On an incompressible forced two-dimensional flow on a β plane with periodic data

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Abstract

An incompressible two-dimensional flow on a β plane is considered. The β plane is a tangent plane of a sphere to approximately describe fluid motion on a rotating sphere assuming that the Coriolis parameter is a linear function of the latitude. Rossby waves are expected to dominate the β plane dynamics, and here in this paper, a mathematical support for the crucial role of the resonant pairs of the Rossby waves is given.

Key words: Navier-Stokes equations, β plane, Rossby wave, mild solution

1 Introduction

We consider an incompressible two-dimensional flow on a β plane [1],

$$\partial_t w + J(\Psi, w) + \beta\partial_x\Psi = \nu\Delta w, \tag{1}$$

where $w = w(t) = w(t,x)$ $(t > 0, x \in \mathbb{R}^2)$, $J(A,B) = (\partial_x A)(\partial_y B) - (\partial_y A)(\partial_x B)$. Ψ is the streamfunction of the fluid, and w is the vorticity ($\Psi = -(-\Delta)^{-1}w$).

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[1] A similar equation, the Charney-Hasegawa-Mima equation,

$$\partial_t w - \partial_t\Psi + J(\Psi, w) + \beta\partial_x\Psi = \nu\Delta w$$

will be discussed in a separate paper.

The velocity of the fluid is given by $(u, v) = (-\partial_y \Phi, \partial_x \Phi)$, and the initial data is $w|_{t=0} = w_0(x_1, x_2)$.

The β plane was first introduced by meteorologists (see [3,4]) as a tangent plane of a sphere to approximately describe fluid motion on a rotating sphere, assuming that the Colioris parameter is a linear function of the latitude. Conventionally the x- and y-axes are taken eastward and northward, respectively, and the x-direction is often called *zonal* direction in earth and planetary sciences. We employ this intuitive terminology when convenient in this paper. A formal derivation of the β plane approximation is given in [10]. The equation (1) describes a two-dimensional motion of an incompressible fluid on the β plane, and it has been known that in its solution, as time goes on, a stripe pattern emerges, consisting of alternating eastward or westward zonal flows (see [11]), similar to the zonal band structure observed on Jupiter [2].

The equation (1) has been widely employed in earth and planetary sciences to study the effect of differential rotation (latitudinal variation of the Coriolis parameter) and the mechanism of zonal flow formation. From a physical point of view, one of the most important properties of equation (1) is that there is a linear wave solution, in contrast with non-rotating two-dimensional Navier-Stokes fluid. The linear wave solution originates from the third term of (1), and its dispersion relation is

$$\omega = -\frac{\beta k_1}{k_1^2 + k_2^2}, \tag{2}$$

where ω and (k_1, k_2) are the angular frequency and the wavenumber vector of the linear wave, respectively. This wave is called *Rossby wave*, which is known to play quite an important role in fluid motion of atmosphere and ocean. Generally in nonlinear dynamics of linear waves, resonant pairs are expected to give a dominant contribution to the nonlinear interactions. In the case of the Rossby waves, therefore, the resonant waves are expected to dominate the dynamics, and here in this paper, we will give a mathematical support for the crucial role of the resonant pairs of the Rossby waves. For the rotating Navier-Stokes equations in pure mathematical analysis, refer to [1,2,5–9,12].

The paper is organized as follows: In section 2, we show existence of local-in-time unique solution to (1). The existence time is independent of the parameter β. In section 3, we state the main result. To state the main result, we need to set a filtered equation and give a resonant-nonresonant decomposition. We consider the resonant-nonresonant decomposition precisely in Section 4. In the last section, we give a proof of the main theorem.

[2] The origin of the zonal band structure on Jupiter is still controversial. Three dimensional deep convection is another possible origin of the surface zonal bands.

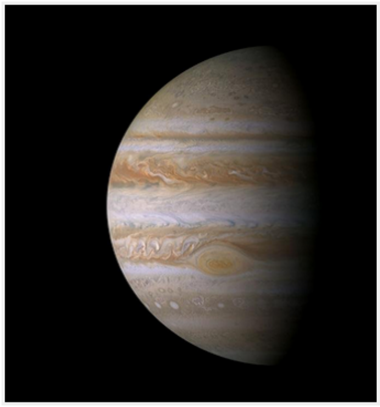

Fig. 1. Zonal bands on Jupiter (NASA).

