Uniqueness and non-degeneracy of ground states of quasilinear Schrödinger equations

Tatsuya Watanabe (Kyoto Sangyo University) [1]

1 Introduction

We consider the following quasilinear elliptic problem:

$$-\Delta u + \lambda u - \kappa \Delta(|u|^\alpha)|u|^{\alpha-2}u = |u|^{p-1}u \text{ in } \mathbb{R}^N, \tag{1}$$

where $\lambda > 0$, $\kappa > 0$, $\alpha > 1$, $p > 1$ and $N \geq 1$. Equation (1) can be obtained as a stationary problem of the following modified Schrödinger equation:

$$i\frac{\partial z}{\partial t} = -\Delta z - \kappa \Delta(|z|^\alpha)|z|^{\alpha-2}z - |z|^{p-1}z, \ (t,x) \in (0,\infty) \times \mathbb{R}^N. \tag{2}$$

Equation (2) appears in the study of plasma physics. See [7], [10] for the derivation and the background. Especially if we consider the standing wave of (2) of the form $z(t,x) = u(x)e^{i\lambda t}$, then $u(x)$ satisfies (1).

Equation (1) has a variational structure, that is, one can obtain solutions of (1) as critical points of the associated functional I defined by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2(1 + \alpha\kappa |u|^{2\alpha-2}) + \lambda u^2 \, dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1} \, dx.$$

We remark that nonlinear functional $\displaystyle\int_{\mathbb{R}^N} |\nabla u|^2 |u|^{2\alpha-2} \, dx$ is not defined on all $H^1(\mathbb{R}^N)$ except for $N = 1$. Thus the natural function space for $N \geq 2$ is given by

$$X := \{u \in H^1(\mathbb{R}^N); \int_{\mathbb{R}^N} |\nabla u|^2 |u|^{2\alpha-2} \, dx < \infty\}.$$

Existence of a solution of (1) has been studied in [1], [8], [11], [12], [14]. We are interested in the *ground state* of (1). We define the ground state energy level and the set of ground states by

$$m := \inf\{I(u); \, I'(u) = 0, \ u \in X \setminus \{0\}\},$$

$$\mathcal{G} := \{u \in X \setminus \{0\}; \, I(u) = m, \ I'(u) = 0\}.$$

As to the existence of a ground state, we have the following result.

[1] This talk is based on joint works [2], [3], [4] with Shinji Adachi (Shizuoka University) and Masataka Shibata (Tokyo Institute of Technology).

Theorem 1.1. *([2], [9]) Let* $\lambda > 0,\ \kappa > 0,\ \alpha > 1$ *and* $1 < p < \frac{(2\alpha-1)N+2}{N-2}$ *for* $N \geq 3$, $1 < p < \infty$ *for* $N = 1, 2$. *Then* $\mathcal{G} \neq \emptyset$. *Moreover any ground state* $w \in \mathcal{G}$ *is of the class* $C^2(\mathbb{R}^N)$, *positive, radially symmetric and decreasing with respect to* $r = |x|$ *(up to translation).*

We note that the ground state of (1) exists even if p is H^1-supercritical because $\frac{(2\alpha-1)N+2}{N-2} > \frac{N+2}{N-2}$. We can also see that $p = \frac{(2\alpha-1)N+2}{N-2}$ is the critical exponent for (1) by the Pohozaev type identity.

Remark 1.2. *As to the existence of a ground state, we have more general result. More precisely, we consider the following equation:*

$$-\Delta u - \kappa\Delta(|u|^{\alpha})|u|^{\alpha-2}u = g(u) \text{ in } \mathbb{R}^N. \tag{3}$$

We impose the following conditions on the nonlinear term g*:*
(g1) $g(s)$ *is real-valued and locally Hölder continuous on* $[0, \infty)$.
(g2) $-\infty < \liminf_{s\to 0} \dfrac{g(s)}{s} \leq \limsup_{s\to 0} \dfrac{g(s)}{s} = -\lambda < 0$ *for some* $\lambda > 0$.
(g3) $\lim_{s\to\infty} \dfrac{|g(s)|}{s^{\frac{(2\alpha-1)N+2}{N-2}}} = 0$.
(g4) There exists $s_0 > 0$ *such that* $G(s_0) > 0$, *where* $G(s) = \int_0^s g(t)\,dt$.

Under (g1)-(g4), we can prove the existence of a ground state of (3).

On the other hand, the uniqueness and the non-degeneracy of the ground state are less investigated. When $N = 1$, Ambrosetti and Wang [5] showed that there exists $\kappa^* > 0$ such that the non-degeneracy holds for any $\kappa > -\kappa^*$, $\lambda > 0$ and $p > 1$. In [9], the authors studied the case $N = 1, \kappa = 1$ and proved that the uniqueness holds for any $\lambda > 0$ and $p > 1$. Their argument is based on the ODE analysis. The aim of this talk is to give the uniqueness and non-degeneracy in the higher dimensional case. We believe it is important for applications, for example, the stability of the standing wave.

