Total lightcone curvatures of spacelike submanifolds in Lorentz-Minkowski space

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Abstract

We introduce the totally absolute lightcone curvature for a spacelike submanifold with general codimension and investigate global properties of this curvature. One of the consequences is that the Chern-Lashof type inequality holds. Then the notion of lightlike tightness is naturally induced. Moreover, the lightcone Willmore conjecture is proposed.

1 Introduction

In this paper we consider global properties of spacelike submanifolds in Lorentz-Minkowski space. The study of the extrinsic differential geometry of submanifolds in Lorentz-Minkowski space is of interest in the special relativity theory. Moreover, it is a natural generalization of the extrinsic geometry of submanifolds in Euclidean space. In [11], it was considered the case for codimension two spacelike submanifolds. The normalized lightcone Gauss map was introduced which plays the similar role to the Gauss map of a hypersurface in the Euclidean space. For example, the Gauss-Bonnet type theorem holds for the corresponding Gauss-Kronecker curvature (cf., [11, Theorem 6.5]). Moreover, we recently discovered a new geometry on the hyperbolic space which is different from the Gauss-Bolyai-Lobachevskii geometry (i.e., the hyperbolic geometry) [1, 2, 6, 9]. We call this new geometry the *horospherical geometry*. The horospherical Gauss map (or, the hyperbolic Gauss map) is one of the key notions in the horospherical geometry. We also showed that the Gauss-Bonnet type theorem holds for the horospherical Gauss-Kronecker curvature[9]. The notion of the normalized lightcone Gauss map unifies the both of the notion of Gauss maps in the Euclidean space and the notion of horospherical Gauss maps in the hyperbolic space.

In this paper we generalize the normalized lightcone Gauss map and the corresponding curvatures for general spacelike submanifolds in Lorentz-Minkowski space. If we try to develop this theory as a direct analogy to the Euclidean case, there exist several problems. The main problem is that the fiber of the unit normal bundle of a spacelike submanifold is a union of the

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pseudo-spheres which is not only non-compact but also non-connected. So, we can not integrate the curvatures along the fiber at each point. Therefore, we cannot define the Lipschitz-Killing curvature analogous to the Euclidean case directly [5]. In order to avoid this problem, we arbitrary choose a future directed unit normal vector field along the submanifold and consider the pseudo-orthonormal space of this timelike vector on each fiber of the normal bundle. Then we obtain a spacelike codimension two unit normal sphere bundle in the normal bundle over the submanifold whose fiber is the Euclidean sphere. As a consequence, we define the lightcone Lipschitz-Killing curvature and the total absolute lightcone curvature at each point. We can show that the total absolute lightcone curvature is independent of the choice of the unit future directed timelike normal vector field (cf., Lemma 6.2). Then we show that the Chern-Lashof type inequality holds for this curvature (cf, §7). In §8 we consider codimension two spacelike submanifolds. In this case the situation is different from the higher codimensional case. We have two different lightcone Gauss-Kronecker curvatures at each point. The corresponding total absolute lightcone Gauss-Kronecker curvatures are also different (cf., the remark after Theorem 8.3). However, we also have the Chern-Lashof type inequality for each total absolute Gauss-Kronecker curvature. Moreover, we consider the Willmore type integral (cf., [15, Theorem 7.2.2]) of the lightcone mean curvature for spacelike surface in Lorentz-Minkowski 4-space. Then we propose the lightcone Willmore conjecture for spacelike embedded torus which is a generalized version of the original Willmore conjecture (cf., Remark 8.7). Finally, we introduce the notion of the lightlike tightness which characterize the minimal value of the total absolute lightcone curvature. As a special case, we have the horo-spherical Chern-Lashof type inequality and horo-tight immersions in the hyperbolic space [1, 2, 14]. Motivated by those arguments, we can introduce the notion of several kinds of tightness and tautness depending on the causal characters which will be one of the subjects of a future program of the research.

2 Basic concepts in Lorentz-Minkowski space

We introduce in this section some basic notions on Lorentz-Minkowski $n+1$-space. For basic concepts and properties, see [13].

Let $\mathbb{R}^{n+1} = \{(x_0, x_1, \dots, x_n) \mid x_i \in \mathbb{R}\ (i = 0, 1, \dots, n)\ \}$ be an $n+1$-dimensional cartesian space. For any $\boldsymbol{x} = (x_0, x_1, \dots, x_n),\ \boldsymbol{y} = (y_0, y_1, \dots, y_n) \in \mathbb{R}^{n+1}$, the *pseudo scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0 y_0 + \sum_{i=1}^{n} x_i y_i.$$

We call $(\mathbb{R}^{n+1}, \langle, \rangle)$ *Lorentz-Minkowski* $n+1$*-space* (or, simply *Minkowski* $n+1$*-space*. We write $\mathbb{R}^{n+1}_1$ instead of $(\mathbb{R}^{n+1}, \langle, \rangle)$. We say that a non-zero vector $\boldsymbol{x} \in \mathbb{R}^{n+1}_1$ is *spacelike, lightlike or timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$, $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ or $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$ respectively. The norm of the vector $\boldsymbol{x} \in \mathbb{R}^{n+1}_1$ is defined to be $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$. We have the canonical projection $\pi : \mathbb{R}^{n+1}_1 \longrightarrow \mathbb{R}^n$ defined by $\pi(x_0, x_1, \dots, x_n) = (x_1, \dots, x_n)$. Here we identify $\{\boldsymbol{0}\} \times \mathbb{R}^n$ with $\mathbb{R}^n$ and it is considered as Euclidean n-space whose scalar product is induced from the pseudo scalar product $\langle, \rangle$. For a vector $\boldsymbol{v} \in \mathbb{R}^{n+1}_1$ and a real number c, we define a *hyperplane with pseudo normal* $\boldsymbol{v}$ by

$$HP(\boldsymbol{v}, c) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{x}, \boldsymbol{v} \rangle = c\ \}.$$

We call $HP(\boldsymbol{v}, c)$ a *spacelike hyperplane*, a *timelike hyperplane* or a *lightlike hyperplane* if $\boldsymbol{v}$ is timelike, spacelike or lightlike respectively.

We now define *Hyperbolic n-space* by

$$H^n(-1) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1\}$$

and *de Sitter n-space* by

$$S^n_1 = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\ \}.$$

We define

$$LC^* = \{\boldsymbol{x} = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}_1\ | x_0 \neq 0,\ \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0\}$$

and we call it *the* (*open*) *lightcone* at the origin.

If $\boldsymbol{x} = (x_0, x_1, \ldots, x_2)$ is a non-zero lightlike vector, then $x_0 \neq 0$. Therefore we have

$$\widetilde{\boldsymbol{x}} = \left(1, \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) \in S^{n-1}_+ = \{\boldsymbol{x} = (x_0, x_1, \ldots, x_n) \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0,\ x_0 = 1\}.$$

We call S^{n-1}_+ the *lightcone unit* $n-1$*-sphere.*

For any $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n \in \mathbb{R}^{n+1}_1$, we define a vector $\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_n$ by

$$\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_n = \begin{vmatrix} -\boldsymbol{e}_0 & \boldsymbol{e}_1 & \cdots & \boldsymbol{e}_n \\ x^1_0 & x^1_1 & \cdots & x^1_n \\ x^2_0 & x^2_1 & \cdots & x^2_n \\ \vdots & \vdots & \cdots & \vdots \\ x^n_0 & x^n_1 & \cdots & x^n_n \end{vmatrix},$$

where $\boldsymbol{e}_0, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ is the canonical basis of $\mathbb{R}^{n+1}_1$ and $\boldsymbol{x}_i = (x^i_0, x^i_1, \ldots, x^i_n)$. We can easily check that

$$\langle \boldsymbol{x}, \boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_n \rangle = \det(\boldsymbol{x}, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n),$$

so that $\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_n$ is pseudo orthogonal to any $\boldsymbol{x}_i$ $(i = 1, \ldots, n)$.

3 Differential geometry on spacelike submanifolds

In this section we introduce the basic geometrical framework for the study of spacelike submanifolds in Minkowski $n+1$-space analogous to the case of codimension two in [11]. Let $\mathbb{R}^{n+1}_1$ be an oriented and time-oriented space. We choose $\boldsymbol{e}_0 = (1, 0, \ldots, 0)$ as the future timelike vector field. Let $\boldsymbol{X} : U \longrightarrow \mathbb{R}^{n+1}_1$ be a spacelike embedding of codimension k, where $U \subset \mathbb{R}^s$ $(s+k = n+1)$ is an open subset. We also write $M = \boldsymbol{X}(U)$ and identify M and U through the embedding $\boldsymbol{X}$. We say that $\boldsymbol{X}$ is *spacelike* if the tangent space T_pM of M at p is a spacelike subspace (i.e., consists of spacelike vectors) for any point $p \in M$. For any $p = \boldsymbol{X}(u) \in M \subset \mathbb{R}^{n+1}_1$, we have

$$T_pM = \langle \boldsymbol{X}_{u_1}(u), \ldots, \boldsymbol{X}_{u_s}(u) \rangle_{\mathbb{R}}.$$

Let $N_p(M)$ be the pseudo-normal space of M at p in $\mathbb{R}^{n+1}_1$. Since T_pM is a spacelike subspace of $T_p\mathbb{R}^{n+1}_1$, $N_p(M)$ is a k-dimensional Lorentzian subspace of $T_p\mathbb{R}^{n+1}_1$ (cf.,[13]). On the pseudo-normal space $N_p(M)$, we have two kinds of pseudo spheres:

$$\begin{aligned} N_p(M; -1) &= \{\boldsymbol{v} \in N_p(M) \mid \langle \boldsymbol{v}, \boldsymbol{v} \rangle = -1\ \} \\ N_p(M; 1) &= \{\boldsymbol{v} \in N_p(M) \mid \langle \boldsymbol{v}, \boldsymbol{v} \rangle = 1\ \}, \end{aligned}$$

so that we have two unit spherical normal bundles over M:

$$N(M;-1)=\bigcup_{p\in M} N_p(M;-1) \text{ and } N(M;1)=\bigcup_{p\in M} N_p(M;1).$$

Then we have the Whitney sum decomposition

$$T\mathbb{R}^{n+1}_1|M = TM \oplus N(M).$$

Since $M=\boldsymbol{X}(U)$ is spacelike, $\boldsymbol{e}_0$ is a transversal future directed timelike vector field along M. For any $\boldsymbol{v}\in T_p\mathbb{R}^{n+1}_1|M$, we have $\boldsymbol{v}=\boldsymbol{v}_1+\boldsymbol{v}_2$, where $\boldsymbol{v}_1\in T_pM$ and $\boldsymbol{v}_2\in N_p(M)$. If $\boldsymbol{v}$ is timelike, then $\boldsymbol{v}_2$ is timelike. Let $\pi_{N(M)}: T\mathbb{R}^{n+1}_1|M\longrightarrow N(M)$ be the canonical projection. Then $\pi_{N(M)}(\boldsymbol{e}_0)$ is a future directed timelike normal vector field along M. So we always have a future directed unit timelike normal vector field along M (even globally). We now arbitrarily choose a future directed unit timelike normal vector field $\boldsymbol{n}^T(u)\in N_p(M;-1)$, where $p=\boldsymbol{X}(u)$. Therefore we have the pseudo-orthonormal compliment $(\langle \boldsymbol{n}^T(u)\rangle_{\mathbb{R}})^{\perp}$ in $N_p(M)$ which is a $k-1$-dimensional subspace of $N_p(M)$. We can also choose a pseudo-normal section $\boldsymbol{n}^S(u)\in(\langle \boldsymbol{n}^T(u)\rangle_{\mathbb{R}})^{\perp}\cap N(M;1)$ at least locally, then we have $\langle \boldsymbol{n}^S,\boldsymbol{n}^S\rangle=1$ and $\langle \boldsymbol{n}^S,\boldsymbol{n}^T\rangle=0$. We define a $k-1$-dimensional spacelike unit sphere in $N_p(M)$ by

$$N_1(M)_p[\boldsymbol{n}^T]=\{\boldsymbol{\xi}\in N_p(M;1)\mid \langle\boldsymbol{\xi},\boldsymbol{n}(p)\rangle=0\ \}.$$

Then we have a *spacelike unit $k-1$-spherical bundle over M with respect to $\boldsymbol{n}^T$* defined by

$$N_1(M)[\boldsymbol{n}^T]=\bigcup_{p\in M} N_1(M)_p[\boldsymbol{n}^T].$$

Since we have $T_{(p,\xi)}N_1(M)[\boldsymbol{n}^T]=T_pM\times T_\xi N_1(M)_p[\boldsymbol{n}^T]$, we have the canonical Riemannian metric on $N_1(M)[\boldsymbol{n}^T]$. We denote the Riemannian metric on $N_1(M)[\boldsymbol{n}^T]$ by $(G_{ij}(p,\boldsymbol{\xi}))_{1\leqslant i,j\leqslant n-1}$.

For any future directed unit normal $\boldsymbol{n}^T$ along M, we arbitrary choose the unit spacelike normal vector field $\boldsymbol{n}^S$ with $\boldsymbol{n}^S(u)\in N_1(M)_p[\boldsymbol{n}^T]$, where $p=\boldsymbol{X}(u)$. We call $(\boldsymbol{n}^T,\boldsymbol{n}^S)$ a *future directed pair* along M. Clearly, the vectors $\boldsymbol{n}^T(u)\pm\boldsymbol{n}^S(u)$ are lightlike. Here we choose $\boldsymbol{n}^T+\boldsymbol{n}^S$ as a lightlike normal vector field along M. We define a mapping

$$\mathbb{LG}(\boldsymbol{n}^T,\boldsymbol{n}^S):U\longrightarrow LC^*$$

by $\mathbb{LG}(\boldsymbol{n}^T,\boldsymbol{n}^S)(u)=\boldsymbol{n}^T(u)+\boldsymbol{n}^S(u)$. We call it the *lightcone Gauss image* of $M=\boldsymbol{X}(U)$ with respect to $(\boldsymbol{n}^T,\boldsymbol{n}^S)$. We also define a mapping

$$\widetilde{\mathbb{LG}}(\boldsymbol{n}^T,\boldsymbol{n}^S):U\longrightarrow S^{n-1}_+$$

by $\widetilde{\mathbb{LG}}(\boldsymbol{n}^T,\boldsymbol{n}^S)(u)=\boldsymbol{n}^T\widetilde{(u)+\boldsymbol{n}}^S(u)$ which is called the *lightcone Gauss map* of $M=\boldsymbol{X}(U)$ with respect to $(\boldsymbol{n}^T,\boldsymbol{n}^S)$. Under the identification of M and U through $\boldsymbol{X}$, we have the linear mapping provided by the derivative of the lightcone Gauss image $\mathbb{LG}(\boldsymbol{n}^T,\boldsymbol{n}^S)$ at each point $p\in M$,

$$d_p\mathbb{LG}(\boldsymbol{n}^T,\boldsymbol{n}^S):T_pM\longrightarrow T_p\mathbb{R}^{n+1}_1=T_pM\oplus N_p(M).$$

Consider the orthogonal projections $\pi^t:T_pM\oplus N_p(M)\to T_p(M)$ and $\pi^n:T_p(M)\oplus N_p(M)\to N_p(M)$. We define

$$d_p\mathbb{LG}(\boldsymbol{n}^T,\boldsymbol{n}^S)^t=\pi^t\circ d_p(\boldsymbol{n}^T+\boldsymbol{n}^S)$$

and

$$d_p\mathbb{LG}(\boldsymbol{n}^T,\boldsymbol{n}^S)^n = \pi^n \circ d_p(\boldsymbol{n}^T+\boldsymbol{n}^S).$$

We respectively call the linear transformations $S_p(\boldsymbol{n}^T,\boldsymbol{n}^S) = -d_p\mathbb{LG}(\boldsymbol{n}^T,\boldsymbol{n}^S)^t$ and $d_p\mathbb{LG}(\boldsymbol{n}^T,\boldsymbol{n}^S)^n$ of T_pM, the $(\boldsymbol{n}^T,\boldsymbol{n}^S)$-*shape operator* of $M=\boldsymbol{X}(U)$ at $p=\boldsymbol{X}(u)$ and the *normal connection with respect to* $(\boldsymbol{n}^T,\boldsymbol{n}^S)$ of $M=\boldsymbol{X}(U)$ at $p=\boldsymbol{X}(u)$. The eigenvalues of $S_p(\boldsymbol{n}^T,\boldsymbol{n}^S)$, denoted by $\{\kappa_i(\boldsymbol{n}^T,\boldsymbol{n}^S)(p)\}_{i=1}^s$, are called the *lightcone principal curvatures with respect to* $(\boldsymbol{n}^T,\boldsymbol{n}^S)$ at $p=\boldsymbol{X}(u)$. Then the *lightcone Gauss-Kronecker curvature with respect to* $(\boldsymbol{n}^T,\boldsymbol{n}^S)$ at $p=\boldsymbol{X}(u)$ is defined by

$$K_\ell(\boldsymbol{n}^T,\boldsymbol{n}^S)(p) = \det S_p(\boldsymbol{n}^T,\boldsymbol{n}^S).$$

We say that a point $p=\boldsymbol{X}(u)$ is an $(\boldsymbol{n}^T,\boldsymbol{n}^S)$-*umbilical point* $S_p(\boldsymbol{n}^T,\boldsymbol{n}^S)=\kappa(\boldsymbol{n}^T,\boldsymbol{n}^S)(p)1_{T_pM}$ for some function κ. We say that $M=\boldsymbol{X}(U)$ is *totally* $(\boldsymbol{n}^T,\boldsymbol{n}^S)$-*umbilical* if all points on M are $(\boldsymbol{n}^T,\boldsymbol{n}^S)$-umbilical.