2 Local existence and uniqueness of the solution

We introduce a weighted ℓ^1 (in Fourier side) space given by

$$X^s := \{w = \sum_{n\in\mathbb{Z}^2} a_n e^{in\cdot x} \in \mathcal{S}'(\mathbb{T}^2) \quad \text{for} \quad a = \{a_n\}_{n\in\mathbb{Z}^2} :$$
$$a_n = a_n^* \ (n \in \mathbb{Z}^2),\ \|a\|_s := \|a\|_{\ell_1^s} := \sum_{n\in\mathbb{Z}^2} (1+|n|^2)^{s/2}|a_n| < \infty\}$$

for $s \geq 0$, where a_n^* is the complex conjugate of the Fourier coefficients a_n. $\|a\|_0$ is also written as $\|a\|$. It is well known that X^0 is an algebra which is continuously embedded in BUC, the space of bounded uniformly continuous functions. For the periodic case, we can point out the following relationship between $X^s(\mathbb{T}^2)$ and the Hölder space $C^s(\mathbb{T}^2)$.

Proposition 1 *We have $w \in C^\infty(\mathbb{T}^2)$ if and only if $w \in \cap_{s\geq 1} X^s$. Obviously, $X^s \subset C^s$ $(s \geq 0)$.*

Throughout this paper we use ℓ^1-norm of amplitudes. To treat the Coriolis term, we need to define rigorously the multipliers $n_j/|n|^2$ $(j = 1, 2)$ as

$$\lim_{\epsilon\to 0} \frac{n_j}{|n|^2 + \epsilon}.$$

Thus we see that

$$\left.\frac{n_j}{|n|^2}\right|_{n=0} = 0.$$

Let $\omega_n := in_1/|n|^2$. Without loss of generality, we set $\nu = 1$. We consider equation (1) on $\mathbb{T}^2$, and then we can rewrite (1) by amplitude functions as follows (we set $\sum_{n\in\mathbb{Z}^2} a_{0,n}e^{in\cdot x} := w_0$ and $\sum_{n\in\mathbb{Z}^2} a_n(t)e^{in\cdot x} := w(t)$):

$$
\begin{aligned}
\partial_t a_n(t)+|n|^2a_n(t)+\beta\omega_n a_n(t) &= \sum_{n=k+m}\left(\frac{k_1m_2}{|k|^2}a_k(t)a_m(t)-\frac{k_2m_1}{|k|^2}a_k(t)a_m(t)\right)\\
&= \sum_{n=k+m}\left(\frac{k_1n_2}{|k|^2}a_k(t)a_m(t)-\frac{k_2n_1}{|k|^2}a_k(t)a_m(t)\right)\\
&=: J_n(a,a), \quad \textit{and} \quad a_n(0)=a_{0,n}.
\end{aligned}
$$

We now obtain a local existence result in $\ell_1(\mathbb{Z}^2)$ as follows:

Theorem 2 *Assume that $a(0):=\{a_n(0)\}_{n\in\mathbb{Z}^2}\in\ell_1(\mathbb{Z}^2)$. Then there is a local-in-time unique solution $a(t):=\{a_n(t)\}_{n\in\mathbb{Z}^2}\in C([0,T_L]:\ell_1(\mathbb{Z}^2))$ satisfying*

$$
T_L\geq\frac{C}{\|a_0\|_0^2},\qquad \sup_{0<t<T_L}\|a(t)\|_0\leq 10\|a_0\|_0,
$$

where C is a positive constant independent of β. Moreover if $\|a(0)\|_s<\infty$, we have the following pointwise estimate:

$$
|a_n(t)|\leq\frac{C_1}{(1+|n|^2)^{s/2}}\quad \textit{for}\quad 0<t<T_L, \tag{3}
$$

where $C_1>0$ is independent of β.

Proof. The solution can be rewritten as follows (we say "mild solution"):

$$
a_n(t)=e^{-(|n|^2+\beta\omega_n)t}a_n(0)+\int_0^t e^{-(|n|^2+\beta\omega_n)(t-\tau)}J_n(a(\tau),a(\tau))d\tau,\quad n\in\mathbb{Z}^2.
$$

A direct calculation shows that

$$
|a_n(t)|\leq|a_n(0)|+\int_0^t\frac{C}{(t-\tau)^{1/2}}|J_n(a(\tau),a(\tau))|d\tau,\quad n\in\mathbb{Z}^2,
$$

here we used

$$
\sup_{t>0,n\in\mathbb{Z}^2}t^{s/2}|n|^se^{-t|n|^2}\leq C\quad\text{for}\quad s>0. \tag{4}
$$

By the convolution in $\ell_1(\mathbb{Z}^2)$, we have the following estimate:

$$
\|a(t)\|_0\leq\|a(0)\|_0+Ct^{1/2}\sup_{0<\tau\leq t}\|a(\tau)\|_0\sup_{0<\tau\leq t}\|a(\tau)\|_0\quad\text{for}\quad t>0,
$$

