$. Proposition 1.2 and 1.3 state that the asymptotic expansion of the solution as $t\to\infty$ and the approximation of the solution as $|x|\to\infty$ are given by the same form. We derive the higher-order asymptotic expansion of the solution by employing this proposition.

Notation. Throughout this manuscript, we use the following notation. For any $x=(x_1,\dots,x_n)$ and $y=(y_1,\dots,y_n)\in\mathbb{R}^n$, we denote $x\cdot y := \sum_{j=1}^n x_j y_j$, $|x|^2 := x\cdot x$ and $\langle x\rangle := \sqrt{1+|x|^2}$. For $1\le p\le\infty$ and $\theta>0$, $L^p(\mathbb{R}^n)$ denotes the Lebesgue spaces and $W^{\theta,p}(\mathbb{R}^n)$ denotes the Sobolev spaces. The norm of $L^p(\mathbb{R}^n)$ is represented by $\|\cdot\|_p$. For $s>0$, we define the weighted $L^1(\mathbb{R}^n)$ space by $L^1_s(\mathbb{R}^n) := \{\varphi\in L^1(\mathbb{R}^n) \mid \|\varphi\|_{L^1_s} := \int_{\mathbb{R}^n}\langle x\rangle^s|\varphi(x)|dx < \infty\}$. For $f=f(x)$ and $g=g(x)$, we denote the convolution by $f * g(x) := \int_{\mathbb{R}^n} f(x-y)g(y)dy$. The gamma function $\Gamma = \Gamma(p)$ for $p > 0$ is provided by $\Gamma(p) := \int_0^\infty e^{-t}t^{p-1}dt$. Varias constants are simply denoted by C.

2. Preliminaries

Before stating our results, we study some basic properties of $G_\theta(t) := \mathcal{F}^{-1}[e^{-t|\xi|^\theta}]$.

Lemma 2.1. *For any $l \in \mathbb{Z}_+$ and $\alpha \in \mathbb{Z}_+^n$, there exists a positive constant $C > 0$ such that*

$$\left|\partial_t^l \nabla^\alpha G_\theta(t,x)\right| \le Ct^{-\frac{n}{\theta}-l-\frac{|\alpha|}{\theta}} \left\langle t^{-1/\theta}x\right\rangle^{-n-\theta-\theta l-|\alpha|}$$

for any $(t,x) \in (0,\infty) \times \mathbb{R}^n$.

Proof. This lemma is proved by employing [11, Theorem 3.1.]. Here we omit the proof. □

By applying Taylor's formula, Lemma 2.1 and Hausdorf-Young's inequality, we obtain the following lemma.

Lemma 2.2. *Let $N \in \mathbb{Z}_+$ and $u_0 \in L^1_N(\mathbb{R}^n)$. Then the following estimate holds for any $1 \le p \le \infty$:*

$$\left\|G_\theta(t) * u_0 - \sum_{(l,\beta)\in\bar{\Lambda}_N} \frac{\partial_t^l \nabla^\beta G_\theta(1+t)}{l!\beta!} \int_{\mathbb{R}^n} (-1)^l (-y)^\beta u_0(y)dy\right\|_p = o\left(t^{-\frac{n}{\theta}(1-\frac{1}{p})-\frac{N}{\theta}}\right)$$

as $t \to \infty$, where $\bar{\Lambda}_N$ is defined by (1.8).

When we study the decay of $G_\theta(t) * u_0(x)$ as $|x| \to \infty$, we obtain the approximation of this by the same form as in Lemma 2.2.

Lemma 2.3. *Let $N \in \mathbb{Z}_+$ and $u_0 \in L^1_N(\mathbb{R}^n)$. Then the following inequality holds for any $t > 0$ and $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \le N$:*

$$\begin{aligned}&\left\|x^\alpha\left(G_\theta(t) * u_0 - \sum_{(l,\beta)\in\Lambda_N} \frac{\partial_t^l \nabla^\beta G_\theta(1+t)}{l!\beta!} \int_{\mathbb{R}^n} (-1)^l (-y)^\beta u_0(y)dy\right)\right\|_1 \\ \le& C(1+t)^{-\frac{N-|\alpha|}{\theta}},\end{aligned}$$

where C is a positive constant and Λ_N is defined by (1.9).

3. Main Results

In order to derive the asymptotic expansion of the solution, we introduce the following corresponding integral equation:

$$u(t) = G_\theta(t) * u_0 - \int_0^t G_\theta(t-s) * (au)(s)ds. \tag{3.1}$$

The solution to (3.1) is called the mild solution of (1.3). Generally speaking a mild solution solves an original Cauchy problem if it has sufficiently high regularity. Hereafter we consider the mild solution for deriving the asymptotic expansion of the solution of (1.3). For $N = \min\{m \in \mathbb{Z}_+ \mid m < (\mu-1)\theta, \nu+\theta\}$, the nonlinear term on the right hand side of (3.1) is split as

$$\int_0^t G_\theta(t-s) * (au)(s)ds = \sum_{(l,\beta)\in\Lambda_N} J_{l,\beta}(t) + \int_0^t G_\theta(t-s) * \left(a(u - U_{\Lambda_N})\right)(s)ds,$$