We deduce now the lightcone Weingarten formula. Since $\boldsymbol{X}_{u_i}$ $(i=1,\dots s)$ are spacelike vectors, we have a Riemannian metric (the *lightcone first fundamental form*) on $M=\boldsymbol{X}(U)$ defined by $ds^2=\sum_{i=1}^s g_{ij}du_idu_j$, where $g_{ij}(u)=\langle \boldsymbol{X}_{u_i}(u),\boldsymbol{X}_{u_j}(u)\rangle$ for any $u\in U$. We also have a *lightcone second fundamental invariant with respect to the normal vector field* $(\boldsymbol{n}^T,\boldsymbol{n}^S)$ defined by $h_{ij}(\boldsymbol{n}^T,\boldsymbol{n}^S)(u)=\langle -(\boldsymbol{n}^T+\boldsymbol{n}^S)_{u_i}(u),\boldsymbol{X}_{u_j}(u)\rangle$ for any $u\in U$. By the similar arguments to those in the proof of [11, Proposition 3.2], we have the following proposition.

Proposition 3.1 *We choose a pseudo-orthonormal frame* $\{\boldsymbol{n}^T,\boldsymbol{n}^S_1,\dots,\boldsymbol{n}^S_{k-1}\}$ *of* $N(M)$ *with* $\boldsymbol{n}^S_{k-1}=\boldsymbol{n}^S$. *Then we have the following lightcone Weingarten formula with respect to* $(\boldsymbol{n}^T,\boldsymbol{n}^S)$:

(a) $\mathbb{LG}(\boldsymbol{n}^T,\boldsymbol{n}^S)_{u_i} = \langle \boldsymbol{n}^S_{u_i},\boldsymbol{n}^T\rangle(\boldsymbol{n}^T-\boldsymbol{n}^S)+\sum_{\ell=1}^{k-2}\langle(\boldsymbol{n}^T+\boldsymbol{n}^S)_{u_i},\boldsymbol{n}^S_\ell\rangle\boldsymbol{n}^S_\ell-\sum_{j=1}^s h_i^j(\boldsymbol{n}^T,\boldsymbol{n}^S)\boldsymbol{X}_{u_j}$

(b) $\pi^t\circ\mathbb{LG}(\boldsymbol{n}^T,\boldsymbol{n}^S)_{u_i} = -\sum_{j=1}^s h_i^j(\boldsymbol{n}^T,\boldsymbol{n}^S)\boldsymbol{X}_{u_j}$.

Here $\left(h_i^j(\boldsymbol{n}^T,\boldsymbol{n}^S)\right)=\left(h_{ik}(\boldsymbol{n}^T,\boldsymbol{n}^S)\right)\left(g^{kj}\right)$ *and* $\left(g^{kj}\right)=\left(g_{kj}\right)^{-1}$.

As a corollary of the above proposition, we have an explicit expression of the lightcone curvature in terms of the Riemannian metric and the lightcone second fundamental invariant.

Corollary 3.2 *Under the same notations as in the above proposition, the lightcone Gauss-Kronecker curvature relative to* $(\boldsymbol{n}^T,\boldsymbol{n}^S)$ *is given by*

$$K_\ell(\boldsymbol{n}^T,\boldsymbol{n}^S)=\frac{\det\left(h_{ij}(\boldsymbol{n}^T,\boldsymbol{n}^S)\right)}{\det\left(g_{\alpha\beta}\right)}.$$

Since $\langle -(\boldsymbol{n}^T+\boldsymbol{n}^S)(u),\boldsymbol{X}_{u_j}(u)\rangle=0$, we have $h_{ij}(\boldsymbol{n}^T,\boldsymbol{n}^S)(u)=\langle \boldsymbol{n}^T(u)+\boldsymbol{n}^S(u),\boldsymbol{X}_{u_iu_j}(u)\rangle$. Therefore the lightcone second fundamental invariant at a point $p_0=\boldsymbol{X}(u_0)$ depends only on the values $\boldsymbol{n}^T(u_0)+\boldsymbol{n}^S(u_0)$ and $\boldsymbol{X}_{u_iu_j}(u_0)$, respectively Thus, the lightcone curvatures also depend only on $\boldsymbol{n}^T(u_0)+\boldsymbol{n}^S(u_0)$, $\boldsymbol{X}_{u_i}(u_0)$ and $\boldsymbol{X}_{u_iu_j}(u_0)$, independent of the derivation of the vector fields $\boldsymbol{n}^T$ and $\boldsymbol{n}^S$. We write $\kappa_i(\boldsymbol{n}^T_0,\boldsymbol{n}^S_0)(p_0)$ $(i=1,\dots,s)$ and $K_\ell(\boldsymbol{n}^T_0,\boldsymbol{n}^S_0)(u_0)$ as the lightcone curvatures at $p_0=\boldsymbol{X}(u_0)$ with respect to $(\boldsymbol{n}^T_0,\boldsymbol{n}^S_0)=(\boldsymbol{n}^T(u_0),\boldsymbol{n}^S(u_0))$. We might also say that a point $p_0=\boldsymbol{X}(u_0)$ is $(\boldsymbol{n}^T_0,\boldsymbol{n}^S_0)$-*umbilical* because the lightcone $(\boldsymbol{n}^T,\boldsymbol{n}^S)$-shape operator at p_0 depends only on the normal vectors $(\boldsymbol{n}^T_0,\boldsymbol{n}^S_0)$. So we denote that $h_{ij}(\boldsymbol{n}^T,\boldsymbol{\xi})(u_0)=h_{ij}(\boldsymbol{n}^T,\boldsymbol{n}^S)(u_0)$ and $K_\ell(\boldsymbol{n}^T,\boldsymbol{\xi})(p_0)=K_\ell(\boldsymbol{n}^T_0,\boldsymbol{n}^S_0)(p_0)$, where $\boldsymbol{\xi}=\boldsymbol{n}^S(u_0)$ for some local extension $\boldsymbol{n}^T(u)$ of $\boldsymbol{\xi}$. Analogously, we say that a point $p_0=\boldsymbol{X}(u_0)$ is an $(\boldsymbol{n}^T_0,\boldsymbol{n}^S_0)$-*parabolic point* of $\boldsymbol{X}:U\longrightarrow\mathbb{R}^{n+1}_1$ if $K_\ell(\boldsymbol{n}^T_0,\boldsymbol{n}^S_0)(u_0)=0$. And we say that a point $p_0=\boldsymbol{X}(u_0)$ is a $(\boldsymbol{n}^T_0,\boldsymbol{n}^S_0)$-*flat point* if it is an $(\boldsymbol{n}^T_0,\boldsymbol{n}^S_0)$-umbilical point and $K_\ell(\boldsymbol{n}^T_0,\boldsymbol{n}^S_0)(u_0)=0$.

On the other hand, the lightcone Gauss map $\widetilde{\mathbb{LG}}(\boldsymbol{n}^T,\boldsymbol{n}^S)$ with respect to $(\boldsymbol{n}^T,\boldsymbol{n}^S)$ also induces a linear mapping $d_p\widetilde{\mathbb{LG}}(\boldsymbol{n}^T,\boldsymbol{n}^S) : T_pM \longrightarrow T_p\mathbb{R}^{n+1}_1$ under the identification of U and M, where $p = \boldsymbol{X}(u)$. We have the following proposition.

Proposition 3.3 *Under the above notations, we have the following normalized lightcone Weingarten formula with respect to $(\boldsymbol{n}^T,\boldsymbol{n}^S)$:*

$$\pi^t \circ \widetilde{\mathbb{LG}}(\boldsymbol{n}^T,\boldsymbol{n}^S)_{u_i} = -\sum_{j=1}^{s} \frac{1}{\ell_0(u)} h_i^j(\boldsymbol{n}^T,\boldsymbol{n}^S)\boldsymbol{X}_{u_j},$$

where $\mathbb{LG}(\boldsymbol{n}^T,\boldsymbol{n}^S)(u) = (\ell_0(u),\ell_1(u),\dots,\ell_n(u))$.

Proof. By definition, we have $\ell_0\widetilde{\mathbb{LG}}(\boldsymbol{n}^T,\boldsymbol{n}^S) = \mathbb{LG}(\boldsymbol{n}^T,\boldsymbol{n}^S)$. It follows that $\ell_0\widetilde{\mathbb{LG}}(\boldsymbol{n}^T,\boldsymbol{n}^S)_{u_i} = \mathbb{LG}(\boldsymbol{n}^T,\boldsymbol{n}^S)_{u_i} - \ell_{0u_i}\widetilde{\mathbb{LG}}(\boldsymbol{n}^T,\boldsymbol{n}^S)$. Since $\widetilde{\mathbb{LG}}(\boldsymbol{n}^T,\boldsymbol{n}^S)(u) \in N_p(M)$, we have

$$\pi^t \circ \widetilde{\mathbb{LG}}(\boldsymbol{n}^T,\boldsymbol{n}^S)_{u_i} = \frac{1}{\ell_0}\pi^t \circ \mathbb{LG}(\boldsymbol{n}^T,\boldsymbol{n}^S)_{u_i}.$$

By the lightcone Weingarten formula with respect to $(\boldsymbol{n}^T,\boldsymbol{n}^S)$ (Proposition 3.1), we have the desired formula. □

We call the linear transformation $\widetilde{S}(\boldsymbol{n}^T,\boldsymbol{n}^S)_p = -\pi^t \circ d_p\widetilde{\mathbb{LG}}(\boldsymbol{n}^T,\boldsymbol{n}^S)$ the *normalized lightcone shape operator* of M at p with respect to $(\boldsymbol{n}^T,\boldsymbol{n}^S)$. The eigenvalues $\{\widetilde{\kappa}_i(\boldsymbol{n}^T,\boldsymbol{n}^S)(p)\}_{i=1}^s$ of $\widetilde{S}(\boldsymbol{n}^T,\boldsymbol{n}^S)_p$ are called the *normalized lightcone principal curvatures*. By the above poposition, we have $\widetilde{\kappa}_i(\boldsymbol{n}^T,\boldsymbol{n}^S)(p) = (1/\ell_0(u))\kappa_i(\boldsymbol{n}^T,\boldsymbol{n}^S)(p)$. The *normalized Gauss-Kronecker curvature* of M with respect to $(\boldsymbol{n}^T,\boldsymbol{n}^S)$ is defined to be $\widetilde{K}_\ell(\boldsymbol{n}^T,\boldsymbol{n}^S)(u) = \det \widetilde{S}(\boldsymbol{n}^T,\boldsymbol{n}^S)_p$. Then we have the following relation between the normalized lightcone Gauss-Kronecker curvature and the lightcone Gauss-Kronecker curvature:

$$\widetilde{K}_\ell(\boldsymbol{n}^T,\boldsymbol{n}^S)(u) = \left(\frac{1}{\ell_0(u)}\right)^s K_\ell(\boldsymbol{n}^T,\boldsymbol{n}^S)(u).$$

On the other hand, we consider a submanifold $\Delta = \{(\boldsymbol{v},\boldsymbol{w}) \mid \langle \boldsymbol{v},\boldsymbol{w}\rangle = 0\ \} \subset H^n_+(-1) \times S^n_1$ and the canonical projection $\bar{\pi} : \Delta \longrightarrow H^n_+(-1)$. It is well known that Δ can be identified with the unit tangent bundle $S(TH^n_+(-1))$ over $H^n_+(-1)$. We define a function $\mathcal{N}_h : \Delta \longrightarrow \mathbb{R}$ by $\mathcal{N}_h(\boldsymbol{v},\boldsymbol{w}) = 1/(v_0 + w_0)$, where $\boldsymbol{v} = (v_0,v_1,\dots,v_n), \boldsymbol{w} = (w_0,w_1,\dots,w_n)$. Then we have

$$\mathcal{N}_h(\boldsymbol{n}^T(u),\boldsymbol{n}^S(u)) = \frac{1}{\ell_0(u)}.$$

Therefore we can rewrite the above formula as follows:

$$\widetilde{K}_\ell(\boldsymbol{n}^T,\boldsymbol{n}^S)(u) = \mathcal{N}_h(\boldsymbol{n}^T(u),\boldsymbol{n}^S(u))^s K_\ell(\boldsymbol{n}^T,\boldsymbol{n}^S)(u).$$

By definition, $p_0 = \boldsymbol{X}(u_0)$ is the $(\boldsymbol{n}^T_0,\boldsymbol{n}^S_0)$-umbilicl point if and only if $\widetilde{S}(\boldsymbol{n}^T,\boldsymbol{n}^S)_{p_0} = \widetilde{\kappa}_i(\boldsymbol{n}^T,\boldsymbol{n}^S)(p)1_{T_{p_0}M}$. We have the following proposition.

Proposition 3.4 *Let $(\boldsymbol{n}^T,\boldsymbol{n}^S)$ be a future directed normal pair along $M = \boldsymbol{X}(U)$. Then the following conditions are equivalent*:

(1) *The normalized lgihtcone Gauss map* $\widetilde{\mathbb{LG}}(\boldsymbol{n}^T, \boldsymbol{n}^S)$ *of* $M = \boldsymbol{X}(U)$ *with respect to* $(\boldsymbol{n}^T, \boldsymbol{n}^S)$ *is constant*

(2) *There exists* $\boldsymbol{v} \in S_+^{n-1}$ *and a real number* c *such that* $M \subset HP(\boldsymbol{v}, c)$.

Suppose that the above condition holds. Then

(3) $M = \boldsymbol{X}(U)$ *is totally* $(\boldsymbol{n}^T, \boldsymbol{n}^S)$*-flat.*

Proof. Suppose that the normalized lightcone Gauss Map $\widetilde{\mathbb{LG}}(\boldsymbol{n}^T, \boldsymbol{n}^S)(u) = \boldsymbol{v}$ is constant. We consider a function $F : U \longrightarrow \mathbb{R}$ defined by $F(u) = \langle \boldsymbol{X}(u), \boldsymbol{v} \rangle$. By definition, we have

$$\frac{\partial F}{\partial u_i}(u) = \langle \boldsymbol{X}_{u_i}(u), \boldsymbol{v} \rangle = \langle \boldsymbol{X}_{u_i}(u), \widetilde{\mathbb{LG}}(\boldsymbol{n}^T, \boldsymbol{n}^S)(u) \rangle,$$

for any $i = 1, \dots, s$. Therefore, $F(u) = \langle \boldsymbol{X}(u), \boldsymbol{v} \rangle = c$ is constant. It follows that $M \subset HP(\boldsymbol{v}, c)$ for $\boldsymbol{v} \in S_+^{n-1}$.