Using the above estimates, we easily have the local existence result (see [6] for example). To show the pointwise bound, it suffices to show that

$$
\|a(t)\|_s\leq C\quad\text{for}\quad t\in[0,T_L],\quad\text{if}\quad\|a(0)\|_s\leq C\quad(C\text{ is independent of }\beta). \tag{5}
$$

Indeed, by (5) and the following estimate:

$$\sup_n (1+|n|^2)^{s/2}|a_n(t)| \le \|a(t)\|_s \quad (0<t<T_L),$$

we have the pointwise bound. To show (5), we use a bootstrapping argument. Namely, we first control $\sup_{0<t\le T_L}\|a(t)\|_{1/2}$ from the estimate of the mild solution. Second, we control $\sup_t \|a(t)\|_1$, third $\sup_{0<t\le T_L}\|a(t)\|_{3/2}$ and so on. Here we only control $\sup_{0<t\le T_L}\|a(t)\|_{1/2}$. By (4), we have

$$\|a(t)\|_{1/2} \le \|a_n(0)\|_{1/2} + Ct^{1/4}\sup_{0<\tau\le t}\|a(\tau)\|_0^2 \quad \text{for} \quad 0<t<T_L.$$

Thus $\sup_{0<t\le T_L}\|a(t)\|_{1/2}$ is controllable. The same calculations for $\|a(t)\|_1$, $\|a(t)\|_{3/2},\cdots$ give the desired estimate.

3 Filtered equation, resonant-nonresonant decomposition and the main theorem

In this section, we give a filtered equation, a resonant-nonresonant decomposition and state the main theorem. We set $c_n(t) := e^{-t\beta\omega_n}a_n(t)$ for $n\in\mathbb{Z}^2$. A direct calculation shows that $\{c_n(t)\}$ satisfy the following equation:

$$\partial_t c_n(t) = -|n|^2 c_n(t) + B_n(c(t),c(t)) \quad \textit{and} \quad c_n(0)=a_n(0), \tag{6}$$

which we call the "filtered equation", where

$$B_n(c_1,c_2) := e^{-t\beta\omega(n)}J_n(e^{t\beta\omega(\cdot)}c_1, e^{t\beta\omega(\cdot)}c_2)$$

with $\omega(n) := \omega_n = in_1/|n|^2$. From now on we handle $\{c_n(t)\}_{n\in\mathbb{Z}^2}$ instead of the coefficients $\{a_n(t)\}_{n\in\mathbb{Z}^2}$, since they are equivalent in $\ell_1(\mathbb{Z}^2)$ to each other. The nonlinear operator B_n can be decomposed into two parts, the resonant part $B^0(\cdot,\cdot)$ (independent of β) and the non-resonant part $B^{0+}(\beta t,\cdot,\cdot)$ (depending on β). More precisely,

$$\partial_t c_n(t) = -|n|^2 c_n(t) + B_n^0(c(t),c(t)) + B^{0+}(\beta t, c(t), c(t)) \quad \textit{and} \quad c_n(0)=a_n(0).$$

We will give its detail in the next section. Now we define the solution $b(t)$ to the resonant part of the equation and the remainder term $r_n(t) := c_n(t) - b_n(t)$ as follows:

$$\partial_t b_n(t) = -|n|^2 b_n(t) + B_n^0(b(t),b(t)) \quad \textit{and} \quad b_n(0)=c_n(0)=a_n(0), \tag{7}$$

$$\partial_t r_n(t) = -|n|^2 r_n(t) + B_n^0(c(t),r(t)) + B_n^0(r(t),b(t)) + B_n^{0+}(\beta t, c(t), c(t))$$

and $r_n(0)=0$. We call equation (7) the limit equation. Formally, the limit equation is the case of $\beta\to\infty$ for (1). This kind of decomposition has already

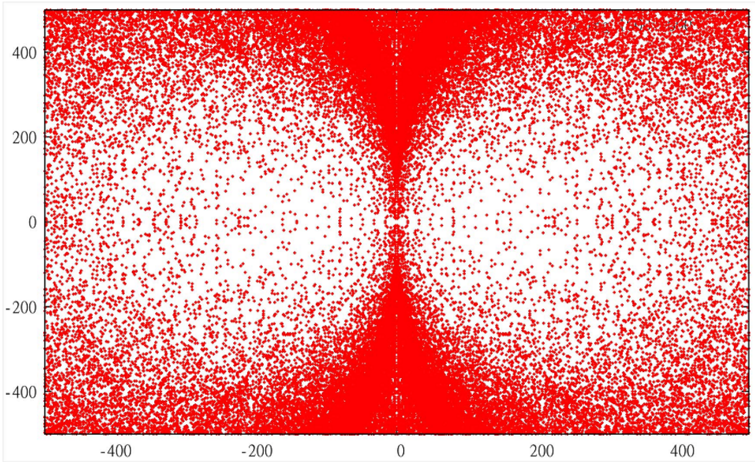

Fig. 2. The frequency set of the three-wave interaction. In this plot, trivial three wave interactions, $(n_1, n_2) = (n_1, -n_2) + (0, 2n_2)$, are ignored.