2 Main results

Theorem 2.1. *(Uniqueness for large* κ*)*

Suppose $N \geq 3$, $\alpha > 1$ *and* $1 < p < \frac{(2\alpha-1)N+2}{N-2}$ *if* $1 < \alpha \leq 2$, $\alpha - 1 \leq p < \frac{(2\alpha-1)N+2}{N-2}$ *if* $\alpha > 2$. *There exists* $c_0 = c_0(p, \alpha) > 0$ *such that if* $\kappa\lambda^{\frac{2\alpha-2}{p-1}} \geq c_0$, *then* (1) *has at most one positive radial solution* w *and hence the ground state of* (1) *is unique up to translation. In other words, it follows that*

$$\mathcal{G} = \{w(\cdot - y); y \in \mathbb{R}^N\}.$$

Remark 2.2. *For a solution u of (1), we rescale $\tilde{u}(x)$ as $u(x) = \lambda^{\frac{1}{p-1}}\tilde{u}(\lambda^{\frac{1}{2}}x)$. Then we can see that (1) is reduced to*

$$-\Delta\tilde{u} + \tilde{u} - \kappa\lambda^{\frac{2\alpha-2}{p-1}}\Delta(|\tilde{u}|^{\alpha})|\tilde{u}|^{\alpha-2}\tilde{u} = |\tilde{u}|^{p-1}\tilde{u} \quad \text{in } \mathbb{R}^N.$$

Thus it seem to be natural to describe the condition for the uniqueness in terms of $\kappa\lambda^{\frac{2\alpha-2}{p-1}}$.

Theorem 2.3. *Suppose $N = 2$, $\alpha > 1$ and $2\alpha - 1 \le p < \infty$. There exists $c_1 = c_1(p,\alpha) > 0$ such that if $\kappa\lambda^{\frac{2\alpha-2}{p-1}} \ge c_1$, then the ground state of (1) is unique up to translation.*

Theorem 2.4. *(Non-degeneracy for large κ)*
Suppose $N \ge 3$, $\alpha > 1$ and $2\alpha - 1 \le p < \frac{(2\alpha-1)N+2}{N-2}$. Assume further $\kappa\lambda^{\frac{2\alpha-2}{p-1}} \ge c_0$ where c_0 is given in Theorem 2.1. Then w is non-degenerate in $H^1_{rad}(\mathbb{R}^N)$, that is, if $L_w(\phi) = 0$ in $\mathbb{R}^N$ and $\phi \in H^1_{rad}(\mathbb{R}^N)$, then $\phi \equiv 0$.
Here L_w is the linearized operator of (1) defined by

$$\begin{aligned} L_w(\phi) = & -\Delta\phi + \lambda\phi - pw^{p-1}\phi - \kappa \operatorname{div}(\alpha w_\kappa^{2\alpha-2}\nabla\phi) \\ & - \kappa(2\alpha(\alpha-1)w_\kappa^{2\alpha-3}\Delta w_\kappa + \alpha(\alpha-1)(2\alpha-3)w_\kappa^{2\alpha-4}|\nabla w_\kappa|^2)\phi. \end{aligned}$$

Theorem 2.5. *(Uniqueness and non-degeneracy for small κ)*
Suppose $N \ge 2$, $\alpha > 1$ and $1 < p < \frac{N+2}{N-2}$ if $N \ge 3$, $1 < p < \infty$ if $N = 2$. There exists $c_2(p,\alpha) > 0$ such that if $0 < \kappa\lambda^{\frac{2\alpha-2}{p-1}} \le c_2$, then the ground state of (1) is unique up to translation and non-degenerate in $H^1_{rad}(\mathbb{R}^N)$.

Here we briefly explain the ideas of the proof. Firstly we adapt the *dual variational formulation.* Let f be a unique solution of the following ODE:

$$f'(s) = \frac{1}{\sqrt{1 + \alpha\kappa f(s)^{2\alpha-2}}} \quad \text{on } s \in [0,\infty), \quad f(0) = 0.$$

Using the function f, we consider the following semilinear problem:

$$-\Delta v + \lambda f(v)f'(v) = |f(v)|^{p-1}f(v)f'(v) \quad \text{in } \mathbb{R}^N. \tag{4}$$

The functional associated to (4) is defined by

$$J(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \lambda f(v)^2\,dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |f(v)|^{p+1}\,dx.$$

Lemma 2.6. *It follows*
(i) $X = f(H^1(\mathbb{R}^N))$, that is, $X = \{f(v); v \in H^1(\mathbb{R}^N)\}$.
(ii) For any $v \in H^1(\mathbb{R}^N)$, we put $u = f(v)$. Then it follows

$$I(u) = J(v), I'(u)u = J'(v)\frac{f(v)}{f'(v)}.$$

By Lemma 2.6, we can see that the set of ground states $\mathcal{G}$ has one-to-one correspondence to that of the semilinear problem (4). This enables us to apply the uniqueness and non-degeneracy result [6], [13], [15] for semilinear elliptic equations. We require that $\kappa\lambda^{\frac{2\alpha-2}{p-1}}$ is large in order to guarantee some monotonicity condition on the nonlinear term.

