ON THE PARAMETER DEPENDENCE OF THE BERGMAN KERNELS OF RECTANGLES

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Abstract: We study the parameter dependence of the Bergman kernels on planar domains depending on complex parameters ζ in nontrivial "pseudoconvex" ways. It turns out that, in an example where the domains are rectangles R_ζ, the Levi form of $\log K_{R_\zeta}(z,z)$ with respect to ζ approaches to 0 as (ζ, z) tends to the boundary point $(1+i, 0)$. In contrast to this, if (ζ, z) tends to the other boundary point, completely different phenomena are observed.

1. Preliminaries

We briefly present here certain results underlying the proofs of Theorems. This exposition is adapted to our special cases.

1.1. **Bergman kernel.** The Bergman kernel of a domain $\Omega \subset \mathbb{C}^n$ is a reproducing kernel for the Hilbert space of all square integrable holomorphic functions on Ω. In what follows, let Ω be a bounded domain in $\mathbb{C}^n$, let $A^2(\Omega)$ be the space of square integrable holomorphic functions on Ω. It is a closed subspace of $L^2(\Omega)$ with respect to Lebesgue measure. And let $\{\phi_j(z)\}_{j=1}^{\infty}$ be a complete orthonormal basis for $A^2(\Omega)$. Then the Bergman kernel $K_\Omega(z,w)$ is identified with the following series:

$$K_\Omega(z,w) = \sum_{j=1}^{\infty} \phi_j(z)\overline{\phi_j(w)},$$

which is independent of the choice of orthonormal basis. For $z = w$, one has $K_\Omega(z,z) > 0$.

The Bergman kernel satisfies the following transformation formula.

Proposition 1.1. *Let $f: \Omega \longrightarrow D$ be a biholomorphic mapping between Ω and D. Then,*

$$K_\Omega(z,w) = K_D(f(z), f(w)) \det f'(z)\overline{\det f'(w)}.$$

By Cauchy's estimate it is easy to see that $K_\Omega(z,w)$ is a C^∞function on $\Omega \times \Omega$ and that, on the diagonal, it can be represented as

$$K_\Omega(z,z) = \sup\{|f(z)|^2 \mid f \in A^2(\Omega), \|f(z)\|_{A^2(\Omega)} = 1\} \quad \text{for} \quad \forall\, z \in \Omega.$$

Riemann's mapping Theorem states that any simply connected domain of the complex z plane can be mapped with a univalent transformation onto the unit disk or onto the upper half of the complex w plane. Unfortunately, the proof of this celebrated theorem is not constructive, that is, given a special domain in the z plane, there is no general constructive approach for find the univalent transformation. Nevertheless, as we will see, there are many particular domains, such as the interior of a polygon, for which the univalent function can be constructed explicitly.

1.2. **Schwarz-Christoffel transformation.** Let Γ be a piecewise linear boundary of a polygon in the w-plane and let the interior angles at successive vertices be $\alpha_1\pi, \alpha_2\pi, \cdots, \alpha_n\pi$. The transformation defined by the equation

$$w = F(z) = C \int_0^z (\xi - a_1)^{\alpha_1 - 1}(\xi - a_2)^{\alpha_2 - 1} \cdots (\xi - a_n)^{\alpha_n - 1} d\xi + C', \tag{1.2.1}$$

where C, C' are complex numbers and $a_1, a_2, \cdots, a_n$ are real numbers, maps Γ into the real axis of the complex z plane and the interior of the polygon to the upper half of the z plane.

The vertices of the polygon $A_1, A_2, \cdots A_n$ are mapped to the points $a_1, a_2, \cdots, a_n$. This map is an analytic one-to-one conformal transformation between the upper half of the z plane and the interior of the polygon.

Remark 1.2. Actually, for any univalent transformation, the correspondence of three points on the boundaries of two simply connected domains can be prescribed arbitrarily. In particular, any of the three vertices of the polygon can be associated with any three points on the real axis.

2. Results and Proofs

The considered parameter rectangles are

$$R_\zeta := \{z = s + it \in \mathbb{C}_z \mid 0 < s < \mathrm{Re}\zeta,\ 0 < t < \mathrm{Im}\zeta\}$$

where $\zeta \in B$ with $B := \{\zeta \in \mathbb{C} || \zeta - (1+i)| < \eta\}$ and define $\mathcal{R} := \bigcup_{\zeta \in B}\{\zeta\} \times R_\zeta$. Then the following results hold.