been done for the rotating 3D-Navier-Stokes/Euler equations (see [1,2,9,12] for example). It is known by numerical integration that for large values of β, the flow field is non-isotropic as there arise zonal (i.e. in x_1-direction) flows, meaning that most energy is concentrated to the Fourier components of $n \sim (0, n_2)$. Numerical computation thus suggests that the solution to the limit equation is a non-isotropic flow (namely, zonal jet flow).

The main result is as follows: the solution $c(t)$ to (6) tends to the solution $b(t)$ to the limit equation provided that β tends to infinity. More precisely,

Theorem 3 *For all $\epsilon > 0$, there is $\beta_0 > 0$ s.t. $\|r(t)\|_0 \le \epsilon$ for $0 < t < T_L$ and $|\beta| > \beta_0$, where T_L is the local existence time (see Theorem 2).*

To control the remainder term, we use oscillatory integral and estimate the non-resonant operator B^{0+} (see Section 5).

4 Decomposition of the nonlinear term

In this section we decompose the nonlinear term into two parts: the resonant part and the non-resonant part, and we estimate them in $\ell_1(\mathbb{Z}^2)$-norm.

Theorem 4 *Let $c_1(t) := \{c_{1,n}(t)\}_{n \in \mathbb{Z}^2}$, $c_2(t) := \{c_{2,n}(t)\}_{n \in \mathbb{Z}^2} \in \ell^1(\mathbb{Z}^2)$, $\omega_{nkm} := n_1/|n|^2 - k_1/|k|^2 - m_1/|m|^2$, and*

$$\begin{aligned} b_{nkm}(c_k, c_m) :=& \frac{k_1 m_2}{|k|^2} c_k(t) c_m(t) - \frac{k_2 m_1}{|k|^2} c_k(t) c_m(t) \\ =& \frac{k_1 n_2}{|k|^2} c_k(t) c_m(t) - \frac{k_2 n_1}{|k|^2} c_k(t) c_m(t). \end{aligned}$$

Recall that $\omega(n) = in_1/|n|^2$. Then the non-linear term

$$B_n(c_1, c_2) := e^{-t\beta\omega(n)} J_n(e^{t\beta\omega(\cdot)} c_1, e^{t\beta\omega(\cdot)} c_2) = \sum_{n=k+m} e^{it\beta\omega_{nkm}} b_{nkm}(c_{1,k}, c_{2,m})$$

can be decomposed into two parts:

$$B_n^0(c_1, c_2) = \sum_{\substack{n=k+m \\ \omega_{nkm}=0}} b_{nkm}(c_{1,k}, c_{2,m})$$

and

$$B_n^{0+}(\beta t, c_1, c_2) := \sum_{\substack{n=k+m \\ \omega_{nkm}\neq 0}} e^{i\beta t\omega_{nkm}} b_{nkm}(c_{1,k}, c_{2,m}).$$

The resonant part $B_n^0(c_1, c_2)$ can also be decomposed as follows:

$$B_n^0(c_1, c_2) := \sum_{\mu \in D} B_n^\mu(c_1, c_2) \tag{8}$$

for $D = \{(0,0,0), (0,1,1), (1,0,1), (1,1,0), (1,1,1)\}$.

The bilinear forms $B^\mu := \{B_n^\mu\}_{n\in\mathbb{Z}^2}$ ($\mu \in D$) are defined as follows:

$$B_n^{(0,0,0)}(c_1, c_2) := \sum_{\substack{n=k+m \\ n_1=k_1=m_1=0}} b_{nkm}(c_{1,k}, c_{2,m})$$

$$B_n^{(0,1,1)}(c_1, c_2) := \sum_{\substack{n=k+m \\ |n|=|m|\neq 0, -k_1=m_1\neq 0}} b_{nkm}(c_{1,k}, c_{2,m}), \tag{9}$$

$$B_n^{(1,0,1)}(c_1, c_2) := \sum_{\substack{n=k+m \\ |n|=|m|\neq 0, n_1=m_1\neq 0}} b_{nkm}(c_{1,k}, c_{2,m}), \tag{10}$$

$$B_n^{(1,1,0)}(c_1, c_2) := \sum_{\substack{n=k+m \\ |n|=|k|\neq 0, n_1=k_1\neq 0}} b_{nkm}(c_{1,k}, c_{2,m}), \tag{11}$$

$$B_n^{(1,1,1)}(c_1,c_2) := \sum_{\substack{n=k+m\\ \omega_{nkm}=0, n_1,k_1,m_1\neq 0}} b_{nkm}(c_{1,k},c_{2,m}). \tag{12}$$