Suppose that M is a subset of alightlike hyperplane $H(\boldsymbol{v}, c)$ for $\boldsymbol{v} \in S_+^{N-1}$. Since $M \subset HP(\boldsymbol{v}, c)$, we have $T_pM \subset H(\boldsymbol{v}, 0)$. If $\langle \boldsymbol{n}^(u), \boldsymbol{v} \rangle = 0$, then $\boldsymbol{n}^T(u) \in HP(\boldsymbol{v}, 0)$. We remark that $HP(\boldsymbol{v}, 0)$ does not include timelike vectors. This is a contradiction. So we have $\langle \boldsymbol{n}^T(u), \boldsymbol{v} \rangle \neq 0$. We now define a normal vector field along $M = \boldsymbol{X}(U)$ by

$$\boldsymbol{n}^S(u) = \frac{-1}{\langle \boldsymbol{n}^T(u), \boldsymbol{v} \rangle} \boldsymbol{v} - \boldsymbol{n}^T(u).$$

We can easily show that $\boldsymbol{n}^S(u) \in N_1(M)_p[\boldsymbol{n}^T]$ for $p = \boldsymbol{X}(u)$. Therefore $(\boldsymbol{n}^T, \boldsymbol{n}^S)$ is a future directed normal pair such that $\widetilde{\mathbb{LG}}(\boldsymbol{n}^T, \boldsymbol{n}^S)(u) = \boldsymbol{v}$.

On the other hand, by Proposition 3.3, if $\widetilde{\mathbb{LG}}(\boldsymbol{n}^T, \boldsymbol{n}^S)$ is constant, then $(h_i^j(\boldsymbol{n}^T, \boldsymbol{n}^S)(u)) = O$, so that $M = \boldsymbol{X}(U)$ is lightcone $(\boldsymbol{n}^T, \boldsymbol{n}^S)$-flat. □

4 The lightcone Lipschitz-Killing cuvature

In this section we define the lightcone Gauss map of $N_1(M)[\boldsymbol{n}^T]$ and investigate the geometric properties. We define a map

$$\widetilde{\mathbb{LG}}(\boldsymbol{n}^T) : N_1(M)[\boldsymbol{n}^T] \longrightarrow S_+^{n-1}$$

by $\widetilde{\mathbb{LG}}(\boldsymbol{n}^T)(u, \boldsymbol{\xi}) = \widetilde{\boldsymbol{n}^T(u) + \boldsymbol{\xi}}$, which we call the *lightcone Gauss map* of $N_1(M)[\boldsymbol{n}^T]$. The lghtcone Gauss map leads us to a curvature similar to the codimension two case[11]. Let $T_{(p,\xi)}N_1(M)[\boldsymbol{n}^T]$ be the tangent space of $N_1(M)[\boldsymbol{n}^T]$ at $(p, \boldsymbol{\xi})$. We have the canonical identification

$$T_{(p,\xi)}N_1(M)[\boldsymbol{n}^T] = T_pM \oplus T_\xi S^{k-2} \subset T_pM \oplus N_p(M) = T_p\mathbb{R}_1^{n+1},$$

where $T_\xi S^{k-2} \subset T_\xi N_p(M) \equiv N_p(M)$ and $p = \boldsymbol{X}(u)$. Let

$$\Pi^t : \widetilde{\mathbb{LG}}(\boldsymbol{n}^T)^* T\mathbb{R}_1^{n+1} = TN_1(M)[\boldsymbol{n}^T] \oplus \mathbb{R}^{k+1} \longrightarrow TN_1(M)[\boldsymbol{n}^T]$$

be the canonical projection. It follows that we have a linear transformation

$$\Pi^t_{\widetilde{\mathbb{LG}(n^T)(p,\xi)}} \circ d_{(p,\xi)}\widetilde{\mathbb{LG}}(\boldsymbol{n}^T) : T_{(p,\xi)}N_1(M)[\boldsymbol{n}^T] \longrightarrow T_{(p,\xi)}N_1(M)[\boldsymbol{n}^T].$$

The *lightcone Lipschitz-Killing curvature* of $N_1(M)[\boldsymbol{n}^T]$ at $(p, \boldsymbol{\xi})$ is defined to be

$$\widetilde{K}_\ell(\boldsymbol{n}^T)(p, \boldsymbol{\xi}) = \det\left(-\Pi^t_{\widetilde{\mathbb{LG}}(n^T)(p,\xi)} \circ d_{(p,\xi)}\widetilde{\mathbb{LG}}(\boldsymbol{n}^T)\right).$$

In order to investigate the lightcone Gauss map $\widetilde{\mathbb{LG}}(\boldsymbol{n}^T)$ of $N_1(M)[\boldsymbol{n}^T]$, we define a map

$$\mathbb{LG}(\boldsymbol{n}^T) : N_1(M)[\boldsymbol{n}^T] \longrightarrow LC^*$$

by $\mathbb{LG}(\boldsymbol{n}^T)(u, \boldsymbol{\xi}) = \boldsymbol{n}^T(u) + \boldsymbol{\xi}$, which is called the *lightcone Gauss image* of $N_1(M)[\boldsymbol{n}^T]$. We now write $\mathbb{LG}(\boldsymbol{n}^T)(p, \boldsymbol{\xi}) = (\ell_0(p, \boldsymbol{\xi}), \ell_1(p, \boldsymbol{\xi}), \dots, \ell_n(p, \boldsymbol{\xi}))$. For any future directed timelike unit normal vector field $\boldsymbol{n}^T$ along M, there exists a pseudo-orthonormal frame $\{\boldsymbol{n}^T, \boldsymbol{n}^S_1, \dots, \boldsymbol{n}^S_{k-1}\}$ of $N(M)$ with $\boldsymbol{n}^S_{k-1}(u_0) = \boldsymbol{\xi}$ and $p = \boldsymbol{X}(u_0)$, so that we have a frame field

$$\{\boldsymbol{X}_{u_1}, \dots, \boldsymbol{X}_{u_s}, \boldsymbol{n}^T, \boldsymbol{n}^S_1, \dots, \boldsymbol{n}^S_{k-1}\}$$

of $\mathbb{R}^{n+1}_1$ along M. We define an S^{k-1}-family of spacelike unit normal vetor field

$$\boldsymbol{N}(u, \mu) = \sum_{j=1}^{k-1} \mu_j \boldsymbol{n}^S_j(u) \in N(M; 1)$$

along M for $\mu = (\mu_1, \dots, \mu_{k-1}) \in S^{k-2} \subset \mathbb{R}^{k-1}$. We also define a map

$$\Psi : U \times S^{k-2} \longrightarrow N_1(M)[\boldsymbol{n}^T]$$

by $\Psi(u, \mu) = (\boldsymbol{X}(u), \boldsymbol{N}^S(u, \mu))$, which gives a local parametrization of $N_1(M)[\boldsymbol{n}^T]$. Then we have $(p, \boldsymbol{\xi}) = (\boldsymbol{X}(u_0), \boldsymbol{N}^S(u_0, \mu_0))$, where $\mu_0 = (0, \dots, 0, 1)$. It follows that $\mathbb{LG}(\boldsymbol{n}^T) \circ \Psi(u, \mu) = \boldsymbol{n}^T(u) + \boldsymbol{N}^S(u, \mu)$. We now write that $\mathbb{LG}(\boldsymbol{n}^T, \boldsymbol{N}^S)(u, \mu) = \mathbb{LG}(\boldsymbol{n}^T) \circ \Psi(u, \mu)$. We consider the local coordinate neighbourhood of S^{k-1}:

$$U^+_{k-1} = \{(\mu_1, \dots, \mu_{k-1}) \in S^{k-1} \mid \mu_{k-1} > 0\ \}.$$

Then we have $\mu_{k-1} = \sqrt{1 - \sum_{j=1}^{k-2} \mu_j^2}$. For $i = 1, \dots, s, j = 1, \dots k-2$, we have the following calculation:

$$\begin{aligned}
\frac{\partial \mathbb{LG}(\boldsymbol{n}^T, \boldsymbol{N}^S)}{\partial u_i}(u, \mu) &= \boldsymbol{n}^T_{u_i}(u) + \sum_{\ell=1}^{k-1} \mu_\ell \boldsymbol{n}^S_{\ell, u_i}(u), \\
\frac{\partial \mathbb{LG}(\boldsymbol{n}^T, \boldsymbol{N}^S)}{\partial \mu_j}(u, \mu) &= \boldsymbol{n}^S_j(u) - \frac{\mu_j}{\mu_{k-1}} \boldsymbol{n}^S_{k-1}(u).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\frac{\partial \mathbb{LG}(\boldsymbol{n}^T, \boldsymbol{N}^S)}{\partial u_i}(u_0, \mu_0) &= \boldsymbol{n}^T_{u_i}(u_0) + \boldsymbol{n}^S_{k-1, u_i}(u_0) = (\boldsymbol{n}^T + \boldsymbol{n}^S_{k-1})_{u_i}(u_0), \\
\frac{\partial \mathbb{LG}(\boldsymbol{n}^T, \boldsymbol{N}^S)}{\partial \mu_j}(u_0, \mu_0) &= \boldsymbol{n}^S_j(u_0).
\end{aligned}$$

We now remark that $\{\boldsymbol{X}_{u_1}, \dots, \boldsymbol{X}_{u_s}, \boldsymbol{n}^S_1, \dots, \boldsymbol{n}^S_{k-2}\}$ is a basis of $T_{(p,\xi)} N_1(M)[\boldsymbol{n}^T]$ at $u = u_0$. By Proposition 3.1, we have

$$\begin{aligned}
(\boldsymbol{n}^T + \boldsymbol{n}^S_{k-1})_{u_i}(u_0) &= \langle \boldsymbol{n}^S_{u_i}, \boldsymbol{n}^T \rangle (\boldsymbol{n}^T - \boldsymbol{n}^S_{k-1})(u_0) \\
&\quad + \sum_{\ell=1}^{k-2} \langle (\boldsymbol{n}^T + \boldsymbol{n}^S_{k-1})_{u_i}, \boldsymbol{n}^S_\ell \rangle \boldsymbol{n}^S_\ell(u_0) - \sum_{j=1}^{s} h^j_i(\boldsymbol{n}^T, \boldsymbol{n}^S) \boldsymbol{X}_{u_j}(u_0).
\end{aligned}$$

Since $\langle \boldsymbol{n}^T - \boldsymbol{n}^S_{k-1}, \boldsymbol{X}_{u_i}\rangle = \langle \boldsymbol{n}^T - \boldsymbol{n}^S_{k-1}, \boldsymbol{n}^S_j\rangle = \langle \boldsymbol{n}^S_\ell, \boldsymbol{X}_{u_i}\rangle = 0$ and $\langle \boldsymbol{n}^S_j, \boldsymbol{n}^S_\ell\rangle = \delta_{j\ell}$, we have

$$\begin{aligned}&\det\left(-\Pi^t_{\mathbb{LG}(n^T)(p,\xi)} \circ d_{(p,\xi)}\mathbb{LG}(\boldsymbol{n}^T)\right)\\ &= \det\left(\begin{pmatrix} \underset{1\leqslant i,j\leqslant s}{\langle -(\boldsymbol{n}^T + \boldsymbol{n}^S_{k-1})_{u_i}, \boldsymbol{X}_{u_j}\rangle} & \underset{1\leqslant i\leqslant s; 1\leqslant j\leqslant k-2}{\langle -(\boldsymbol{n}^T + \boldsymbol{n}^S_{k-1})_{u_i}, \boldsymbol{n}^S_j\rangle} \\ \boldsymbol{0}_{(k-2)\times s} & -I_{(k-2)}\end{pmatrix}\begin{pmatrix} g^{ij} & \boldsymbol{0} \\ \boldsymbol{0} & I_{(k-2)}\end{pmatrix}\right)(u_0).\end{aligned}$$

Since $\ell_0\widetilde{\mathbb{LG}}(\boldsymbol{n}^T, \boldsymbol{N}^S) = \mathbb{LG}(\boldsymbol{n}^T, \boldsymbol{N}^S)$, we have

$$\begin{aligned}(\ell_0)_{u_i}\widetilde{\mathbb{LG}}(\boldsymbol{n}^T, \boldsymbol{N}^S) + \ell_0\widetilde{\mathbb{LG}}(\boldsymbol{n}^T, \boldsymbol{N}^S)_{u_i} &= \mathbb{LG}(\boldsymbol{n}^T, \boldsymbol{N}^S)_{u_i},\\ (\ell_0)_{\mu_j}\widetilde{\mathbb{LG}}(\boldsymbol{n}^T, \boldsymbol{N}^S) + \ell_0\widetilde{\mathbb{LG}}(\boldsymbol{n}^T, \boldsymbol{N}^S)_{\mu_j} &= \mathbb{LG}(\boldsymbol{n}^T, \boldsymbol{N}^S)_{\mu_j}.\end{aligned}$$

Moreover, we have $\langle\widetilde{\mathbb{LG}}(\boldsymbol{n}^T, \boldsymbol{N}^S)(u_0,\mu_0), \boldsymbol{X}_{u_i}(u_0)\rangle = \langle\widetilde{\mathbb{LG}}(\boldsymbol{n}^T, \boldsymbol{N}^S)(u_0,\mu_0), \boldsymbol{n}^S_j(u_0)\rangle = 0$. It follows that

$$\begin{aligned}&\widetilde{K}_\ell(\boldsymbol{n}^T)(p,\boldsymbol{\xi}) = \det\left(-\Pi^t_{\widetilde{\mathbb{LG}}(n^T)(p,\xi)} \circ d_{(p,\xi)}\widetilde{\mathbb{LG}}(\boldsymbol{n}^T)\right)\\ &= \det\left(\begin{pmatrix} \underset{1\leqslant i,j\leqslant s}{\frac{1}{\ell_0}\langle -(\boldsymbol{n}^T + \boldsymbol{n}^S_{k-1})_{u_i}, \boldsymbol{X}_{u_j}\rangle} & \underset{1\leqslant i\leqslant s; 1\leqslant j\leqslant k-2}{\frac{1}{\ell_0}\langle -(\boldsymbol{n}^T + \boldsymbol{n}^S_{k-1})_{u_i}, \boldsymbol{n}^S_j\rangle} \\ \boldsymbol{0}_{(k-2)\times s} & -\frac{1}{\ell_0}I_{(k-2)}\end{pmatrix}\begin{pmatrix} g^{ij} & \boldsymbol{0} \\ \boldsymbol{0} & I_{(k-2)}\end{pmatrix}\right)(u_0).\end{aligned}$$

On the other hand, Corollary 3.2 implies that

$$\begin{aligned}&K_\ell(\boldsymbol{n}^T, \boldsymbol{\xi})(p) = K_\ell(\boldsymbol{n}^T, \boldsymbol{n}^S_{k-1})(u_0)\\ &= \det((\langle -(\boldsymbol{n}^T + \boldsymbol{n}^S_{k-1})_{u_i}, \boldsymbol{X}_{u_j}\rangle)(g^{ij}))(u_0)\\ &= \det\left(\begin{pmatrix} \underset{1\leqslant i,j\leqslant s}{\langle -(\boldsymbol{n}^T + \boldsymbol{n}^S_{k-1})_{u_i}, \boldsymbol{X}_{u_j}\rangle} & \underset{1\leqslant i\leqslant s; 1\leqslant j\leqslant k-2}{\langle -(\boldsymbol{n}^T + \boldsymbol{n}^S_{k-1})_{u_i}, \boldsymbol{n}^S_j\rangle} \\ \boldsymbol{0}_{(k-2)\times s} & I_{(k-2)}\end{pmatrix}\begin{pmatrix} g^{ij} & \boldsymbol{0} \\ \boldsymbol{0} & I_{(k-2)}\end{pmatrix}\right)(u_0).\end{aligned}$$

Thereofore we have the following theorem.