where

$$J_{l,\beta}(t,x) := \int_0^t G_\theta(t-s) * (aU_{l,\beta})(s)ds \tag{3.2}$$

for $(l,\beta) \in \Lambda_N$ and $U_{l,\beta}(t)$ is defined by (1.6). We introduce the function $\tilde{U}_{l,\beta} = \tilde{U}_{l,\beta}(t,x)$ by

$$\begin{aligned}\tilde{U}_{l,\beta}(t,x) :=& \frac{\partial_t^l \nabla^\beta G_\theta(1+t,x)}{l!\beta!}\bigg(\int_{\mathbb{R}^n} (-1)^l (-y)^\beta u_0(y)dy \\ & - \int_0^t \int_{\mathbb{R}^n} (-1-s)^l (-y)^\beta \left(a(u-U_{\Lambda_N})\right)(s,y)dyds\bigg)\end{aligned} \tag{3.3}$$

for some $(l,\beta) \in \mathbb{Z}_+ \times \mathbb{Z}_+^n$, where $U_{\Lambda_N}(t)$ is defined by (1.10). By employing the assumption (1.4) and Proposition 1.3, we see that $\tilde{U}_{l,\beta}(t)$ is well-defined when $(l,\beta) \in \bar{\Lambda}_{2N}$ holds, where $\bar{\Lambda}_{2N}$ is defined as (1.8). Then we give our main result in the following theorems.

Theorem 3.1. *Let $n \in \mathbb{N}$, $1 < \theta \le 2$, $\mu > 1 + 1/\theta$ and $\nu > 0$. Let (1.4) and (1.5) be satisfied. Assume that $u_0 \in L^1_{2N}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ for $N := \max\{m \in \mathbb{Z}_+ \mid m < (\mu-1)\theta, \nu+\theta\}$ and $u(t)$ is the solution to (1.3), where $L^1_{2N}(\mathbb{R}^n)$ is defined as (1.7). Let Λ_N and $\bar{\Lambda}_{2N}$ be defined as (1.8) and (1.9), and the functions $\{J_{k,\alpha}(t)\}_{(k,\alpha)\in\Lambda_N}$ and $\{\tilde{U}_{l,\beta}(t)\}_{(l,\beta)\in\bar{\Lambda}_{2N}}$ be given by (3.2) and (3.3). Then the following estimate holds:*

$$\left\| u(t) - \sum_{(k,\alpha)\in\Lambda_N} J_{k,\alpha}(t) - \sum_{(l,\beta)\in\bar{\Lambda}_{2N}} \tilde{U}_{l,\beta}(t) \right\|_p = o\left(t^{-\frac{n}{\theta}(1-\frac{1}{p})-\frac{2N}{\theta}}\right) \quad as \quad t \to \infty$$

for $1 \le p \le \infty$.

Theorem 3.2. *Let $n \in \mathbb{N}$, $1 < \theta \le 2$, $\mu > 1 + 1/\theta$ and $\nu > 0$. Let (1.4) and (1.5) be satisfied. Assume that $u_0 \in L^1_{2N}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ for $N := \max\{m \in \mathbb{Z}_+ \mid m < (\mu-1)\theta, \nu+\theta\}$ and $u(t)$ is the solution to (1.3), where $L^1_{2N}(\mathbb{R}^n)$ is defined as (1.7). Let Λ_N and Λ_{2N} be defined as (1.9) and the functions $\{J_{k,\alpha}(t)\}_{(k,\alpha)\in\Lambda_N}$ and $\{\tilde{U}_{l,\beta}(t)\}_{(l,\beta)\in\Lambda_{2N}}$ be given by (3.2) and (3.3). Then*

$$u(t) - \sum_{(k,\alpha)\in\Lambda_N} J_{k,\alpha}(t) - \sum_{(l,\beta)\in\Lambda_{2N}} \tilde{U}_{l,\beta}(t) \in L^1_{2N}(\mathbb{R}^n)$$

holds for any $t > 0$. Moreover there exists a positive constant C such that:

$$\left\| x^\alpha \left(u(t) - \sum_{(k,\alpha)\in\Lambda_N} J_{k,\alpha}(t) - \sum_{(l,\beta)\in\Lambda_{2N}} \tilde{U}_{l,\beta}(t) \right) \right\|_1 \le C(1+t)^{-\frac{2N-|\alpha|}{\theta}}$$

for any $t > 0$ and $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \le 2N$.

For $N := \max\{m \in \mathbb{Z}_+ \mid m < (\mu-1)\theta, \nu+\theta\}$, Proposition 1.2 and 1.3 give the Nth-order asymptotic expansion of the solution. Theorem 3.1 and 3.2 provide the $2N$th-order asymptotic expansion. We prove those theorems by employing Proposition 1.2 and 1.3. Similarly, by applying Theorem 3.1 and 3.2, we obtain a $3N$th-order asymptotic expansion. By repeating this procedure, we can derive an mNth-order asymptotic expansion for arbitrary large $m \in \mathbb{Z}_+$.

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