Moreover we have the following estimates (see (15)and (16)).

$$\|\{e^{-|n|^2t}B_n(c_1,c_2)\}_{n\in\mathbb{Z}^2}\|_0 \le (C/t^{1/2})\|c_1\|_0\|c_2\|_0 \tag{13}$$

and

$$\|\{e^{-|n|^2t}B^{0+}(\beta t,c_1,c_2)\}_{n\in\mathbb{Z}^2}\|_0 \le (C/t^{1/2})\|c_1\|_0\|c_2\|_0, \tag{14}$$

where $C>0$ is independent of β. (From these inequalities, we can show the local existence for b and r. The proof is quite similar to that of Theorem 2, thus we omit its detail.)

The following remarks and the definition are important to see that the solution to the limit equation is a zonal flow. Let us define a frequency set of the Fourier coefficients $b=\{b_n\}_{n\in\mathbb{Z}^2}\in\ell_1(\mathbb{Z}^2)$ as

$$\begin{aligned}\Lambda^b :=& \{n\in\mathbb{Z}^2 : b_{n_1,n_2}\neq 0\}\cup\{n\in\mathbb{Z}^2 : b_{-n_1,n_2}\neq 0\}\\ &\cup\{n\in\mathbb{Z}^2 : b_{n_1,-n_2}\neq 0\}\cup\{n\in\mathbb{Z}^2 : b_{-n_1,-n_2}\neq 0\}.\end{aligned}$$

Remark. (Trivial resonances.) For any $b_1,\ b_2\in\ell_1(\mathbb{Z}^2)$, we have

$$\Lambda^{B^{(1,1,0)}(b_1,b_2)}\subset\Lambda^{b_1},\ \Lambda^{B^{(1,0,1)}(b_1,b_2)}\subset\Lambda^{b_2},$$

$$\Lambda^{B^{(0,0,0)}(b_1,b_2)}\subset\{n\in\mathbb{Z}^2 : n_1=0\}\quad\text{and}\quad\Lambda^{B^{(0,1,1)}}\subset\{n\in\mathbb{Z}^2 : n_1=0\}.$$

The following definition is the key in this paper.

Definition 5 *(Three wave interaction frequencies.) Let Λ be such that for any $n\in\mathbb{Z}^2$ with $n_1\neq 0$, if there is k and m with $k_1\neq 0$ and $m_1\neq 0$ such that $\omega_{nkm}=0$, then $n\in\Lambda$, if not, then $n\notin\Lambda$. (The wavenumber set Λ is the whole red area in Figure 2.)*

Remark. If b is such that $\Lambda^b\subset\Lambda$, then $\Lambda^{B^{(1,1,1)}(b,b)}\subset\Lambda$.

We note that every wavenumber $n=(n_1,n_2)$ has a trivial resonance with $\{k,m\}=\{(n_1,-n_2),(0,2n_2)\}$, and is therefore not resonantly independent of other wavenumbers. However, taking into account the Hermite conjugate relation $a_n=a^*_{-n}$, if the four wavenumbers $n=(\pm n_1,\pm n_2)$ have only the trivial resonances and have no energy, i.e. $|a_n|=0$, then we can conclude that these wavenumbers do not gain the energy through the three wave resonant interactions. In other words, for $c=\{c_n\}_{n\in\mathbb{Z}^2}$, if $\Lambda^c\subset\Lambda$, then the resonant interactions $B^{(1,0,1)}(c,c), B^{(1,1,0)}(c,c), B^{(1,1,1)}(c,c)$ vanish at wavenumbers outside Λ, $B^{(0,0,0)}(c,c)$ and $B^{(0,1,1)}(c,c)$ are included in $\{n\in\mathbb{Z}^2: n_1=0\}$. Generally in a turbulent solution of the governing equation, every wavenumber component has more or less energy. But the number of the resonant triads gives a rough estimate of the strength of the nonlinear interactions between Fourier modes, and the wavenumbers in the white area in Figure 2 are then expected to have less energy exchange compared with wavenumbers in the red area, resulting in a non-isotropic energy distribution in wavenumber space.