Theorem 4.1 *Under the same notations as those of the above paragraph, we have*

$$\begin{aligned}\widetilde{K}_\ell(\boldsymbol{n}^T)(p_0,\boldsymbol{\xi}_0) &= (-1)^{k-2}\mathcal{N}_h(\boldsymbol{n}^T(u_0),\boldsymbol{\xi}_0)^{n-1}K_\ell(\boldsymbol{n}^T, \boldsymbol{n}^S)(u_0)\\ &= (-\mathcal{N}_h(\boldsymbol{n}^T(u_0),\boldsymbol{\xi}_0))^{k-2}\widetilde{K}_\ell(\boldsymbol{n}^T, \boldsymbol{n}^S)(u_0),\end{aligned}$$

where $p_0 = \boldsymbol{X}(u_0)$ and $\boldsymbol{n}^S(u)$ is a local section of $N_1(M)[\boldsymbol{n}^T]$ such that $\boldsymbol{n}^S(u_0) = \boldsymbol{\xi}_0$.

We have the following corollary of the above theorem.

Corollary 4.2 *The following conditions are equivalent*:
(1) *$p_0 = \boldsymbol{X}(u_0)$ is a $(\boldsymbol{n}^T_0, \boldsymbol{\xi}_0)$-parabolic point ($K_\ell(\boldsymbol{n}^T, \boldsymbol{n}^S)(u_0) = 0$),*
(2) *$\widetilde{K}_\ell(\boldsymbol{n}^T)(p_0,\boldsymbol{\xi}_0) = 0$.*
Here, $\boldsymbol{n}^S(u)$ is a local section of $N_1(M)[\boldsymbol{n}^T]$ such that $\boldsymbol{n}^S(u_0) = \boldsymbol{\xi}_0$.

5 Lightcone height functions

In order to ivetigate the geometric meanings of the lightcone Lipschitz-Killing curvature of $N_1(M)[\boldsymbol{n}^T]$, we introduce a family of functions on $M = \boldsymbol{X}(U)$. We define the family of *lightcone height functions*

$$H : U \times S_+^{n-1} \longrightarrow \mathbb{R}$$

on $M = \boldsymbol{X}(U)$ by $H(u, \boldsymbol{v}) = \langle \boldsymbol{X}(u), \boldsymbol{v} \rangle$. We denote the Hessian matrix of the lightcone height-function $h_{\boldsymbol{v}_0}(u) = H(u, \boldsymbol{v}_0)$ at u_0 by $\mathrm{Hess}(h_{\boldsymbol{v}_0})(u_0)$. The following proposition characterizes the lightlike parabolic points and lightlike flat points in terms of the family of lightcone height functions.

Proposition 5.1 *Let $H : U \times S_+^{n-1} \longrightarrow \mathbb{R}$ be the family of lightcone height functions on M. Then*
(1) $(\partial H/\partial u_i)(u_0, \boldsymbol{v}_0) = 0$ $(i = 1, \dots, s)$ if and only if there exists $\boldsymbol{\xi}_0 \in N_1(M)_{p_0}[\boldsymbol{n}^T]$ such that $\boldsymbol{v}_0 = \widetilde{\mathbb{LG}}(\boldsymbol{n}^T)(p_0, \boldsymbol{\xi}_0)$, where $p_0 = \boldsymbol{X}(u_0)$.

Suppose that $p_0 = \boldsymbol{X}(u_0)$, $\boldsymbol{v}_0 = \widetilde{\mathbb{LG}}(\boldsymbol{n}^T)(p_0, \boldsymbol{\xi}_0)$. Then
(2) p_0 is a $(\boldsymbol{n}_0^T, \boldsymbol{\xi}_0)$-parabolic point if and only if $\det \mathrm{Hess}(h_{\boldsymbol{v}_0})(u_0) = 0$, where $\boldsymbol{n}_0^T = \boldsymbol{n}^T(u_0)$,
(3) p_0 is a flat $(\boldsymbol{n}_0^T, \boldsymbol{\xi})$-umbilic point if and only if $\mathrm{rank}\, \mathrm{Hess}(h_{\boldsymbol{v}_0})(u_0) = 0$,
(4) u_0 is a non-degenerate critical point of $h_{\boldsymbol{v}_0}$ if and only if $(p_0, \boldsymbol{\xi}_0)$ is a regular point of $\widetilde{\mathbb{LG}}(\boldsymbol{n}^T)$.

Proof. (1) Since $(\partial H/\partial u_i)(u_0, \boldsymbol{v}_0) = \langle \boldsymbol{X}_{u_i}(u_0), \boldsymbol{v}_0 \rangle$, $(\partial H/\partial u_i)(u_0, \boldsymbol{v}_0) = 0$ $(i = 1, \dots, s)$ if and only if $\boldsymbol{v}_0 \in N_{p_0}(M)$ and $\boldsymbol{v}_0 \in S_+^{n-1}$. If $\langle \boldsymbol{v}_0, \boldsymbol{n}^T(u_0) \rangle = 0$, then $\boldsymbol{n}^T(u_0) \in HP(\boldsymbol{v}_0, 0)$. But $HP(\boldsymbol{v}_0, 0)$ is a lightlike hyperplane. This fact contradicts to the fact that $\boldsymbol{n}^T(u_0)$ is timelike. Thus, $\langle \boldsymbol{v}_0, \boldsymbol{n}^T(u_0) \rangle \neq 0$. Then we can easily show that

$$\boldsymbol{\xi}_0 = -\frac{1}{\langle \boldsymbol{n}^T(u_0), \boldsymbol{v}_0 \rangle} \boldsymbol{v}_0 - \boldsymbol{n}^T(u_0) \in N_1(M)_{p_0}[\boldsymbol{n}^T].$$

It follows that

$$\boldsymbol{v}_0 = \widetilde{\boldsymbol{n}^T(u_0) + \boldsymbol{\xi}_0} = \widetilde{\mathbb{LG}}(\boldsymbol{n}^T)(p_0, \boldsymbol{\xi}_0).$$

The converse also holds.

For the proof of the assertions (2) and (3), as a consequence of Proposition 3.1, we have

$$\begin{aligned}
\mathrm{Hess}(h_{\boldsymbol{v}_0})(u_0) &= \left(\langle \boldsymbol{X}_{u_i u_j}(u_0), \widetilde{\mathbb{LG}}(\boldsymbol{n}^T)(p_0, \boldsymbol{\xi}_0) \rangle \right) = \left(\frac{1}{\ell_0} \langle \boldsymbol{X}_{u_i u_j}(u_0), \boldsymbol{n}^T(u_0) + \boldsymbol{\xi}_0 \rangle \right) \\
&= \left(\frac{1}{\ell_0} \langle \boldsymbol{X}_{u_i}(u_0), (\boldsymbol{n}^T + \boldsymbol{n}^S)_{u_j}(u_0) \rangle \right) \\
&= \left(\frac{1}{\ell_0} \langle \boldsymbol{X}_{u_i}(u_0), -\sum_{k=1}^{s} h_j^k(\boldsymbol{n}^T, \boldsymbol{\xi}_0)(u_0) \boldsymbol{X}_{u_k}(u_0) \rangle \right) \\
&= \left(-\mathcal{N}_h(\boldsymbol{n}^T(\underline{u_0}), \boldsymbol{\xi}_0) h_{ij}(\boldsymbol{n}^T, \boldsymbol{\xi}_0)(u_0) \right),
\end{aligned}$$

where $\boldsymbol{n}^S(u)$ is a local section of $N_1(M)[\boldsymbol{n}^T]$ such that $\boldsymbol{n}^S(u_0) = \boldsymbol{\xi}_0$. By definition, $K_\ell(\boldsymbol{n}^T, \boldsymbol{\xi})(p_0) = 0$ if and only if $\det\left(h_{ij}(\boldsymbol{n}^T, \boldsymbol{\xi})(u_0)\right) = 0$. The assertion (2) holds. Moreover, p_0 is a flat $(\boldsymbol{n}_0^T, \boldsymbol{\xi}_0)$-umbilical point if and only if $(h_{ij}(\boldsymbol{n}^T, \boldsymbol{\xi}_0)(u_0)) = O$. So we have the assertion (3).

By the above calculation u_0 is a non-degenerate critical point of $h_{\boldsymbol{v}_0}$ if and only if

$$\widetilde{K}_\ell(\boldsymbol{n}^T, \boldsymbol{\xi}_0)(u_0) = \frac{\det\left(-\mathcal{N}_h(\boldsymbol{n}^T(\underline{u_0}), \boldsymbol{\xi}_0) h_{ij}(\boldsymbol{n}^T, \boldsymbol{\xi}_0)(u_0) \right)}{\det(g_{ij}(u_0))} \neq 0.$$

By Corollary 4.2, the last condition is equivalent to the condition $\widetilde{K}_\ell(\boldsymbol{n}^T)(p_0,\boldsymbol{\xi}_0) \neq 0$. By the definition of $\widetilde{K}_\ell(\boldsymbol{n}^T)(p_0,\boldsymbol{\xi}_0)$, the above condition means that $(p_0,\boldsymbol{\xi}_0)$ is a regular point of $\widetilde{\mathbb{LG}}(\boldsymbol{n}^T)$. □

6 The total absolute lightcone curvature

We have the following theorem.

Theorem 6.1 *Let $d\mathfrak{v}_{N_1(M)[\boldsymbol{n}^T]}$ be the canonical volume form of $N_1(M)[\boldsymbol{n}^T]$ and $d\mathfrak{v}_{S^{n-1}_+}$ the canonical volume form of S^{n-1}_+. Then we have*

$$(\widetilde{\mathbb{LG}}(\boldsymbol{n}^T)^* d\mathfrak{v}_{S^{n-1}_+})_{(p,\xi)} = |\widetilde{K}_\ell(\boldsymbol{n}^T)(p,\boldsymbol{\xi})| d\mathfrak{v}_{N_1(M)[\boldsymbol{n}^T](p,\xi)}.$$

Proof. Without the loss of generality, we may assume that a point $(p,\boldsymbol{\xi})$ is a non-singular point of $\widetilde{\mathbb{LG}}(\boldsymbol{n}^T,\boldsymbol{\xi})$, We consider the same frame $\{\boldsymbol{X}_{u_1},\dots,\boldsymbol{X}_{u_s},\boldsymbol{n}^T,\boldsymbol{n}^S_1,\dots,\boldsymbol{n}^S_{k-1}\}$ as in the previous sections such that $\boldsymbol{n}^S_{k-1}(u_0)=\boldsymbol{\xi}$ and $p=\boldsymbol{X}(u_0)$. We also consider the local coordinate neighbourhood of S^{k-1}

$$U^+_{k-1} = \{(\mu_1,\dots,\mu_{k-1}) \in S^{k-1} \mid \mu_{k-1} > 0\ \},$$

so that we have $\mu_{k-1} = \sqrt{1-\sum_{j=1}^{k-2}\mu_j^2}$ and

$$\begin{aligned}
\frac{\partial \mathbb{LG}(\boldsymbol{n}^T,\boldsymbol{N}^S)}{\partial u_i}(u_0,\mu_0) &= \boldsymbol{n}^T_{u_i}(u_0)+\boldsymbol{n}^S_{k-1,u_i}(u_0) = (\boldsymbol{n}^T+\boldsymbol{n}^S_{k-1})_{u_i}(u_0),\\
\frac{\partial \mathbb{LG}(\boldsymbol{n}^T,\boldsymbol{N}^S)}{\partial \mu_j}(u_0,\mu_0) &= \boldsymbol{n}^S_j(u_0).
\end{aligned}$$

Then we have

$$\begin{aligned}
\mathbb{LG}(\boldsymbol{n}^T,\boldsymbol{\xi})_{u_i}(u_0) &= (\boldsymbol{n}^T+\boldsymbol{n}^S_{k-1})_{u_i}(u_0)\\
&= \langle \boldsymbol{n}^S_{u_i},\boldsymbol{n}^T\rangle(\boldsymbol{n}^T-\boldsymbol{n}^S_{k-1})(u_0)+\sum_{\ell=1}^{k-2}\langle(\boldsymbol{n}^T+\boldsymbol{n}^S_{k-1})_{u_i},\boldsymbol{n}^S_\ell\rangle\boldsymbol{n}^S_\ell(u_0)\\
&\quad -\sum_{j=1}^{s} h^j_i(\boldsymbol{n}^T,\boldsymbol{n}^S)\boldsymbol{X}_{u_j}(u_0).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\langle\, \mathbb{LG}(\boldsymbol{n}^T,\boldsymbol{\xi})_{u_i}(u_0),\mathbb{LG}(\boldsymbol{n}^T,\boldsymbol{\xi})_{u_j}(u_0)\rangle = \langle \pi^t\circ(\boldsymbol{n}^T+\boldsymbol{n}^S_{k-1})_{u_i}(u_0),\pi^t\circ(\boldsymbol{n}^T+\boldsymbol{n}^S_{k-1})_{u_j}(u_0)\rangle\\
&+\sum_{\ell=1}^{k-2}\langle(\boldsymbol{n}^T+\boldsymbol{n}^S_{k-1})_{u_i}(u_0),\boldsymbol{n}^S_\ell(u_0)\rangle\langle(\boldsymbol{n}^T+\boldsymbol{n}^S_{k-1})_{u_j}(u_0),\boldsymbol{n}^S_\ell(u_0)\rangle.
\end{aligned}$$

Since we have

$$\begin{aligned}
\widetilde{\mathbb{LG}}(\boldsymbol{n}^T,\boldsymbol{\xi})_{u_i}(u_0) &= -\frac{(\ell_0)_{u_i}}{\ell_0}\widetilde{\mathbb{LG}}(\boldsymbol{n}^T,\boldsymbol{\xi})(u_0)+\frac{1}{\ell_0}\mathbb{LG}(\boldsymbol{n}^T,\boldsymbol{\xi})_{u_i}(u_0),\\
\widetilde{\mathbb{LG}}(\boldsymbol{n}^T,\boldsymbol{\xi})_{\mu_j}(u_0) &= -\frac{(\ell_0)_{\mu_j}}{\ell_0}\widetilde{\mathbb{LG}}(\boldsymbol{n}^T,\boldsymbol{\xi})(u_0)+\frac{1}{\ell_0}\boldsymbol{n}^S_j(u_0),
\end{aligned}$$

we can calculate that

$$
\begin{aligned}
\langle \widetilde{\mathbb{LG}}(\boldsymbol{n}^T,\boldsymbol{\xi})_{u_i}, \widetilde{\mathbb{LG}}(\boldsymbol{n}^T,\boldsymbol{\xi})_{u_j}\rangle &= \frac{1}{\ell_0^2}\langle \mathbb{LG}(\boldsymbol{n}^T,\boldsymbol{\xi})_{u_i}, \mathbb{LG}(\boldsymbol{n}^T,\boldsymbol{\xi})_{u_j}\rangle,\\
\langle \widetilde{\mathbb{LG}}(\boldsymbol{n}^T,\boldsymbol{\xi})_{u_i}, \widetilde{\mathbb{LG}}(\boldsymbol{n}^T,\boldsymbol{\xi})_{\mu_j}\rangle &= \frac{1}{\ell_0^2}\langle \mathbb{LG}(\boldsymbol{n}^T,\boldsymbol{\xi})_{u_i}, \boldsymbol{n}^S_j\rangle,\\
\langle \widetilde{\mathbb{LG}}(\boldsymbol{n}^T,\boldsymbol{\xi})_{\mu_i}, \widetilde{\mathbb{LG}}(\boldsymbol{n}^T,\boldsymbol{\xi})_{\mu_j}\rangle &= \frac{1}{\ell_0^2}\langle \boldsymbol{n}^S_i, \boldsymbol{n}^S_j\rangle
\end{aligned}
$$

at $(u_0,\mu_0)\in U\times S^{k-2}$. We consider the matrix A defined by

$$
A=\begin{pmatrix}
\underset{1\leqslant i,j\leqslant s}{\langle \widetilde{\mathbb{LG}}(\boldsymbol{n}^T,\boldsymbol{\xi})_{u_i}, \widetilde{\mathbb{LG}}(\boldsymbol{n}^T,\boldsymbol{\xi})_{u_j}\rangle} & \underset{1\leqslant i\leqslant s;1\leqslant j\leqslant k-2}{\langle \widetilde{\mathbb{LG}}(\boldsymbol{n}^T,\boldsymbol{\xi})_{u_i}, \widetilde{\mathbb{LG}}(\boldsymbol{n}^T,\boldsymbol{\xi})_{\mu_j}\rangle}\\
\underset{1\leqslant i\leqslant k-1;1\leqslant j\leqslant s}{\langle \widetilde{\mathbb{LG}}(\boldsymbol{n}^T,\boldsymbol{\xi})_{u_j}, \widetilde{\mathbb{LG}}(\boldsymbol{n}^T,\boldsymbol{\xi})_{\mu_i}\rangle} & \underset{1\leqslant i,j\leqslant k-2}{\langle \widetilde{\mathbb{LG}}(\boldsymbol{n}^T,\boldsymbol{\xi})_{\mu_i}, \widetilde{\mathbb{LG}}(\boldsymbol{n}^T,\boldsymbol{\xi})_{\mu_j}\rangle}
\end{pmatrix}(u_0).
$$