Theorem 2.1. *The Bergman kernels of R_ζ on the diagonal are*

$$K_{R_\zeta}(z,z) = \frac{1}{\pi(\mathrm{Im}\,\mathrm{sn}^2(u,k(\zeta)))^2} |\mathrm{sn}(u,k(\zeta))\mathrm{cn}(u,k(\zeta))\mathrm{dn}(u,k(\zeta))\frac{\mathrm{sn}^{-1}(1,k(\zeta))}{\mathrm{Re}\zeta}|^2$$

where $u = \mathrm{sn}^{-1}(1,k(\zeta))z/\mathrm{Re}\zeta$ and $\mathrm{sn}(u,k), \mathrm{cn}(u,k), \mathrm{dn}(u,k)$ are the Jacobi's elliptic funtions of the first kind, $\mathrm{sn}^{-1}(1,k)$ is the complete elliptic integral of the first kind. $k(\zeta)$ is a real valued analytic function with respect to ζ, its Taylor expansion to the second order near the point $\zeta = 1 + i$ is:

$$k(\zeta) = k_0 + 2\mathrm{Re}((a+ib)\varepsilon) + 2\mathrm{Re}((c+id)\varepsilon^2) + 2e|\varepsilon|^2 + \cdots,$$

where $k_0 = 1/\sqrt{2}, a = b = -2c = K/\left(4\sqrt{2}(2E-K)\right)$, $d = e = -\sqrt{2}a^2$, here K is the value of the complete elliptic integral of the first kind at the point $k = 1/\sqrt{2}$, and E is the value of the complete elliptic integral of the second kind at the point $k = 1/\sqrt{2}$.

Proof. Firstly, for symmetry, we consider the transformation $F(w,\zeta)$ which maps the upper half of the w plane $\mathbb{H}$ onto R'_ζ which is a rectangle with vertices $A_1(\mathrm{Re}\zeta)$, $A_2(\zeta)$, $A_3(-\overline{\zeta})$, $A_4(-\mathrm{Re}\zeta)$ for each ζ in the z plane. We associate $A_1(\mathrm{Re}\zeta)$ with $a_1(1)$, $A_2(\zeta)$ with $a_2(1/k(\zeta))$, and $w = 0$ with $z = 0$. Then by symmetry, $A_3(-\overline{\zeta})$, $A_4(-\mathrm{Re}\zeta)$ are associated with $a_3(-1/k(\zeta))$, $a_4(-1)$ respectively. Our goal is to determine both the transformation $z = F(w,\zeta)$ and the constant k as an analytic function with respect to the variable ζ. In this case $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1/2$, $a_1 = 1$, $a_2 = 1/k$, $a_3 = -1/k$, $a_4 = -1$. Furthermore, because $F(0,\zeta) = 0$ (symmetry), the constant C' of integration (2.2.1) is zero; thus Equation (2.2.1) yields

$$z = F(w,\zeta) = C(\zeta)\int_0^w ((1-t^2)(1-k^2(\zeta)t^2))^{-\frac{1}{2}}dt. \tag{2.1.1}$$

The integral appearing in (2.1.1), with the choice of a single branch defined by the requirement that $0 < arg(w - a_i) < \pi$, $i = 1,2,3,4$, is the so-called elliptic integral of the first kind. The association of $A_1(\mathrm{Re}\zeta)$ with $a_1(1)$ and $A_2(\zeta)$ with $a_2(1/k(\zeta))$ imply that

$$\mathrm{Re}\zeta = C(\zeta)\int_0^1 ((1-t^2)(1-k^2(\zeta)t^2))^{-\frac{1}{2}}dt, \tag{2.1..2}$$

$$\mathrm{Im}\zeta = C(\zeta)\int_0^1 ((1-t^2)(1-(1-k^2(\zeta))t^2))^{-\frac{1}{2}}dt. \tag{2.1.3}$$

Since $k(\zeta)$ is a real valued analytic function with respect to ζ, the power series of $k(\zeta)$ is

$$k(\zeta) = k_0 + 2\mathrm{Re}((a+ib)\varepsilon) + 2\mathrm{Re}((c+id)\varepsilon^2) + 2e|\varepsilon|^2 + \cdots$$

in a neighborhood of the point $\zeta = 1 + i$. Then, (2.1.2) and (2.1.3) yield that $k_0 = 1/\sqrt{2}$,

$$a = b = -2c = \frac{1 + \sum_{n\geqslant 1}(\frac{(2n-1)!!}{(2n)!!})^2(\frac{1}{2})^n}{4\sqrt{2}\sum_{n\geqslant 1}(\frac{(2n-1)!!}{(2n)!!})^2 n(\frac{1}{2})^{n-1}} = \frac{K}{4\sqrt{2}(2E-K)},$$