Proof of Theorem 4. We now define the resonant frequency set $\mathcal{K}\subset(\mathbb{Z}^2)^3$ (the non-resonant frequency set is its complementary set) and $(\mathbb{Z}^2)^3_\mu$. Recall that $\omega_{nkm}=n_1/|n|^2-k_1/|k|^2-m_1/|m|^2$. Let

$$\mathcal{K}:=\{(n,k,m)\in\left(\mathbb{Z}^2\right)^3:\omega_{nkm}=0\}$$

and

$$(\mathbb{Z}^2)^3_\mu:=\{(n,k,m)\in\left(\mathbb{Z}^2\right)^3: n_1\in\mathbb{Z}^2_{\mu_1}, k_1\in\mathbb{Z}^2_{\mu_2}\quad and\quad m_1\in\mathbb{Z}^2_{\mu_3}\}$$

for $\mu=(\mu_1,\mu_2,\mu_3)\in\{0,1\}^3$, where $\mathbb{Z}^2_0:=\{(0,0)\}$ and $\mathbb{Z}^2_1:=\mathbb{Z}^2\setminus\{(0,0)\}$. Thus we have the following decomposition:

$$(\mathbb{Z}^2)^3=\mathcal{K}\cup\mathcal{K}^c=\bigcup_{\mu\in\{0,1\}^3}(\mathcal{K}\cap(\mathbb{Z}^2)^3_\mu)\cup\mathcal{K}^c.$$

Moreover, we see that

$$\left(\mathcal{K}\cap(\mathbb{Z}^2)^3_\mu\right)\cap\left(\mathcal{K}\cap(\mathbb{Z}^2)^3_{\mu'}\right)=\emptyset\quad if\quad \mu\neq\mu'.$$

By using $\mathcal{K}$ and $(\mathbb{Z}^2)^3_\mu$, we can define bilinear forms as

$$\begin{aligned} B^\mu_n(c_1,c_2):=&\sum_{\substack{n=k+m\\(n,k,m)\in\mathcal{K}\cap(\mathbb{Z}^2)^3_\mu}} b_{nkm}(c_{1,k},c_{2,m})\quad(resonant\ part),\\ B^{0+}_n(\beta t,c_1,c_2):=&\sum_{\substack{n=k+m\\(n,k,m)\in\mathcal{K}^c}} e^{i\beta t\omega_{nkm}}b_{nkm}(c_{1,k},c_{2,m})\quad(non-resonant\ part)\end{aligned}$$

for $\mu\in\{0,1\}^3$. We then have the following estimate of each coefficient:

$$|B^\mu_n(c_1,c_2)|\leq\sum_{n=k+m}|n||c_{1,k}||c_{2,m}| \tag{15}$$

and

$$|B^{0+}_n(\beta t,c_1,c_2)|\leq\sum_{n=k+m}|n||c_{1,k}||c_{2,m}|. \tag{16}$$

Inequalities (15) and (16) give (13) and (14) by the use of (4). The definitions of B_n^μ, B_n^{0+} and a direct calculation yields

$$B_n(c(t), c(t)) = \sum_{\mu \in \{0,1\}^3} B_n^\mu(c(t), c(t)) + B_n^{0+}(\beta t, c(t), c(t)). \tag{17}$$

Lastly, to show (8), it suffices to have the following three equalities:

$$B_n^{(1,0,0)}(c, c) = 0, B_n^{(0,1,0)}(c, c) = 0,\ B_n^{(0,0,1)}(c, c) = 0 \quad for \quad c = \{c_n\}_{n \in \mathbb{Z}^2}$$

which are easily shown to hold because, for example, if $k_1 = m_1 = 0$, n_1 vanishes, i.e. $B_n^{(1,0,0)} = 0$ since $n = k + m$. The other two equalities are similarly shown. Thus we complete the proof.

5 Proof of the main theorem

Proof of Theorem 3. First let us define an oscillatory integral of the non-resonant part as follows:

$$\tilde{B}_n^{0+}(\beta t, c_1, c_2) := \sum_{\substack{n=k+m \\ \omega_{nkm} \neq 0}} \frac{1}{i\beta\omega_{nkm}} e^{i\beta t \omega_{nkm}} b_{nkm}(c_{1,k}, c_{2,m}),$$

where

$$\begin{aligned} b_{nkm}(c_k, c_m) :=& \frac{k_1 m_2}{|k|^2} c_k(t) c_m(t) - \frac{k_2 m_1}{|k|^2} c_k(t) c_m(t) \\ =& \frac{k_1 n_2}{|k|^2} c_k(t) c_m(t) - \frac{k_2 n_1}{|k|^2} c_k(t) c_m(t). \end{aligned}$$