By the previous calculation, we have

$$
\begin{aligned}
A &= \frac{1}{(\ell_0)^2}\begin{pmatrix}
\underset{1\leqslant i,j\leqslant s}{\alpha_{ij}} & \underset{1\leqslant i\leqslant s;1\leqslant j\leqslant k-2}{\langle(\boldsymbol{n}^T+\boldsymbol{n}^S_{k-1})_{u_i},\boldsymbol{n}^S_j\rangle}\\
\underset{1\leqslant i\leqslant k-1;1\leqslant j\leqslant s}{\langle(\boldsymbol{n}^T+\boldsymbol{n}^S_{k-1})_{u_j},\boldsymbol{n}^S_i\rangle} & \underset{1\leqslant i,j\leqslant k-2}{\langle \boldsymbol{n}^S_i,\boldsymbol{n}^S_j\rangle}
\end{pmatrix}(u_0)\\
&= \frac{1}{(\ell_0)^2}\begin{pmatrix}
\underset{1\leqslant i,j\leqslant s}{\alpha_{ij}} & \underset{1\leqslant i\leqslant s;1\leqslant j\leqslant k-2}{\langle(\boldsymbol{n}^T+\boldsymbol{n}^S_{k-1})_{u_i},\boldsymbol{n}^S_j\rangle}\\
\underset{1\leqslant i\leqslant k-1;1\leqslant j\leqslant s}{\langle(\boldsymbol{n}^T+\boldsymbol{n}^S_{k-1})_{u_j},\boldsymbol{n}^S_i\rangle} & \underset{1\leqslant i,j\leqslant k-2}{I_{k-1}}
\end{pmatrix}(u_0),
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha_{ij} &= \langle \pi^t\circ(\boldsymbol{n}^T+\boldsymbol{n}^S_{k-1})_{u_i}, \pi^t\circ(\boldsymbol{n}^T+\boldsymbol{n}^S_{k-1})_{u_j}\rangle\\
&\quad+\sum_{\ell=1}^{k-2}\langle(\boldsymbol{n}^T+\boldsymbol{n}^S_{k-1})_{u_i},\boldsymbol{n}^S_\ell\rangle\langle(\boldsymbol{n}^T+\boldsymbol{n}^S_{k-1})_{u_j},\boldsymbol{n}^S_\ell\rangle.
\end{aligned}
$$

We consider a matrix

$$
A_0=\frac{1}{(\ell_0)^2}\begin{pmatrix}
\underset{1\leqslant i,j\leqslant s}{\langle \pi^t\circ(\boldsymbol{n}^T+\boldsymbol{n}^S_{k-1})_{u_i}, \pi^t\circ(\boldsymbol{n}^T+\boldsymbol{n}^S_{k-1})_{u_j}\rangle} & \underset{1\leqslant i\leqslant s;1\leqslant j\leqslant k-2}{\langle(\boldsymbol{n}^T+\boldsymbol{n}^S_{k-1})_{u_i},\boldsymbol{n}^S_j\rangle}\\
\mathbf{0}_{(k-1)\times s} & I_{k-1}
\end{pmatrix}(u_0).
$$

We denote that A^j, A_0^j the j-the columns of the aboe two matrices. Then we have the relation that

$$
A^j = A_0^j+\sum_{\ell=1}^{k-1}\langle(\boldsymbol{n}^T+\boldsymbol{n}^S_{k-1})_{u_j},\boldsymbol{n}^S_\ell\rangle A_0^{s+\ell},
$$

for $j=1,\ldots,s$. It follows that

$$
\det(A)=\det(A_0)=\frac{1}{(\ell_0)^{2(n-1)}}\det\begin{pmatrix}\underset{1\leqslant i,j\leqslant s}{\langle \pi^t\circ(\boldsymbol{n}^T+\boldsymbol{n}^S_{k-1})_{u_i}, \pi^t\circ(\boldsymbol{n}^T+\boldsymbol{n}^S_{k-1})_{u_j}\rangle}\end{pmatrix}(u_0).
$$

By Proposition 3.1, we have $\pi^t\circ(\boldsymbol{n}^T+\boldsymbol{n}^S_{k-1})_{u_i}(u_0)=-\sum_{j=1}^s h_i^j(\boldsymbol{n}^T,\boldsymbol{\xi})(u_0)\boldsymbol{X}_{u_j}(u_0)$, so that

$$\langle \pi^t\circ(\boldsymbol{n}^T+\boldsymbol{n}^S_{k-1})_{u_i},\pi^t\circ(\boldsymbol{n}^T+\boldsymbol{n}^S_{k-1})_{u_j}\rangle(u_0)=\sum_{\alpha,\beta}h_i^\alpha(\boldsymbol{n}^T,\boldsymbol{\xi})(u_0)h_j^\beta(\boldsymbol{n}^T,\boldsymbol{\xi})(u_0)g_{\alpha\beta}(u_0).$$

It follows from Corollary 3.2 and Theorem 4.1 that

$$\det(A)=\left(\frac{(-1)^{k-1}}{(\ell_0)^{(n-1)}}\right)^2(K_\ell(\boldsymbol{n}^T,\boldsymbol{\xi})(u_0))^2\det(g_{\alpha\beta})=(\widetilde{K}_\ell(\boldsymbol{n}^T)(p,\boldsymbol{\xi}))^2\det(g_{\alpha\beta}).$$

This completes the proof. □

On the other hand, let $\overline{\boldsymbol{n}}^T$ be another timelike unit normal future directed vector field along $M=\boldsymbol{X}(U)$. Since the canonical action of $SO_0(1,n)$ on $H^n(-1)$ is transitive, there exists $g\in SO_0(1,n)$ such that $g.\boldsymbol{n}^T(u_0)=\overline{\boldsymbol{n}}^T(u_0)$. Then we define a smooth mapping

$$\Phi_g: N_1(M)_p[\boldsymbol{n}^T]\longrightarrow N_1(M)_p[\overline{\boldsymbol{n}}^T]$$

by $\Phi_g(p,\xi)=(p,g.\xi)$, where $p=\boldsymbol{X}(u_0)$. By the definition of the canonical Riemannian metrices on $N_1(M)_p[\boldsymbol{n}^T]$ and $N_1(M)_p[\overline{\boldsymbol{n}}^T]$, Φ_g is an isometry. Therefore, we have

$$\Phi_g^*d\mathfrak{v}_{N_1(M)[\overline{\boldsymbol{n}}^T](p,g.\xi)}=d\mathfrak{v}_{N_1(M)[\boldsymbol{n}^T](p,\xi)}.$$

We define the $k-2$-dimnsional lightcone unit sphere on the fibere as $S_+^{k-2}(N(M)_p)=S_+^{n-1}\cap N_p(M)$. Then we have $\widetilde{\mathbb{LG}}(\boldsymbol{n}^T)(N_1(M)_p[\boldsymbol{n}^T])\subset S_+^{k-2}(N(M)_p)$. Moreover, we can easily show that

$$\widetilde{\mathbb{LG}}(\boldsymbol{n}^T)|_{N_1(M)_p[\boldsymbol{n}^T]}:N_1(M)_p[\boldsymbol{n}^T]\longrightarrow S_+^{k-2}(N(M)_p)$$

is a diffeomorphism.

There exists a differential form $d\sigma_{k-2}(\boldsymbol{n}^T)$ of degree $k-2$ on $N_1(M)[\boldsymbol{n}^T]$ such that its restriction to a fiber is the volume element of the $k-2$-sphere. We remark that

$$d\mathfrak{v}_{N_1(M)[\boldsymbol{n}^T]}=d\mathfrak{v}_M\wedge d\sigma_{k-2}(\boldsymbol{n}^T).$$

Then we have the following key lemma:

Lemma 6.2 *Let $\boldsymbol{X}:U\longrightarrow\mathbb{R}^{n+1}_1$ be a spacelike embedding with codimension k and $\boldsymbol{n}^T,\overline{\boldsymbol{n}}^T$ be future directed unit timelike normal vector fields along $M=\boldsymbol{X}(U)$. For any $(p,\boldsymbol{\xi})\in N_1(M)[\boldsymbol{n}^T]$ with $p=\boldsymbol{X}(u_0)$, $g\in SO_0(1,n)$ and Φ_g are given in the previous paragraphes. Then we have*

$$|\widetilde{K}_\ell(\boldsymbol{n}^T)(p,\boldsymbol{\xi})|d\sigma_{k-2}(\boldsymbol{n}^T)_{\boldsymbol{\xi}}=|\widetilde{K}_\ell(\overline{\boldsymbol{n}}^T)(p,g.\boldsymbol{\xi})|d\sigma_{k-2}(\overline{\boldsymbol{n}}^T)_{g.\boldsymbol{\xi}}$$

and

$$\int_{N_1(M)_p[\boldsymbol{n}^T]}|\widetilde{K}_\ell(\boldsymbol{n}^T)(p,\boldsymbol{\xi})|d\sigma_{k-2}(\boldsymbol{n}^T)=\int_{N_1(M)_p[\overline{\boldsymbol{n}}^T]}|\widetilde{K}_\ell(\overline{\boldsymbol{n}}^T)(p,\overline{\boldsymbol{\xi}})|d\sigma_{k-2}(\overline{\boldsymbol{n}}^T).$$

Proof. Under the previous notations, we have

$$\begin{aligned}&\left(\widetilde{\mathbb{LG}}(\boldsymbol{n}^T)|_{N_1(M)_p[\boldsymbol{n}^T]}\right)^*d\mathfrak{v}_{S_+^{k-2}(N(M)_p)}\\&\quad=\left(\widetilde{\mathbb{LG}}(\boldsymbol{n}^T)^*d\mathfrak{v}_{S_+^{n-2}}\right)|_{N_1(M)_p[\boldsymbol{n}^T]}=|\widetilde{K}_\ell(\boldsymbol{n}^T)|d\sigma_{k-2}(\boldsymbol{n}^T).\end{aligned}$$

We remark that the canonical action of $SO_0(k-1)$ on $S_+^{k-2}(N(M)_p)$ is transitive. For any $h \in SO_0(k-1)$, we denote that $\psi(h)(v) = h.v$ for $v \in S_+^{k-2}(N(M)_p)$, so that we have an isometry $\psi(h) : S_+^{k-2}(N(M)_p) \longrightarrow S_+^{k-2}(N(M)_p)$. Thus, we have $\psi(h)^* d\mathfrak{v}_{S_+^{k-2}(N(M)_p)(v)} = d\mathfrak{v}_{S_+^{k-2}(N(M)_p)(h.v)}$.

On the other hand, we have

$$
\begin{aligned}
&\widetilde{\mathbb{LG}}(\overline{\boldsymbol{n}}^T)|_{N_1(M)_p[\overline{\boldsymbol{n}}^T]} \circ \Phi_g(p, \boldsymbol{\xi}) = \overline{\boldsymbol{n}}^T\widetilde{(u) + g.\boldsymbol{\xi}} \\
&= g.(\widetilde{\boldsymbol{n}^T(u)} + \boldsymbol{\xi}) = \psi(h)((\widetilde{\boldsymbol{n}^T(u) + \boldsymbol{\xi}})) = \psi(h) \circ \widetilde{\mathbb{LG}}(\boldsymbol{n}^T)|_{N_1(M)_p[\boldsymbol{n}^T]}(p, \boldsymbol{\xi}),
\end{aligned}
$$

for some $h \in SO_0(k-1)$. We set $\boldsymbol{v} = (\widetilde{\boldsymbol{n}^T(u) + \boldsymbol{\xi}}) = \widetilde{\mathbb{LG}}(\boldsymbol{n}^T)|_{N_1(M)_p[\boldsymbol{n}^T]}(p, \boldsymbol{\xi}) \in S_+^{k-2}(N(M)_p)$. Then we have

$$
\begin{aligned}
&(\widetilde{\mathbb{LG}}(\overline{\boldsymbol{n}}^T)|_{N_1(M)_p[\overline{\boldsymbol{n}}^T]} \circ \Phi_g)^* d\mathfrak{v}_{S_+^{k-2}(N(M)_p)(\boldsymbol{v})} \\
&= (\Phi_g)^* \left((\widetilde{\mathbb{LG}}(\overline{\boldsymbol{n}}^T)|_{N_1(M)_p[\boldsymbol{n}^T]})^* d\mathfrak{v}_{S_+^{k-2}(N(M)_p)(\boldsymbol{v})} \right) \\
&= (\Phi_g)^* \left(|\widetilde{K}_\ell(\overline{\boldsymbol{n}^T})| d\sigma_{k-2}(\overline{\boldsymbol{n}}^T)_{\boldsymbol{\xi}} \right) \\
&= |\widetilde{K}_\ell(\overline{\boldsymbol{n}^T})(p, g.\boldsymbol{\xi})| d\sigma_{k-2}(\overline{\boldsymbol{n}}^T)_{g.\boldsymbol{\xi}}.
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
&(\widetilde{\mathbb{LG}}(\boldsymbol{n}^T)|_{N_1(M)_p[\boldsymbol{n}^T]})^* \circ \psi(h)^* (d\mathfrak{v}_{S_+^{k-2}(N(M)_p)(\boldsymbol{v})}) \\
&= (\widetilde{\mathbb{LG}}(\boldsymbol{n}^T)|_{N_1(M)_p[\boldsymbol{n}^T]})^* d\mathfrak{v}_{S_+^{k-2}(N(M)_p)(h.\boldsymbol{v})} \\
&= |\widetilde{K}_\ell(\boldsymbol{n}^T)(p, \boldsymbol{\xi})| d\sigma_{k-2}(\boldsymbol{n}^T)_{\boldsymbol{\xi}}.
\end{aligned}
$$

Since $\widetilde{\mathbb{LG}}(\overline{\boldsymbol{n}}^T)|_{N_1(M)_p[\overline{\boldsymbol{n}}^T]} \circ \Phi_g(p, \boldsymbol{\xi}) = \psi(h) \circ \widetilde{\mathbb{LG}}(\boldsymbol{n}^T)|_{N_1(M)_p[\boldsymbol{n}^T]}(p, \boldsymbol{\xi})$, we have

$$
|\widetilde{K}_\ell(\overline{\boldsymbol{n}}^T)(p, g.\boldsymbol{\xi})| d\sigma_{k-2}(\overline{\boldsymbol{n}}^T)_{g.\boldsymbol{\xi}} = |\widetilde{K}_\ell(\boldsymbol{n}^T)(p, \boldsymbol{\xi})| d\sigma_{k-2}(\boldsymbol{n}^T)_{\boldsymbol{\xi}}.
$$

Moreover, we have

$$
\begin{aligned}
\int_{N_1(M)_p[\boldsymbol{n}^T]} |\widetilde{K}_\ell(\boldsymbol{n}^T)(p, \boldsymbol{\xi})| d\sigma_{k-2}(\boldsymbol{n}^T) &= \int_{\Phi_g(N_1(M)_p[\boldsymbol{n}^T])} |\widetilde{K}_\ell(\boldsymbol{n}^T)(p, g.\boldsymbol{\xi})| d\sigma_{k-2}(g.\boldsymbol{n}^T) \\
&= \int_{N_1(M)_p[\overline{\boldsymbol{n}}^T]} |\widetilde{K}_\ell(\overline{\boldsymbol{n}}^T)(p, \overline{\boldsymbol{\xi}})| d\sigma_{k-2}(\overline{\boldsymbol{n}}^T).
\end{aligned}
$$

This completes the proof. □

We call the integral

$$
K_\ell^*(p) = \int_{N_1(M)_p[\boldsymbol{n}^T]} |\widetilde{K}_\ell(\boldsymbol{n}^T)(p, \boldsymbol{\xi})| d\sigma_{k-2}(\boldsymbol{n}^T)
$$

a *total absolute lightcone curvature* of M at $p = \boldsymbol{X}(u_0)$. In the global situation, we consider a closed orientable manifold M with dimension s and a spacelike immersion $f : M \longrightarrow \mathbb{R}_1^{n+1}$. We define the *total absolute lightcone curvature* of M by the integral

$$
\tau_\ell(M, f) = \frac{1}{\gamma_{n-1}} \int_M K_\ell^*(p) d\mathfrak{v}_M = \frac{1}{\gamma_{n-1}} \int_{N_1(M)[\boldsymbol{n}^T]} |\widetilde{K}_\ell(\boldsymbol{n}^T)(p, \boldsymbol{\xi})| d\mathfrak{v}_{N_1(M)[\boldsymbol{n}^T]},
$$

where γ_{n-1} is the volume of the unit $n-1$-sphere S^{n-1}.