$d = e = -\sqrt{2}a^2$, and $C(\zeta) = \mathrm{Re}\zeta/K(k(\zeta))$. In summary, the transformation $F(w,\zeta)$ is given by

$$z = F(w,\zeta) = \frac{\mathrm{Re}\zeta}{K(k)}\int_0^w \Big((1-t^2)\big(1-k^2(\zeta)t^2\big)\Big)^{-\frac{1}{2}}\,dt\,.$$

Secondly, the inverse of the integral in $F(w,\zeta)$ gives w as a function of z via one of the so-called Jacobi's elliptic function $\mathrm{sn}(u,k)$. Then the inverse of $F(w,\zeta)$ with respect to the first variable is given by

$$w = f(z,\zeta) = \mathrm{sn}\left(\frac{K(k)}{\mathrm{Re}\zeta}z, k(\zeta)\right).$$

Moreover,

$$\overline{F(-\overline{w},\zeta)} = -F(w,\zeta)$$

implies that $f(z,\zeta)$ maps R_ζ to $\{w\in\mathbb{C}_w | w = a+ib, a>0, b>0\}$, thus f^2 maps R_ζ to the upper half w plane $\mathbb{H}$. In addition, it is known that the Bergman kernel of the upper half w plane on the diagonal is

$$K_{\mathbb{H}}(w,w) = \frac{1}{4\pi(\mathrm{Im}w)^2},$$

then by Proposition 1.1, the Bergman kernels of R_ζ on the diagonal are

$$\begin{aligned} K_{R_\zeta}(z,z) &= \frac{1}{\pi}\frac{1}{(\mathrm{Im}f^2)^2}|f_z|^2|f|^2 \\ &= \frac{1}{\pi(\mathrm{Im}\,\mathrm{sn}^2(u,k))^2}|\mathrm{sn}^2(u,k)\mathrm{cn}^2(u,k)\mathrm{dn}^2(u,k)|\left|\frac{K(k)}{\mathrm{Re}\zeta}\right|^2 \end{aligned}$$

here $u = K(k)z/\mathrm{Re}\zeta$. □

Theorem 2.2. *For Bergman kernels $K_{R_\zeta}(z,z)$ where $(\zeta,z)\in\mathcal{R}$, it holds that*

$$\lim_{z\to 0,\zeta\to 1+i}\frac{\partial^2\log K_{R_\zeta}(z,z)}{\partial\zeta\partial\overline{\zeta}} = 0,$$

Proof. From the expression of $K_{R_\zeta}(z,z)$,

$$\begin{aligned} \frac{\partial^2\log K_{R_\zeta}(z,z)}{\partial\zeta\partial\overline{\zeta}} &= -\ 2\frac{\partial^2\log(\mathrm{Im}\,\mathrm{sn}^2(u,k))}{\partial\zeta\partial\overline{\zeta}} + 2\mathrm{Re}\frac{\partial^2\log(\mathrm{sn}(u,k)\mathrm{cn}(u,k)\mathrm{dn}(u,k))}{\partial\zeta\partial\overline{\zeta}} \\ &\quad +\ 2\frac{\partial^2(-\log(\mathrm{Re}\zeta)+\log|K(k)|)}{\partial\zeta\partial\overline{\zeta}} \\ &= -\ 2A + 2B + 2C. \end{aligned}$$