Note that

$$\begin{aligned} &\partial_t \left(\tilde{B}_n^{0+}(\beta t, c_1(t), c_2(t)) \right) = \\ &\quad B_n^{0+}(\beta t, c_1(t), c_2(t)) + \tilde{B}_n^{0+}(\beta t, \partial_t c_1(t), c_2(t)) + \tilde{B}_n^{0+}(\beta t, c_1(t), \partial_t c_2(t)) \end{aligned} \tag{18}$$

and

$$\|\{e^{-t|n|^2} \tilde{B}_n^{0+}(\beta t, c_1, c_2)\}_{n \in \mathbb{Z}^2}\|_0 \leq \frac{C}{t^{1/2}\tau\beta} \|c_1\|_0 \|c_2\|_0, \tag{19}$$

where τ is an infimum of $\{|\omega_{nkm}|\}$ over all combinations of n, k and m with $n = k + m$. By using $\tilde{B}_n^{0+}$, we can control the remainder term since β is in the denominator. However ω_{nkm} is also in the denominator and there is a possibility that a subsequence of $\{|\omega_{nkm}|\}$ converges to 0. It means that we cannot control $\|\{e^{-t|n|^2}\tilde{B}^{0+}_n(\beta t, c_1, c_2)\}_{n\in\mathbb{Z}^2}\|_0$ directly. Thus we need to handle finite elements of $b(t) = \{b_n(t)\}_{n\in\mathbb{Z}^2}$, $c(t) = \{c_n(t)\}_{n\in\mathbb{Z}^2}$ and $r(t) = \{r_n(t)\}_{n\in\mathbb{Z}^2}$ (approximated terms). Now first we give a formal calculation, and after that we give a rigorous calculation using the approximated terms. In order to estimate the remainder term $r(t) = \{r_n(t)\}_{n\in\mathbb{Z}^2}$, it suffices to estimate $y_n(t) := r_n(t) - \tilde{B}^{0+}_n(\beta t, c(t), c(t))$, since $\tilde{B}^{0+} := \{\tilde{B}^{0+}_n\}_n$ tends to zero in ℓ^1-norm when $\beta \to \infty$ (if $\tilde{B}^{0+}$ have only finite elements). We see from (18) that the functions $\{y_n(t)\}_n$ satisfy the following equations:

$$\partial_t y_n(t) + |n|^2 y_n(t) - L_n(c(t), b(t), y(t)) = \sum_{j=1}^{3} E^j_n, \tag{20}$$

where

$$\begin{aligned} L_n(c,b,y) &:= B^0_n(c,y) + B^0_n(y,b),\\ E^1_n &:= -\tilde{B}^{0+}_n(\beta t, \partial_t c(t), c(t)) - \tilde{B}^{0+}_n(\beta t, c(t), \partial_t c(t)),\\ E^2_n &:= -|n|^2\tilde{B}^{0+}_n(\beta t, c(t), c(t)),\\ E^3_n &:= L_n(c(t), b(t), \tilde{B}^{0+}(\beta t, c, c)). \end{aligned}$$

Note that we can also estimate $\{\partial_t c_n(t)\}_n$ and $\{E^2_n\}_n$ in ℓ^1-norm if they have only finite elements. Using (19) together with estimate of the resonant part (13), (14) and an absorbing argument, we can control the remainder term.

Now we give more detail computation. To control r, we split it into two parts: finitely many terms and small (in $\ell^1(\mathbb{Z}^2)$) remainder terms, respectively (cf. [1, Theorem 6.3] and [12]). For $r = \{r_n\}_{n\in\mathbb{Z}^2}$, let

$$\mathcal{P}_\eta r := \left\{ r_n : |n| \leq \eta \right\}.$$

Remark. We have $\|(I - \mathcal{P}_\eta)r\|_0 \to 0 \quad (\eta \to \infty)$.

Then we divide r into two parts: finitely many terms (low frequency part) $\mathcal{P}_\eta r$ and small remainder terms (high frequency part) $(I - \mathcal{P}_\eta)r$.

It should be remarked that we have the following estimates:

$$
\begin{aligned}
\|\mathcal{P}_\eta \tilde{B}^{0+}(\beta t, \mathcal{P}_\eta c_1, \mathcal{P}_\eta c_2)\|_0 &\leq \frac{\alpha(\eta)}{\beta}(1+\eta^2)^{1/2}\|\mathcal{P}_\eta c_1\|_0 \|\mathcal{P}_\eta c_2\|_0, \qquad (21)\\
\|\mathcal{P}_\eta B^{0+}(\beta t, \mathcal{P}_\eta c_1, \mathcal{P}_\eta c_2)\|_0 &\leq (1+\eta^2)^{1/2}\|\mathcal{P}_\eta c_1\|_0 \|\mathcal{P}_\eta c_2\|_0,\\
\|\mathcal{P}_\eta(|\cdot|^2 y)\|_0 &\leq (1+\eta^2)\|\mathcal{P}_\eta y\|_0,\\
\|\partial_t \mathcal{P}_\eta c\|_0 &\leq \|\mathcal{P}_\eta(|\cdot|^2 c)\|_0 + \|\mathcal{P}_\eta B(c,c)\|_0\\
&\leq (1+\eta^2)\|c\|_0 + (1+\eta^2)^{1/2}\|c\|_0^2
\end{aligned}
$$

for $0 < t < T_L$ (T_L is a local existence time, see Theorem 2), where

$$
\alpha(\eta) := \max\{|\omega_{nkm}|^{-1} : |n|, |k|, |m| \leq \eta \quad \text{with} \quad n = k + m\}.
$$