7 The Chern-Lashof type theorem

Let $f : M \longrightarrow \mathbb{R}^{n+1}_1$ be a spacelike immersion from an s-dimensional closed orientable manifold M. We have the family of lightcone height functions $H : M \times S^{n-1}_+ \longrightarrow \mathbb{R}$ defined by $H(x, \boldsymbol{v}) = \langle f(x), \boldsymbol{v} \rangle$. By Proposition 5.1, $\boldsymbol{v} \in S^{n-1}_+$ is a critical value of $\widetilde{\mathbb{LG}}(\boldsymbol{n}^T)$ if and only if there exists a point $p \in M$ such that p is a degenerate critical point $h_{\boldsymbol{v}}$. Therefore, we have the following proposition.

Proposition 7.1 *The height function $h_{\boldsymbol{v}}$ is a Morse function if and ony if $\boldsymbol{v}$ is a regular value of $\widetilde{\mathbb{LG}}(\boldsymbol{n}^T)$.*

Proof. By Proposition 5.1, $x \in M$ is a non-degenerate critical point of $h_{\boldsymbol{v}}$ if and only if there exists $\boldsymbol{\xi} \in N_1(M)_{f(p)}[\boldsymbol{n}^T]$ such that $\boldsymbol{v} = \widetilde{\mathbb{LG}}(\boldsymbol{n}^T)(f(p), \boldsymbol{\xi})$ and $(f(p), \boldsymbol{\xi})$ is a regular point of $\widetilde{\mathbb{LG}}(\boldsymbol{n}^T)$. By definition, all critical points of a Morse function are non-degenerate, so that the proof is completed. □

Let $D \subset S^{n-1}_+$ be the set of regular values of $\widetilde{\mathbb{LG}}(\boldsymbol{n}^T)$. Since M is compact, D is open and, by Sard's theorem, the compliment of D in S^{n-1}_+ has null measure. We define an intergral valued function $\eta : D \longrightarrow \mathbb{N}$ by

$$\eta(\boldsymbol{v}) = \text{the number of elemtns of } \widetilde{\mathbb{LG}}(\boldsymbol{n}^T)^{-1}(\boldsymbol{v}),$$

which turns out to be continuous.

Proposition 7.2

$$\tau_\ell(M, f) = \frac{1}{\gamma_{n-1}} \int_D \eta(\boldsymbol{v}) d\mathfrak{v}_{S^{n-1}_+}.$$

Proof. For any $\boldsymbol{v} \in D$, there exists a neighborhood U of $\boldsymbol{v}$ in D such that $\widetilde{\mathbb{LG}}(\boldsymbol{n}^T)^{-1}(U)$ is the disjoint union of connected open sets $V_1, \dots, V_k$, $k = \eta(\boldsymbol{v})$, on which $\widetilde{\mathbb{LG}}(\boldsymbol{n}^T) : V_i \longrightarrow U$ is a diffeomorphism. By Theorem 4.1, we have

$$\int_{V_i} |\widetilde{K}_\ell(\boldsymbol{n}^T)| d\mathfrak{v}_{N_1(M)[\boldsymbol{n}^T]} = \int_{V_i} \widetilde{\mathbb{LG}}(\boldsymbol{n}^T)^* d\mathfrak{v}_{S^{n-1}_+} = \deg\left(\widetilde{\mathbb{LG}}(\boldsymbol{n}^T)|V_i\right) \int_U d\mathfrak{v}_{S^{n-1}_+} = \int_U d\mathfrak{v}_{S^{n-1}_+}.$$

Since $\gamma_{n-1} = \int_{S^{n-1}_+} d\mathfrak{v}_{S^{n-1}_+}$ and $\widetilde{K}_\ell(\boldsymbol{n}^T)$ is zero at a singular point of $\widetilde{\mathbb{LG}}(\boldsymbol{n}^T)$, we have

$$\frac{1}{\gamma_{n-1}} \int_{N_1(M)[\boldsymbol{n}^T]} |\widetilde{K}_\ell(\boldsymbol{n}^T)| d\mathfrak{v}_{N_1(M)[\boldsymbol{n}^T]} = \frac{1}{\gamma_{n-1}} \int_D \eta(\boldsymbol{v}) d\mathfrak{v}_{S^{n-1}_+}.$$

□

We recall that the Morse number of a compact manifold M, $\gamma(M)$, is defined to be the minimum number of ciritical points for any Morse function $\phi : M \longrightarrow \mathbb{R}$.

Theorem 7.3 (The Chern-Lashof type theorem) *Let $f : M \longrightarrow \mathbb{R}^{n+1}_1$ be a spacelike immersion of a compact s-dimensional manifold M. Then*

(1) $\tau_\ell(M, f) \geq \gamma(M) \geq 2$,

(2) *If $\tau_\ell(M, f) < 3$, then M^s is homeomorphic to an s-sphere.*

Proof. Since each Morse function $h_{\boldsymbol{v}}$ certainly satisfies $\eta(\boldsymbol{v}) \geq \gamma(M)$, we have $\tau_\ell(M, f) \geq \gamma(M)$. Since M is compact, there exist at least two critical points for any smooth function on M, so that $\gamma(M) \geq 2$. If $\tau_\ell(M, f) < 3$, there must be a set U of positive measure on which $\eta(\boldsymbol{v}) = 2$. So there is a non-degenerate $h_{\boldsymbol{v}}$ with two cirtical points, and M is homeomorphic to S^s by Reeb's theorem (see, [12]). □

If $\tau_\ell(M, f) = \gamma(M)$, then every non-degenerate lightlike height function $h_{\boldsymbol{v}}$ has the minimum number of citical points allowed by the Morse inequalities. In this case we say that f is a *lightlike-tight spacelike immersion* (or, simply, *L-tight spacelike immersion*). In §9, we consider the problem to characterize the L-tightness for spacelike immersed spheres.

8 Codimension two spacelike submanifolds

In the case when $s = n - 1$, $N_1(M)[\boldsymbol{n}^T]$ is a double covering of M. If M is orientable, we can choose global section $\boldsymbol{\sigma}(p) = (p, \boldsymbol{n}^S(p))$ of $N_1(M)[\boldsymbol{n}^T]$. Let $\pi : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n_0$ be the canonical projection defined by $\pi(x_0, x_1, \ldots, x_n) = (0, x_1, \ldots, x_n)$, where $\mathbb{R}^n_0$ is the Euclidean space given by $x_0 = 0$. Since $\operatorname{Ker} d\pi_{f(p)}$ is a timelike one-dimensional subspace of $\mathbb{R}^{n+1}_1$ and $\boldsymbol{n}^S$ is spacelike, $d\pi_{f(p)}(\boldsymbol{n}^S(p))$ is transverse to $\pi \circ f(M)$ at $p \in M$. Therefore, if M is closed and $f : M \longrightarrow \mathbb{R}^{n+1}_1$ is a spacelike embedding such that $\pi \circ f : M \longrightarrow \mathbb{R}^n_0$ is an embedding, then we can choose the direction of $\boldsymbol{n}^S$ such that $d\pi \circ \boldsymbol{n}^S$ points the direction to the outward of $\pi \circ f(M)$.

In [11] it has been shown that $(\boldsymbol{n}^T\widetilde{(p) \pm \boldsymbol{n}}^S(p))$ is independent of the choice of $\boldsymbol{n}^T$. Therefore, we have the global lightcone Gauss map

$$\widetilde{\mathbb{LG}}_\pm : M \longrightarrow S^{n-1}_+$$

defined by $\widetilde{\mathbb{LG}}_\pm(p) = (\boldsymbol{n}^T\widetilde{(p) \pm \boldsymbol{n}}^S(p))$. Moreover, we have defined the lightcone Gauss-Kronecker curvature $\widetilde{K}^\pm_\ell(p) = \widetilde{K}_\ell(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)(p)$ of M in [11] . Since $\widetilde{\mathbb{LG}}(\boldsymbol{n}^T)(p, \pm\boldsymbol{n}^S(p)) = \widetilde{\mathbb{LG}}_\pm(p)$, we have

$$\widetilde{K}^\pm_\ell(p) = \widetilde{K}_\ell(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)(p) = \widetilde{K}_\ell(\boldsymbol{n}^T)(p, \pm\boldsymbol{n}^S(p)).$$

In [11] we have shown the following Gauss-Bonnet type theorem:

Theorem 8.1 ([11]) *Suppose that M is a closed orientable $n-1$-dimensional manifold, $n-1$ is even and $f : M \longrightarrow \mathbb{R}^{n+1}_1$ is a spcelike embedding. Then*

$$\int_M \widetilde{K}^\pm_\ell d\mathfrak{v}_{M^{n-1}} = \frac{1}{2}\gamma_{n-1}\chi(M),$$

where $\chi(M^{n-1})$ is the Euler characteristic of M^{n-1}.

In order to prove the above theorem, it has been shown in [11] that $\widetilde{K}^\pm_\ell d\mathfrak{v}_M = (\widetilde{\mathbb{LG}}_\pm)^* d\mathfrak{v}_{S^{n-1}_+}$. Let $D^\pm \subset S^{n-1}_+$ denote the set of regular value of $\widetilde{\mathbb{LG}}_\pm$ and $D = D^+ \cap D^-$. We define a mapping $\eta^\pm : D \longrightarrow \mathbb{N}$ by

$$\eta^\pm(\boldsymbol{v}) = \text{the number of elements of } (\widetilde{\mathbb{LG}}_\pm)^{-1}(\boldsymbol{v}).$$

We have the following proposition:

Proposition 8.2 *Suppose that M is a closed orientable $n-1$-dimensional manifold and $f : M \longrightarrow \mathbb{R}^{n+1}_1$ is a spcelike embedding. Then*

$$\int_M |\widetilde{K}^{\pm}_{\ell}| d\mathfrak{v}_M = \int_D \eta^{\pm}(\boldsymbol{v}) d\mathfrak{v}_{S^{n-1}_+}.$$

Proof. Since $\widetilde{K}^{\pm}_{\ell} d\mathfrak{v}_M = (\widetilde{\mathbb{LG}}_{\pm})^* d\mathfrak{v}_{S^{n-1}_+}$, we can prove by exactly the same arguments as those in the proof of Proposirion 7.2. □

Theorem 8.3 *Suppose that M is a closed orientable $n-1$-dimensional manifold and $f : M \longrightarrow \mathbb{R}^{n+1}_1$ is a spcelike embedding such that $\pi \circ f$ is an embedding. Then*

$$\int_M |\widetilde{K}^{\pm}_{\ell}| d\mathfrak{v}_M \geq \gamma_{n-1}.$$

The equality holds if and only if $\widetilde{\mathbb{LG}}_{\pm}$ is bijective on the regular values.

Proof. Since $\pi \circ f$ is an embedding, we can choose the vector field $\boldsymbol{n}^S$ along M such that $d\pi \circ \boldsymbol{n}^S$ is a transversal inward vector filed over $\pi \circ f(M)$ in $\mathbb{R}^n_0$. It is enough to show that both of $\widetilde{\mathbb{LG}}_{\pm}$ are surjective onto D. By Proposition 3.3, $p \in M$ is a critical point of the lightcone height function $h_{\boldsymbol{v}}$ if and only if $\boldsymbol{v} = \widetilde{\mathbb{LG}}(\boldsymbol{n}^T)(p, \boldsymbol{\xi})$ for some $\boldsymbol{\xi} \in N_1(M)_p[\boldsymbol{n}^T]$. Since the codimension of M is two, the last condition is equivalent to the condition $\boldsymbol{v} = \widetilde{\mathbb{LG}}(\boldsymbol{n}^T)(p, \boldsymbol{\xi}) = \widetilde{\mathbb{LG}}_+(p)$ or $\boldsymbol{v} = \widetilde{\mathbb{LG}}(\boldsymbol{n}^T)(p, -\boldsymbol{\xi}) = \widetilde{\mathbb{LG}}_-(p)$. For any $\boldsymbol{v} \in S^{n-1}_+$, there exists the maximum point p_0 and the minimum point q_0 of the lightcone height function $h_{\boldsymbol{v}}$ on the compact manifold M. These points are critical points of $h_{\boldsymbol{v}}$, so that $\boldsymbol{v} = \widetilde{\mathbb{LG}}_+(p_0)$ or $\boldsymbol{v} = \widetilde{\mathbb{LG}}_-(p_0)$ (and $\boldsymbol{v} = \widetilde{\mathbb{LG}}_+(q_0)$ or $\boldsymbol{v} = \widetilde{\mathbb{LG}}_-(q_0)$). It is enought to show that $\widetilde{\mathbb{LG}}_+(p_0) \neq \widetilde{\mathbb{LG}}_+(q_0)$. Suppose that $\widetilde{\mathbb{LG}}_+(p_0) = \widetilde{\mathbb{LG}}_+(q_0)$. We define a function $\widetilde{h}_{\boldsymbol{v}} : \mathbb{R}^4_1 \longrightarrow \mathbb{R}$ by $\widetilde{h}_{\boldsymbol{v}}(\boldsymbol{x}) = \langle \boldsymbol{v}, \boldsymbol{x} \rangle$. It follows that $\widetilde{h}_{\boldsymbol{v}} \circ f(p) = h_{\boldsymbol{v}}(p)$. We distinguish two cases.

(i) If $\boldsymbol{v} = \widetilde{\mathbb{LG}}_+(p_0)$, then we have $\boldsymbol{v} = \widetilde{\mathbb{LG}}_+(q_0)$. We consider the line from $f(q_0)$ directed by $-\boldsymbol{n}^S(q_0)$, parametrized by

$$\boldsymbol{\gamma}_{q_0}(t) = f(q_0) - t\boldsymbol{n}^S(q_0).$$

Then we have

$$\begin{aligned} \frac{d\widetilde{h}_{\boldsymbol{v}} \circ \boldsymbol{\gamma}_{q_0}}{dt}(t) &= \langle -\boldsymbol{n}^S(q_0), \boldsymbol{v} \rangle = \langle -\boldsymbol{n}^S(q_0), \widetilde{\mathbb{LG}}_+(q_0) \rangle \\ &= \left\langle -\boldsymbol{n}^S(q_0), \frac{1}{\ell^+_0(q_0)}(\boldsymbol{n}^T(q_0) + \boldsymbol{n}^S(q_0)) \right\rangle = -\frac{1}{\ell^+_0(q_0)} < 0. \end{aligned}$$

It follows that $\widetilde{h}_{\boldsymbol{v}} \circ \boldsymbol{\gamma}_{q_0}(t)$ is strictly decreasing. Since q_0 is the minimum point of $h_{\boldsymbol{v}}$ and $f(q_0) = \boldsymbol{\gamma}_{q_0}(0)$, $\boldsymbol{\gamma}_{q_0}(t) \notin f(M)$ for any $t > 0$. Thus, we have $\pi \circ \boldsymbol{\gamma}_{q_0}(t) \notin \pi \circ f(M)$ for any $t > 0$. Since $\pi \circ \boldsymbol{\gamma}_{q_0}$ is a line in $\mathbb{R}^n_0$, there exists a positive real number τ such that $\pi \circ \boldsymbol{\gamma}_{q_0}(\tau)$ is in the outside of $\pi \circ f(M)$. On the other hand, since $d\pi \circ \boldsymbol{n}^S$ is an inward transversal vector field along $\pi \circ f(M)$ in $\mathbb{R}^n_0$, there exists a sufficiently small $\varepsilon > 0$ such that $\pi \circ \boldsymbol{\gamma}_{q_0}(\varepsilon)$ is in the inside of $\pi \circ f(M)$. By the Jordan-Brouwer separation theorem, there exists a real number $t_0 > 0$ such that $\pi \circ \boldsymbol{\gamma}_{q_0}(t_0) \in \pi \circ f(M)$. This is a contradiction.