where

$$
\begin{aligned}
A &:= \frac{\partial^2 \log(\mathrm{Im}\,\mathrm{sn}^2(u,k))}{\partial\zeta\partial\overline{\zeta}} = 2\frac{\mathrm{Im}\left(\mathrm{sn}(u,k)\frac{\partial^2\mathrm{sn}(u,k)}{\partial\zeta\partial\overline{\zeta}} + \frac{\partial\mathrm{sn}(u,k)}{\partial\zeta}\frac{\partial\mathrm{sn}(u,k)}{\partial\overline{\zeta}}\right)}{\mathrm{Im}\,\mathrm{sn}^2(u,k)} \\
&+ \frac{2\mathrm{Re}\left(\mathrm{sn}^2(u,k)\frac{\partial\mathrm{sn}(u,k)}{\partial\zeta}\frac{\partial\mathrm{sn}(u,k)}{\partial\overline{\zeta}}\right) - |\mathrm{sn}(u,k)|^2\left(|\frac{\partial\mathrm{sn}(u,k)}{\partial\zeta}|^2 + |\frac{\partial\mathrm{sn}(u,k)}{\partial\overline{\zeta}}|^2\right)}{(\mathrm{Im}\,\mathrm{sn}^2(u,k))^2}, \\
B &:= \mathrm{Re}\frac{\partial^2 \log(\mathrm{sn}(u,k)\mathrm{cn}(u,k)\mathrm{dn}(u,k))}{\partial\zeta\partial\overline{\zeta}} \\
&= \mathrm{Re}\frac{\partial^2 \log\mathrm{sn}(u,k)}{\partial\zeta\partial\overline{\zeta}} + \mathrm{Re}\frac{\partial^2 \log\mathrm{cn}(u,k)}{\partial\zeta\partial\overline{\zeta}} + \mathrm{Re}\frac{\partial^2 \log\mathrm{dn}(u,k)}{\partial\zeta\partial\overline{\zeta}}, \\
C &:= \frac{\partial^2(-\log(\mathrm{Re}\zeta) + \log|K(k)|)}{\partial\zeta\partial\overline{\zeta}}.
\end{aligned}
$$

Using the expression of $k(\zeta)$ we get that

$$
\lim_{\zeta\to 1+i}\frac{\partial k}{\partial\zeta} = (1+i)a,\ \lim_{\zeta\to 1+i}\frac{\partial k}{\partial\overline{\zeta}} = (1-i)a,\ \lim_{\zeta\to 1+i}\frac{\partial k}{\partial\zeta}\frac{\partial k}{\partial\overline{\zeta}} = 2a^2,\ \lim_{\zeta\to 1+i}\frac{\partial^2 k}{\partial\zeta\partial\overline{\zeta}} = -2\sqrt{2}a^2.
$$

and since $u = \mathrm{sn}^{-1}(1, k(\zeta))z/\mathrm{Re}\zeta$, then,

$$
\lim_{\zeta\to 1+i}\frac{\partial u}{\partial\zeta} = (-1+i)\frac{u}{4},\ \lim_{\zeta\to 1+i}\frac{\partial u}{\partial\overline{\zeta}} = (-1-i)\frac{u}{4},\ \lim_{\zeta\to 1+i}\frac{\partial u}{\partial\zeta}\frac{\partial u}{\partial\overline{\zeta}} = \frac{1}{8}u^2,
$$

$$
\lim_{\zeta\to 1+i}\frac{\partial^2 u}{\partial\zeta\partial\overline{\zeta}} = \left(4a^2 + \frac{1}{4}\right)u,\ \lim_{\zeta\to 1+i}\left(\frac{\partial u}{\partial\zeta}\frac{\partial k}{\partial\overline{\zeta}} + \frac{\partial k}{\partial\zeta}\frac{\partial u}{\partial\overline{\zeta}}\right) = 0.
$$

And, using the power series expansion of $\mathrm{sn}(u,k)$, $\mathrm{cn}(u,k)$ and $\mathrm{dn}(u,k)$ in a neighborhood of $u = 0$, that,
$\mathrm{sn}(u,k) = u - \frac{1}{3!}(1+k^2)u^3 + \frac{1}{5!}(1+14k^2+k^4)u^5 + O(u^7)$,
$\mathrm{cn}(u,k) = 1 - \frac{1}{2!}u^2 + \frac{1}{4!}(1+4k^2)u^4 + O(u^5)$,
$\mathrm{dn}(u,k) = 1 - \frac{1}{2!}k^2u^2 + \frac{1}{4!}(4k^2+k^4)u^4 + O(u^6)$,
the following further results can now be verified,

$$
\lim_{z\to 0,\zeta\to 1+i} A = 8a^2 + \frac{1}{4},\ \lim_{z\to 0,\zeta\to 1+i} B = 4a^2 + \frac{1}{8},\ \lim_{z\to 0,\zeta\to 1+i} C = \frac{1}{8} + 4a^2.
$$

These follow that

$$
\lim_{z\to 0,\zeta\to 1+i}\frac{\partial^2 \log K_{R_\zeta}(z,z)}{\partial\zeta\partial\overline{\zeta}} = \lim_{z\to 0,\zeta\to 1+i}(-2A+2B+2C) = 0.
$$

□

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