Note that $\alpha(\eta)$ is always finite, since it only have finite combinations for the choice of n, k and m. Now we set $y_n := r_n - \tilde{B}_n^{0+}(\beta t, \mathcal{P}_\eta c, \mathcal{P}_\eta c)$. Let $L_n(c, b, y) := B_n^0(c, y) + B_n^0(y, b)$. For $|n| \leq \eta$, we see that

$$
\begin{aligned}
\partial_t \left(y_n + \tilde{B}_n^{0+}\right) &= -|n|^2 \left(y_n + \tilde{B}_n^{0+}\right) + L_n(c, b, \mathcal{P}_\eta(y + \tilde{B}^{0+}))\\
&\quad + L_n(c, b, (I - \mathcal{P}_\eta)(y + \tilde{B}^{0+}))\\
&\quad + B_n^{0+}(\beta t, \mathcal{P}_\eta c, \mathcal{P}_\eta c) + B_n^{0+}(\beta t, (I - \mathcal{P}_\eta)c, \mathcal{P}_\eta c)\\
&\quad + B_n^{0+}(\beta t, c, (I - \mathcal{P}_\eta)c).
\end{aligned}
$$

From (18), we have

$$
\partial_t y_n + |n|^2 y_n - L_n(c, b, \mathcal{P}_\eta y) = \sum_{j=1}^{3} E_n^j + R_n \qquad (22)
$$

for $|n| \leq \eta$, where

$$
\begin{aligned}
E_n^1 &:= -\tilde{B}_n^{0+}(\beta t, \partial_t \mathcal{P}_\eta c, \mathcal{P}_\eta c) - \tilde{B}_n^{0+}(\beta t, \mathcal{P}_\eta c, \partial_t \mathcal{P}_\eta c),\\
E_n^2 &:= -|n|^2 \tilde{B}_n^{0+}(\beta t, \mathcal{P}_\eta c, \mathcal{P}_\eta c),\\
E_n^3 &:= L_n(c, b, \mathcal{P}_\eta \tilde{B}_n^{0+}(\beta t, \mathcal{P}_\eta c, \mathcal{P}_\eta c)),\\
R_n &:= L_n(c, b, (I - \mathcal{P}_\eta)(y + \tilde{B}^{0+}))\\
&\qquad + B_n^{0+}(\beta t, (I - \mathcal{P}_\eta)c, \mathcal{P}_\eta c) + B_n^{0+}(\beta t, c, (I - \mathcal{P}_\eta)c).
\end{aligned}
$$

Note that (22) is a linear heat type equation with external forces E^1, E^2, E^3 and R. We see that for any $\epsilon > 0$, there is η_0 such that if $\eta > \eta_0$, then $\|\mathcal{P}_\eta R\|_0 < \epsilon$. Due to the pointwise bound (5), η_0 is independent of β. By using (21), we can also see that for any $\epsilon > 0$, there is β_0 (depending on η_0) such that if $|\beta| > \beta_0$, then $\sum_{j=1}^{3} \|\mathcal{P}_\eta E^j\| < \epsilon$. Thus we have from the mild solution that

$$
\|\mathcal{P}_\eta y(t)\|_0 \leq \int_0^t \frac{C}{(t-\tau)^{1/2}} \Big((\|c(\tau)\|_0 + \|b(\tau)\|_0) \|\mathcal{P}_\eta y(\tau)\|_0 + \epsilon \Big) d\tau.
$$

By Gronwall's inequality, we have that for any $\epsilon > 0$, there is η_0 and β_0 (depending on η_0) such that if $\eta > \eta_0$ and $|\beta| > \beta_0$, then $\|\mathcal{P}_\eta y\|_0 < \epsilon$ for $0 < t < T_L$. Clearly, we can also control $(I - \mathcal{P}_\eta)y$ with sufficiently large η (independent of β), and $\mathcal{P}_\eta \tilde{B}^{0+}(\beta t, \mathcal{P}_\eta c, \mathcal{P}_\eta c)$ with sufficiently large β for fixed η. Thus we can control r for sufficiently large η and β which completes the proof.

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