(ii) If $\boldsymbol{v} = \widetilde{\mathbb{LG}}_+(p_0)$, then we also consider the line from $f(p_0)$ defined by

$$\boldsymbol{\gamma}_{p_0}(t) = f(p_0) - t\boldsymbol{n}^S(p_0).$$

Then we have

$$\frac{d\widetilde{h}_{\boldsymbol{v}} \circ \boldsymbol{\gamma}_{p_0}}{dt}(t) = \langle -\boldsymbol{n}^S(p_0), \boldsymbol{v}\rangle = \langle -\boldsymbol{n}^S(p_0), \widetilde{\mathbb{LG}}_-(p_0)\rangle$$
$$= \left\langle -\boldsymbol{n}^S(p_0), \frac{1}{\ell_0^+(p_0)}(\boldsymbol{n}^T(p_0) - \boldsymbol{n}^S(p_0)\right\rangle = \frac{1}{\ell_0^+(q_0)} > 0,$$

so that $\widetilde{h}_{\boldsymbol{v}} \circ \boldsymbol{\gamma}_{p_0}(t)$ is strictly increasing. Since p_0 is the maximum point of $h_{\boldsymbol{v}}$ and $f(p_0) = \boldsymbol{\gamma}_{p_0}(0)$, $\boldsymbol{\gamma}_{p_0}(t) \notin f(M)$ for any $t > 0$. By exactly thesame reason as in the case (i), there exists a real number $t_0 > 0$ such that $\boldsymbol{\gamma}_{p_0}(t_0) \in \pi \circ f(M)$. This is a contradiction. □

We now define the *total absolute lightcone curvature* of a spacelike embedding $f : M \longrightarrow \mathbb{R}^{n+1}_1$ from a closed orientable $n-1$-dimnsional manifold by

$$\tau^{\pm}_{\ell}(M, f) = \frac{1}{\gamma_{n-1}} \int_M |\widetilde{K}^{\pm}_{\ell}| d\mathfrak{v}_M.$$

We remark that we have the following weaker inequality from Theorem 7.3:

$$\tau^{+}_{\ell}(M, f) + \tau^{-}_{\ell}(M, f) = \tau_{\ell}(M, f) \geq 2.$$

There are examples such that

$$\tau^{+}_{\ell}(M, f) \neq \tau^{-}_{\ell}(M, f)$$

(see Subsection 8.2).

For an even dimensional manifold M, we have the following theorem.

Theorem 8.4 *Let $f : M \longrightarrow \mathbb{R}^{n+1}_1$ be a spacelike embedding from a closed orientable $n-1$-dimensional manifold. Suppose n is an odd number. Then we have*

$$\int_M |\widetilde{K}^{\pm}_{\ell}| d\mathfrak{v}_M \geq \frac{1}{2}\gamma_{n-1}(4 - \chi(M)),$$

where $\chi(M)$ is the Euler characteristic of M.

Proof. Consider the lightcone Gauss map $\widetilde{\mathbb{LG}}_{\sigma} : M \longrightarrow S^{n-1}_+$, where σ is $+$ or $-$. We define $M^+ = \{p \in M \mid \widetilde{K}^{\sigma}_{\ell} > 0\}$ and $M^- = \{p \in M \mid \widetilde{K}^{\sigma}_{\ell} < 0\}$. Then we can write

$$\int_M |\widetilde{K}^{\sigma}_{\ell}| d\mathfrak{v}_M = \int_{M^+} \widetilde{K}^{\sigma}_{\ell} d\mathfrak{v}_M - \int_{M^-} \widetilde{K}^{\sigma}_{\ell} d\mathfrak{v}_M$$

and

$$\int_M \widetilde{K}^{\sigma}_{\ell} d\mathfrak{v}_M = \int_{M^+} \widetilde{K}^{\sigma}_{\ell} d\mathfrak{v}_M + \int_{M^-} \widetilde{K}^{\sigma}_{\ell} d\mathfrak{v}_M.$$

By Theorem 8.1 and the above equations, we have

$$\int_M |\widetilde{K}^{\sigma}_{\ell}| d\mathfrak{v}_M = 2\int_{M^+} \widetilde{K}^{\sigma}_{\ell} d\mathfrak{v}_M - \frac{1}{2}\gamma_{n-1}\chi(M).$$

Thus, it is enough to show that

$$\int_{M^+} \widetilde{K}^{\sigma}_{\ell} d\mathfrak{v}_M \geq \gamma_{n-1}.$$

Let M_0, M_1, M_2, M_2^+ be the subsets of M defined by $M_0 = (\widetilde{K}_\ell^\sigma)^{-1}(0)$, $M_1 = \{p \in M \setminus M_0 \mid \exists q \in M_0 \; with \; \widetilde{\mathbb{LG}}_\sigma(q) = \widetilde{\mathbb{LG}}_\sigma(p) \}$, $M_2 = M \setminus (M_0 \cup M_1)$ and $M_2^+ = M^+ \cap M_2$. Since M_0 is the singular set of $\widetilde{\mathbb{LG}}_\sigma$, $\widetilde{\mathbb{LG}}_\sigma(M_0)$ hs measure zero by Sard's Theorem and also $\widetilde{\mathbb{LG}}_\sigma(M_0) \cup \widetilde{\mathbb{LG}}_\sigma(M_1)$ is a mesure zero set in S_+^2. For any $\boldsymbol{v} \in S_+^2 \setminus (\widetilde{\mathbb{LG}}_\sigma(M_0) \cup \widetilde{\mathbb{LG}}_\sigma(M_1))$, the lightcone height function $h_{\boldsymbol{v}}$ has at least two critical points: a maximum and a minimum. In [11], it was shown that

$$\widetilde{K}_\ell^\sigma(p) = \frac{\det \mathrm{Hess}\,(h_{\boldsymbol{v}}(p))}{\det\,(g_{ij}(p))},$$

where $\boldsymbol{v} = \widetilde{\mathbb{LG}}_\sigma(p)$. Since $\boldsymbol{v}$ is a regular value of $\widetilde{\mathbb{LG}}_\sigma$, $h_{\boldsymbol{v}}$ has a Morse-type singular point with index 0 or $n-1$ at the minimum point and the maximum point. The lightcone Gauss-Kronecker curvarture $\widetilde{K}_\ell^\sigma$ is positive at such points, so that $\widetilde{\mathbb{LG}}_\sigma | M^+$ is surjective. This completes the proof. □

As a special case for $n = 3$, we have the following corollary.

Corollary 8.5 *For a spacelike embedding $f : M \longrightarrow \mathbb{R}_1^4$ from a closed orientable surface M, we have*

$$\int_M |\widetilde{K}_\ell^\pm| d\mathfrak{v}_M \geq 2\pi(4 - \chi(M)),$$

We define the *lightcone mean curvature* of M at p by

$$\widetilde{H}_\ell^\pm(p) = \frac{1}{2}\mathrm{Trace}\,\widetilde{S}_p^\pm = \frac{1}{2}(\widetilde{\kappa}_1^\pm(p) + \widetilde{\kappa}_2^\pm(p)),$$

where $\widetilde{S}_p^\pm = \widetilde{S}(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)_p$. Then we have the following proposition.

Proposition 8.6 *For a spacelike embedding $f : M \longrightarrow \mathbb{R}_1^4$ from a closed orientable surface M, we have*

$$\int_M (\widetilde{H}_\ell^\pm)^2 d\mathfrak{v}_M \geq 4\pi.$$

The equality holds if and only if $f : M \longrightarrow \mathbb{R}_1^4$ is totally umbilical with a non-zero normalized principal curvature.

Proof. Since $\widetilde{H}_\ell^\pm(p) = (\widetilde{\kappa}_1^\pm(p) + \widetilde{\kappa}_2^\pm(p))/2$ and $\widetilde{K}_\ell^\pm = \widetilde{\kappa}_1^\pm(p)\widetilde{\kappa}_2^\pm(p)$, we have

$$(\widetilde{H}_\ell^\pm)^2(p) - \widetilde{K}_\ell^\pm(p) = (\widetilde{\kappa}_1^\pm(p) - \widetilde{\kappa}_2^\pm(p))^2 \geq 0.$$

It follows that

$$\int_M (\widetilde{H}_\ell^\pm)^2 d\mathfrak{v}_M \geq \int_{M^+} (\widetilde{H}_\ell^\pm)^2 d\mathfrak{v}_M \geq \int_{M^+} \widetilde{K}_\ell^\pm d\mathfrak{v}_M.$$

By the assertion in the proof of Theorem 8.4, we have

$$\int_{M^+} \widetilde{K}_\ell^\pm d\mathfrak{v}_M \geq \gamma_2 = 4\pi.$$

The equality holds if and only if

$$\int_{M^+} \left((\widetilde{H}_\ell^\pm)^2 - \widetilde{K}_\ell^\pm \right) d\mathfrak{v}_M = 0.$$

This means that $\widetilde{\kappa}_1^\pm(p) = \widetilde{\kappa}_2^\pm(p)$ for any $p \in M$, so that M is totally umibilical. This completes the proof. □

Remark 8.7 (1) In [8] it was shown that there exists a parallel timelike future directed unit normal vector fieild $\boldsymbol{n}^T$ along $f : M \longrightarrow \mathbb{R}^4_1$ and totally umbilical with a non-zero lightcone principal curvature if and only if M is embedded in the lightcone. It is well konwn that if a compact surface M is embedded in the lightcone, it is homeomorphic to a sphere. In this case the normalized lgihtcone principal curvature is constant, but the lightcone principal curvature is not constant. So, the suface $f(M)$ is not necessarily a round sphere.

On the other hand, suppose that $f(M)$ is in the Euclidean space or the hyperbolic space. Since the intersection of the lightcone with Euclidean space or the hyperbolic space is a round sphere, the equality of the above theorem holds if and only if $f(M)$ is a round sphere.
(2) The above proposition induces the lgihtcone version of the Willmore conjecture:

Conjecture. Let M be a torus. Then for a spacelike immersion $f : M \longrightarrow \mathbb{R}^4_1$, we expect to have

$$\int_M (\widetilde{H}^{\pm}_\ell)^2 d\mathfrak{v}_M \geq 2\pi^2.$$

If M is immersed into the Euclidean space $\mathbb{R}^3_0$, then we have the original Willmore conjecture (cf.§10). Moreover, any Willmore surface M immersed in $\mathbb{R}^3_0$ satisfy the equality. Moreover, if M is immersed into the hyperbolic space $H^3(-1)$, we have the horospherical Willmore conjecture (cf., §10). Therefore we have the following problem.

Problem. Characterize a spacelike torus in $\mathbb{R}^4_1$ such that the equality holds.

9 Lightlike tight spacelike spheres

In this section we consider the characterizations of L-tightness for spacelike spheres. Let $f : M \longrightarrow \mathbb{R}^{n+1}_1$ be a spacelike immersion of a closed orientable manifold M. We remind the reader that f is called an L-tight if every non-degenerate lightcone height function $h_{\boldsymbol{v}}$ has the minimum number of critical points required by the Morse inequalities. If M is homeomorphic to a sphere, then the Morse number $\gamma(M)$ is equal to 2. We have the following theorem.

Theorem 9.1 *Let $f : M \longrightarrow \mathbb{R}^{n+1}_1$ be a spacelike immersion of a closed orientable manifold M. Then the following conditions are equivalent:*
(1) *M is homeomorphic to a sphere and f is L-tight,*
(2) *$\tau_\ell(M, f) = 2$.*

Proof. We use the function $\eta : D \longrightarrow \mathbb{N}$ defined before Proposition 7.2 in §7. Here, D is the regular value set of $\widetilde{\mathbb{LG}}(\boldsymbol{n}^T)$. Since M is compact, D is open and $S^{n-1}_+ \setminus D$ has null measure by the Sard theorem. By Proposition 7.2, $\tau_\ell(M, f) = 2$ if and only if $\eta(\boldsymbol{v}) = 2$. This condition is equivalent to the following condition:

($*$) The lightcone Gauss map of $N_1(M)[\boldsymbol{n}^T]$ takes every regular value exaxtly twice.

Suppose that the conditon (1) holds. Then $\gamma(M) = 2$. Since f is L-tight, the lightcone height funtion $h_{\boldsymbol{v}}$ for $\boldsymbol{v} \in D$ has exactly $\gamma(M) = 2$ non-degenerate critical points. This is equivalent to the condition ($*$). For the converse, suppose that the conditoin ($*$) holds. Then $h_{\boldsymbol{v}}$ for $\boldsymbol{v} \in D$ has exactly 2 non-dgenerate critical points, so that f is L-tight. By the assertion (2) of Theorem 7.3, M is homeomorphic to a sphere. This completes the proof. $\square$

By the above theorem, if M is a sphere, $\tau_\ell(S^s, f) = 2$ if and only if f is L-tight. In order to give a further characteriztion, we introduce the following notion: Let V be a codimension two spacelike affine subspace of $\mathbb{R}^{n+1}_1$. We define $\overline{V}$ as a spacelike subspace parallel to V. Since $\overline{V}^{\perp}$ is a Lorentz plane, there exists a pseudo-orthonormal basis $\{\boldsymbol{v}^T, \boldsymbol{v}^S\}$ of $\overline{V}^{\perp}$ then we have lightlike vectors $\boldsymbol{v}^+ = \boldsymbol{v}^T + \boldsymbol{v}^S, \boldsymbol{v}^- = \boldsymbol{v}^T - \boldsymbol{v}^S$. There exist $p \in \mathbb{R}^{n+1}_1$ such that $V = p + \overline{V}$. For any $\boldsymbol{w} \in \overline{V}$, $\langle p + \boldsymbol{w}, \boldsymbol{v}^{\pm}\rangle = \langle p, \boldsymbol{v}^{\pm}\rangle = c^{\pm}$ are constant numbers. We consider lightlike hyperplanes $HP(\boldsymbol{v}^{\pm}, c^{\pm})$. Then we have

$$V = HP(\boldsymbol{v}^+, c^+) \cap HP(\boldsymbol{v}^-, c^-).$$

For a point $p \in M$, we say that a codimnsion two spacelike affine subspace V is a *codimension two spacelike tangent space* if $T_pM \subset \overline{V}$. Moreover, $HP(\boldsymbol{v}^{\pm}, c^{\pm})$ are said to be *tangent lightlike hayperplanes* of M at p. Let K be a subset of $\mathbb{R}^{n+1}_1$. A hyperplane HP through a point $\boldsymbol{x} \in K$ is called a *support plane* of K if K lies entirely in one of the closed half-spaces determined by HP. The half-space is called a *support half-space*. Let M be a compact orientable $n-1$-dimensional manifold. Then we have unique two lightlike tangent hyperplanes of $f(M)$ at each point $p \in M$. These hyprplanes are $HP(\boldsymbol{v}^{\pm}, c^{\pm})$, where $\boldsymbol{v}^{\pm} = \boldsymbol{n}^T(p) \pm \boldsymbol{n}^S(p)$ and $c^{\pm} = \langle f(p), \boldsymbol{n}^T(p) \pm \boldsymbol{n}^S(p)\rangle$. In this case, we say that $f(M)$ is *lightlike convex* (or, *L-convex* in short) if for each $p \in M$, the lightlike tangent hyperplanes of $f(M)$ at $f(p)$ are support planes of $f(M)$.