On the other hand if we fix λ and put $\kappa = 0$, (1) becomes

$$-\Delta u + \lambda u = |u|^{p-1}u \text{ in } \mathbb{R}^N. \tag{5}$$

Then for $1 < p < \frac{N+2}{N-2}$, it is well-known that the ground state is unique up to translation. Moreover the corresponding linearized operator $L_0 = -\Delta + \lambda - pu^{p-1}$ satisfies Ker $L_0 = \text{span}\left\{\frac{\partial u}{\partial x_i}\right\}$. The uniqueness and the non-degeneracy for small κ follows by applying the implicit function theorem if we could treat the linearized operator L_w as a perturbation of L_0. To this aim, we have to show L^∞-norm of the ground state is uniformly bounded with respect to κ. The proof of uniform boundedness is based on the Moser type iteration. We also need to show the following uniform estimate whose proof is given by the ODE analysis.

Lemma 2.7. *Suppose $N \geq 2$, $\alpha > 1$ and $1 < p < \frac{N+2}{N-2}$ if $N \geq 3$, $1 < p < \infty$ if $N = 2$. Let $\lambda > 0$ be given. There exist $\kappa_0 > 0$ and $C > 0$ independent of $\kappa \in (0, \kappa_0)$ such that*

$$\|\nabla(\log w)\|_{L^\infty(\mathbb{R}^N)} = \left\|\frac{\nabla w}{w}\right\|_{L^\infty(\mathbb{R}^N)} \leq C \textit{ for all } \kappa \in (0, \kappa_0).$$

References

[1] S. Adachi, T. Watanabe, *G-invariant positive solutions for a quasilinear Schrödinger equation*, Adv. Diff. Eqns. **16** (2011), 289-324.

[2] S. Adachi, T. Watanabe, *Uniqueness of the ground state solutions of quasilinear Schrödinger equations*, Nonlinear Anal. **75** (2012), 819-833.

[3] S. Adachi, T. Watanabe, *Asymptotic properties of ground states of quasilinear Schrödinger equations with H^1-subcritical exponent*, Adv. Nonlinear Stud. **12** (2012), 255-279.

[4] S. Adachi, M. Shibata, T. Watanabe, *Asymptotic behavior of positive solutions for a class of quasilinear elliptic equations with general nonlinearities*, preprint.

[5] A. Ambrosetti and Z. Q. Wang, *Positive solutions to a class of quasilinear elliptic equations on* $\mathbb{R}$, Disc. Cont. Dyn. Syst. **9** (2003), 55-68.

[6] P. Bates, J. Shi, *Existence and instability of spike layer solutions to singular perturbation problems*, J. Funct. Anal. **196** (2002), 429–482.

[7] L. Brizhik, A. Eremko, B. Piette, W. J. Zakrzewski, *Electron self-trapping in a discrete two-dimensional lattice*, Physica D **159** (2001), 71–90.

[8] M. Colin, L. Jeanjean, *Solutions for a quasilinear Schrödinger equation: a dual approach*, Nonlinear Anal. TMA. **56** (2004), 213–226.

[9] M. Colin, L. Jeanjean, M. Squassina, *Stability and instability results for standing waves of quasi-linear Schrödinger equations*, Nonlinearity. **23** (2010), 1353-1385.

[10] S. Kurihara, *Large-amplitude quasi-solitons in superfluid films*, J. Phys. Soc. Japan **50** (1981), 3262–3267.

[11] J.-Q. Liu, Y.-Q. Wang, Z.-Q. Wang, *Soliton solutions for quasilinear Schrödinger equations II*, J. Diff. Eqns. **187** (2003), 473-493.

[12] J.-Q. Liu, Y.-Q. Wang, Z.-Q. Wang, *Solutions for quasi-linear Schrödinger equations via the Nehari method*, Comm. PDE **29** (2004), 879–901.

[13] K. Mcleod, J. Serrin, *Uniqueness of positive radial solutions of* $\Delta u + f(u) = 0$ *in* $\mathbb{R}^N$, Arch. Rat. Mech. Anal. **99** (1987), 115–145.

[14] M. Poppenberg, K. Schmitt, Z.-Q. Wang, *On the existence of soliton solutions to quasilinear Schrödinger equations*, Calc. Var. PDE **14** (2002), 329–344.

[15] J. Serrin, M. Tang, *Uniqueness of ground states for quasilinear elliptic equations*, Indiana Univ. Math. J. **49** (2000), 897–923.