We consider the case that M is a sphere. Let $f : S^s \longrightarrow \mathbb{R}^{n+1}_1$ be a spacelike immersion. If $s = n-1$, we have the following theorem.

Theorem 9.2 *Let $f : S^{n-1} \longrightarrow \mathbb{R}^{n+1}_1$ be a spacelike embedding. Then the following conditions are equivalent:*

(1) *f is L-convex,*

(2) *$\tau_\ell(S^{n-1}, f) = 2$,*

(3) *f is L-tight.*

Generally the following condition (4) *implies the condition* (2)*. If we assume that n is odd or $\pi \circ f : M \longrightarrow \mathbb{R}^n_0$ is an embedding, then the condition* (2) *implies the condition* (4)*.*

(4) *$\tau^+_\ell(S^{n-1}, f) = \tau^-_\ell(S^{n-1}, f) = 1$.*

Proof. By Theorem 9.1, the conditons (2) and (3) are equivalent. By Theorem 8.4, the condition (2) implies (4) for the case when n is odd. If $\pi \circ f$ is an embedding, Theorem 8.3 asserts that the condition (2) implies (4) even for the case when n is even. It is trivial that the condition (4) implies the condition (2).

We now give a proof that the conditions (1) and (3) are equivalent. Suppose that f is L-tight. If f is not L-convex, then there exists $p \in S^{n-1}$ and $\boldsymbol{v} \in S^{n-1}_+$ such that one of the tangent lightlike hyperplanes at p separates $f(S^{n-1})$ into two parts. Then we have $\boldsymbol{v} = \widetilde{\mathbb{L}}^{\sigma}(p)$ for $\sigma = \pm$ (i.e., p is a critical point of $h_{\boldsymbol{v}}$). If p is a non-degenerate critical point, it contradicts to the assumption that f is L-tight. If p is a degenerate critical point, under a small perturbation of $\boldsymbol{v} \in S^{n-1}_+$, we have a non-degenerate critical point which is neither the maximum nor the minimum point. This also contradicts to the assumption that f is L-tight. We now suppose that f is not L-tight. Then there exists a non-degenerate lightcone height function $h_{\boldsymbol{v}}$ which as at least three critical points. If necessary, under a small perturbation of $\boldsymbol{v} \in S^{n-1}$, all critical values of $h_{\boldsymbol{v}}$ are different. It follows that there exists a critical point $p \in S^{n-1}$ of $h_{\boldsymbol{v}}$ such that neither the maximum nor the minimum point of $h_{\boldsymbol{v}}$. This means that one of the tangent lightlike hyperplanes of $f(S^{n-1})$ locally separates $f(S^{n-1})$ into at least two parts. Therefore, f is not lightlike convex. $\square$

We now consider the case when $n + 1 - s > 2$. For any $(p, \xi) \in N_1(M)[\boldsymbol{n}^T]$, we consider the lightlike tangent hyperplanes $HP(\boldsymbol{v}^{\pm}_p, c^{\pm})$, where $\boldsymbol{v}^{\pm}_p = \boldsymbol{n}^T(p) \pm \xi$ and $c^{\pm} = \langle f(p), \boldsymbol{v}^{\pm}_p\rangle$. We denote that $T_S M[\boldsymbol{n}^T, \xi]_p = HP(\boldsymbol{v}_p^+, c^+) \cap HP(\boldsymbol{v}_p^-, c^-)$, which is called a *spacelike tangent affine space with codimension two* of $f(M)$ at $p \in M$. We also define

$$\begin{aligned} F_\ell(\boldsymbol{n}^T(p), \pm\xi) &= \{ \boldsymbol{x} \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{x} - f(p), \boldsymbol{v}_p^\pm \rangle \leq 0 \ \} \\ P_\ell(\boldsymbol{n}^T(p), \pm\xi) &= \{ \boldsymbol{x} \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{x} - f(p), \boldsymbol{v}_p^\pm \rangle \geq 0 \ \}. \end{aligned}$$

We call $F_\ell(\boldsymbol{n}^T(p), \xi)$ (respectively, $P_\ell(\boldsymbol{n}^T(p), \pm\xi)$) the *future regions* (respectively, the *past regions*) with respect to $(\boldsymbol{n}^T(p), \pm\xi)$. We have a closed subset

$$S(\boldsymbol{n}^T(p), \xi) = \mathbb{R}^{n+1}_1 \setminus \mathrm{Int}\left((F_\ell(\boldsymbol{n}^T(p), +\xi) \cap F_\ell(\boldsymbol{n}^T(p), -\xi)) \cup (P_\ell(\boldsymbol{n}^T(p), +\xi) \cap P_\ell(\boldsymbol{n}^T(p), -\xi) \right),$$

which is called the *spacelike region* with respect to $(\boldsymbol{n}^T(p), \xi)$. Here, $\mathrm{Int}X$ is the interior of X. We also consider the following subsets of $S(\boldsymbol{n}^T(p), \xi)$:

$$S^+(\boldsymbol{n}^T(p), \xi) \ = \ \{\boldsymbol{x} | \langle \boldsymbol{x} - f(p), \boldsymbol{v}_p^+ \rangle \geq 0, \langle \boldsymbol{x} - f(p), \boldsymbol{v}_p^- \rangle \leq 0 \text{ and } \langle \boldsymbol{x} - f(p), \xi \rangle \geq 0 \}.$$

We remark that $\xi \in S^+(\boldsymbol{n}^T(p), \xi)$. Then we have the following lemma.

Lemma 9.3 *Let $f : M \longrightarrow \mathbb{R}^{n+1}_1$ be a spacelike embedding of a closed orientable manifold with* $\dim M < n-1$*. If f is L-tight, then there exists a spacelike affine subspace $V \subset \mathbb{R}^{n+1}_1$ with* $\dim V = n-1$ *such that $f(M) \subset V$.*

Proof. Since f is L-tight, the lightlike tangent hyperplanes at any point $p \in M$ are the support plane of $f(M)$.

Suppose that there exists $p \in M$ such that

$$f(M) \subset F_\ell(\boldsymbol{n}^T(p), +\xi) \cap F_\ell(\boldsymbol{n}^T(p), -\xi),$$

for any $\xi \in N_1(M)_p[\boldsymbol{n}^T]$. We arbitrary choose $\xi \in N_1(M)_p[\boldsymbol{n}^T]$. Since $HP(\boldsymbol{v}_p^\pm, c^\pm)$ are the tangent lightlike hyperplanes at $p \in M$, we have $T_{f(p)}f(M) \subset T_S M[\boldsymbol{n}^T, \xi]_p$. By the fact $\dim T_S M[\boldsymbol{n}^T, \xi]_p = n-1$ and the assumption $\dim M < n-1$, there exists $\xi' \in N(M)_p[\boldsymbol{n}^T]$ such that $\xi \neq \xi'$ and

$$f(M) \subset (F_\ell(\boldsymbol{n}^T(p), +\xi') \cap F_\ell(\boldsymbol{n}^T(p), -\xi')) \bigcap (F_\ell(\boldsymbol{n}^T(p), +\xi) \cap F_\ell(\boldsymbol{n}^T(p), -\xi)).$$

Therefore, we have $T_{f(p)}f(M) \subset T_S M[\boldsymbol{n}^T, \xi]_p \cap T_S M[\boldsymbol{n}^T, \xi']_p$. We can inductively proceed this process, so that we have

$$f(p) + T_{f(p)}f(M) \subset \bigcap_{i=1}^{\ell} T_S M[\boldsymbol{n}^T, \xi]_p.$$

However, there exists ℓ such that $\dim \bigcap_{i=1}^{\ell} T_S M[\boldsymbol{n}^T, \xi]_p < \dim M$. This is a contradiction. Therefore, $f(M) \subset S(\boldsymbol{n}^T(p), \xi)$ at any point $p \in M$.

Suppose that $f(M) \subset S^+(\boldsymbol{n}^T(p), \xi)$ at a point $p \in M$. Since $\dim M < n-1$, there exists a closed loop $\boldsymbol{\gamma} : [0,1] \longrightarrow N_1(M)[\boldsymbol{n}^T]_p$ such that $\boldsymbol{\gamma}(0) = \boldsymbol{\gamma}(1) = \xi$ and $\boldsymbol{\gamma}(1/2) = -\xi$. By the assumption that f is L-tight, there exists $\bar{\xi} \in N_1(M)[\boldsymbol{n}^T]_p$ such that

$$f(M) \subset S^+(\boldsymbol{n}^T(p), \bar{\xi}) \cap S^+(\boldsymbol{n}^T(p), -\bar{\xi}) = T_S M[\boldsymbol{n}^T, \xi]_p.$$

Here $T_S M[\boldsymbol{n}^T, \xi]_p$ is a spacelike affine subspace in $\mathbb{R}^{n+1}_1$. □

Then we have the following theorem.

Theorem 9.4 *Let $f : S^s \longrightarrow \mathbb{R}^{n+1}_1$ be a spacelike embedding with $n-1 > s$. Then the following conditions are equivalent:*

(1) $\tau_\ell(S^s, f) = 2$,

(2) *f is L-tight,*

(3) *There exists a spacelike affine subspace $V \subset \mathbb{R}^{n+1}_1$ with $\dim V = s+1$ such that $f(S^s)$ is a convex hypersurface in V.*

Proof. By Theorem 9.2, the conditions (1) and (2) are equivalent. It is trivial that the codition (3) implies the condition (2). We now assume that f is L-tight. By Lemma 9.3, there exists a spalike affine subspace V in $\mathbb{R}^{n+1}_1$ with $\dim V = n-1$ such that $f(S^s) \subset V$. For any $p \in S^s$ and $\xi \in N_1(S^s)[\boldsymbol{n}^T(p)]$, $HP(\boldsymbol{v}^\pm, c^\pm) \cap V = V$ or $HP(\boldsymbol{v}^\pm, c^\pm) \cap V$ is a hyperplane in V. Since f is L-tight, every tangent hyperplane in V is a support plane of $f(S^s)$ in V. Therefore, $f(S^s)$ is tight in V in the Euclidean sense. Then we can apply the result of submanifolds in the Euclidean space [5], so that there exists a a spacelike affine subspace $V \subset \mathbb{R}^{n+1}_1$ with $\dim V = s+1$ such that $f(S^s)$ is a convex hypersurface in V. This completes the proof. □

10 Special cases

In this section we consider submanifolds in Euclidean space and Hyperbolic space as special cases as the previous results.

10.1 Submanifolds in Euclidean space

Let $\mathbb{R}^n_0$ be the Euclidean space which is given by the equation $x_0 = 0$ for $\boldsymbol{x} = (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}_1$. Consider an immersion $f : M \longrightarrow \mathbb{R}^n_0$, where M is a closed orientable manifold. In this case we can adopt $\boldsymbol{n}^T = \boldsymbol{e}_0 = (1, 0, \dots, 0)$ as a future directed timelike unit normal vector field along $f(M)$ in $\mathbb{R}^{n+1}_1$. In this case $N_1(M)[\boldsymbol{n}^T] = N_1(M)[\boldsymbol{e}_0]$ is the unit normal bundle $N^e_1(M)$ of $f(M)$ in $\mathbb{R}^n_0$ in the Euclidean sense. Therefore, the lightcone Gauss map $\widetilde{\mathbb{LG}}(\boldsymbol{n}^T)$ is given by $\widetilde{\mathbb{LG}}(\boldsymbol{n}^T)(p, \boldsymbol{\xi}) = \boldsymbol{e}_0 + \boldsymbol{\xi} = \boldsymbol{e}_0 + \mathbb{G}(p, \boldsymbol{\xi})$, where $\mathbb{G} : N^e_1(M) \longrightarrow S^{n-1}$ is the Gauss map of the unit normal bundle $N^e_1(M)$[5]. Since $\boldsymbol{e}_0$ is a constant vector, we have

$$K^*_\ell(p) = K^*(p),$$

where $K^*(p)$ is the total absolute curvature of M at p (cf., [5]) in the Euclidean sense. Therefore, Theorem 7.3 is the original Chern-Lashof theorem in [5]. If $\dim M = n-1$, then the $\widetilde{K}^\pm_\ell(p) = (\pm 1)^{n-1} K(p)$ where K is the Gauss-Kronecker curvature of M. Thus, if n is odd, then $\widetilde{K}^\pm_\ell(p) = K(p)$. Moreover, $|\widetilde{K}^\pm_\ell|(p) = |K|(p)$ for any n. Therefore, Theorems 8.1, 8.3 and 8.4 are the original integral formulae in the Euclidean sense[5]. Furthermore, if $n = 3$, then the Proposition 8.6 is the original Willmore inequality in Eulclidean space [15, Theorem 7.2.2]. For the torus, we have the original Willmore conjecture in this case.

On the other hand, the intersection of a lightlike hyperplane with $\mathbb{R}^n_0$ is a hyperplane in $\mathbb{R}^n_0$, so that the notion of lightlike tightness is equivalent to the original notion of the tightness[4].

We remark that if $\boldsymbol{n}^T = \boldsymbol{v}$ is a constant timelike unit vector, the spacelike submanifold $f(M)$ is a submanifold in the spacelike hyperplane $HP(\boldsymbol{v}, c)$. Since $HP(\boldsymbol{v}, c)$ is isometric to the Euclidean space $\mathbb{R}^n_0$, all results for the case $\boldsymbol{n} = \boldsymbol{e}_0$ hold in this case.

10.2 Submanifolds in Hyperbolic space

Let $f : M \longrightarrow H^n(-1)$ be an immersion into the hyperbolic space. Then we adopt $\boldsymbol{n}^T(p) = f(p)$. In this case $N_1(M)[\boldsymbol{n}^T] = N_1(M)[f]$ is the unit normal bundle $N_1^h(M)$ of $\widetilde{f(M)}$ in $H^n(-1)$. Therefore, the lightcone Gauss map $\widetilde{\mathbb{LG}}(\boldsymbol{n}^T)$ is given by $\widetilde{\mathbb{LG}}(\boldsymbol{n}^T)(p,\boldsymbol{\xi}) = f(p) + \boldsymbol{\xi} = \widetilde{\mathbb{L}}(p,\boldsymbol{\xi})$, where $\widetilde{\mathbb{L}} : N_1^h(M) \longrightarrow S_+^{n-1}$ is the horospherical Gauss map of the unit normal bundle $N_1^h(M)$(cf., [1]). Thus, we have

$$K_\ell^*(p) = \int_{N_1^h(M)_p} |\widetilde{K}_h(p,\boldsymbol{\xi})| d\sigma_{k-2},$$

which is the total absolute horospherical curvature of M at p (cf., [1]) in $H^n(-1)$. Therefore, $\tau_\ell(M,f) = \tau_h(f)$.

On the other hand, let $f : M \longrightarrow H^n(-1)$ be an embedding such that M is a closed orientable manifold with $\dim M = n-1$. In this case, $f(M)$ is a spacelike submanifold of codimension two in $\mathbb{R}_1^{n+1}$, then we have $\tau_\ell^\pm(M,f) = \tau_h^\pm(f;M)$ (cf., [2]). In [2] we gave an example of a curve in $H^2(-1)$ such that $\tau_h^+(f;M) \neq \tau_h^-(f;M)$. This example can be easily generalized into any higher dimensional case.

On the other hand, the notion of the lightlike tightness is equivalent to the notion of the horo-tightness in $H^n(-1)$[2, 3, 4, 14]. Since the intersection of $H^n(-1)$ with a spacelike affine subspace V is a round hypersphere in V, the condition (3) in Theorem 9.4 can be changed into the following condition:

(3′) $f(S^s)$ is a metric (round) sphere in $H^n(-1)$.

Therefore, Theorems 9.2 and 9.4 are characterizations of the horo-tight spheres in $H^n(-1)$ [2]. Further results on horo-tight immersions into $H^n(-1)$ are presented in [14].

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