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Curves on a spacelike surface in three dimensional Lorentz-Minkowski space

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Abstract

In this paper we consider curves on a spacelike surface in Lorentz-Minkowski 3-space. We introduce new geometric invariants for these curves. As an application of the unfolding theory of functions, we investigate the local and global properetics of these invariants.

1 Introduction

In this paper we consider local and global properties of curves on spacelike surfaces in three dimensional Lorentz-Minkowski space. The study of the extrinsic differential geometry of submanifolds in Minkowski space is of special interest in relativity theory. In [2, 4], it was investigated codimension two spacelike submanifolds in Lorentz-Minkowski space. Inspired by these papers, we are particularly interested in spacelike curves in three dimensional Lorentz-Minkowski space as a special case, that is submanifolds of codimension two in the space. As an application of the idea in [2, 4], we consider curves on a spacelike surface in three dimensional Lorentz-Minkowski space. In the extrinsic differential geometry, one of the principal ideas to study surfaces is to investigate geometric properties of curves on them. Therefore we study curves on a spacelike surface in three dimensional Lorentz-Minkowski space. Since we consider a spacelike surface M, we can choose a future directed unit timelike normal vector field $\boldsymbol{n}$ along the surface. For a curve $\boldsymbol{\gamma}$ on the surface, we restrict the normal vector field $\boldsymbol{n}$ along $\boldsymbol{\gamma}$, so that we have a unit timelike normal vector field $\boldsymbol{n}_\gamma$ along $\boldsymbol{\gamma}$. Moreover, we choose the unit tangent vector field $\boldsymbol{t}$ and another normal vector field $\boldsymbol{b}$ along $\boldsymbol{\gamma}$. As a result, we construct a pseudo-orthonormal frame $\{\boldsymbol{t}, \boldsymbol{n}_\gamma, \boldsymbol{b}\}$ along the curve $\boldsymbol{\gamma}$ and call it a *Lorentzian Darboux frame* (cf., §3). Applying the idea in [4] to the Loretzian Darboux frame, we define smooth mappings $\mathbb{L}^{\pm} = \boldsymbol{n}_\gamma \pm \boldsymbol{b}$ and have the normalized mappings $\widetilde{\mathbb{L}}^{\pm}$ which are called the *Lightcone Gauss maps*. By differentiating $\mathbb{L}^{\pm}$, we obtain new invariants $\kappa_l^{\pm}$ of $\boldsymbol{\gamma}$, which are called *lightcone curvatures*. The lightcone Gauss maps induce the normalized lightcone curvature $\widetilde{\kappa}_l^{\pm}$. We also define other important mappings called *lightlike height functions* and *Lightcone pedal*. We show that the lightcone Gauss map is constant if and only if $\kappa_l^+ \equiv 0$ or $\kappa_l^- \equiv 0$. In this case the curve $\boldsymbol{\gamma}$ is a special curve on the surface M, which is called a *lightlike-slice* (or an *L-slice*) of M. We consider L-slices of M as the model curves on the surface M. The singularities of the lightcone Gauss map is a point where $\kappa_l^+ = 0$ or $\kappa_l^- = 0$, which is also a point where $\boldsymbol{\gamma}$ has higher order contact with one of the tangent L-slices of M. As an application of the theory of unfoldings of functions in [1], we give a classification of singularities of both the lightcone Gauss map and the lightcone pedal in Theorem 5.4, which is one of the main results. In order to apply the unfolding theory, we give an explicit characterization of the cusp singularities in Proposition 5.3.

On the other hand, we also investigate the global properties of the normalized lightcone curvatures. Here we consider a unit speed closed regular curve $\boldsymbol{\gamma}$ on a spacelike surface from the unit circle S^1. We show that the total normalized lightcone curvature is equal to the winding number of the projection of the curve to the Euclidean plane in Theorem 6.5. Moreover we consider the total absolute normalized lightcone curvature of $\boldsymbol{\gamma}$, we have the inequality in Theorem 6.7 that the total absolute normalized lightcone curvature is not less than the maximum of the absolute value of the winding number of the projection to the Euclidean plane or 1. In order to characterize the curve with the equality, we introduce the notion of lightlike-convexity relative to M, or we call it L-convex relative to M, for a regular curve on the surface M. Then in Proposition 6.9, we show that the total absolute normalized lightcone curvature attains the minimum if and only if $\boldsymbol{\gamma}$ is L-convex relative to M.

We explain in §2 the basic notions of Lorentz-Minkowski space that will be used throughout the paper. In §3 we introduce lightcone curvatures and study its basic properties. In §4 and §5 are devoted to the study of height functions, the lightcone Gauss map and the lightcone pedal by considering the relationship with curvatures. Moreover, in §5, we have one of the main results in this paper that local properties of the curve provided by the lightcone curvature. In §6, global properties of the lightcone curvatures are investigated. Finally in §7 we consider Euclidean plane curves and the hyperbolic plane curves as special cases.

2 Notations and definitions

In this section we prepare some notations and definitions which we will use in this paper. Let $\mathbb{R}^3$ be a three-dimensional vector space. For any $\boldsymbol{x} = (x_0, x_1, x_2), \boldsymbol{y} = (y_0, y_1, y_2) \in \mathbb{R}^3$, the pseudo-scalar product of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0y_0 + x_1y_1 + x_2y_2$. We call $(\mathbb{R}^3, \langle , \rangle)$ *Minkowski* 3*-space*. We write $\mathbb{R}^3_1$ instead of $(\mathbb{R}^3, \langle , \rangle)$. We say that a non-zero vector $\boldsymbol{x} \in \mathbb{R}^3_1$ is *spacelike, lightlike* or *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$, $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ or $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$ respectively. The norm of the vector $\boldsymbol{x} \in \mathbb{R}^3_1$ is defined by $\| \boldsymbol{x} \| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$. Here we define the notion of *planes*. For a non-zero vector $\boldsymbol{v} \in \mathbb{R}^3_1$ and a real number c, we define a *plane* with *pseudo-normal* $\boldsymbol{v}$ by

$$P(\boldsymbol{v}, c) = \{ \boldsymbol{x} \in \mathbb{R}^3_1 \mid \langle \boldsymbol{x}, \boldsymbol{v} \rangle = c \}.$$

We call $P(\boldsymbol{v}, c)$ a *spacelike plane*, a *timelike plane* or a *lightlike plane* if $\boldsymbol{v}$ is timelike, spacelike or lightlike respectively. We now define *Hyperbolic plane* by

$$H^2_+(-1) = \{ \boldsymbol{x} \in \mathbb{R}^3_1 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1, x_0 > 0 \}$$

and *de Sitter* 2*-space* by

$$S^2_1 = \{ \boldsymbol{x} \in \mathbb{R}^3_1 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1 \}.$$

We define

$$LC^* = \{ \boldsymbol{x} = (x_0, x_1, x_2) \in \mathbb{R}^3_1 \mid x_0 \neq 0,\ \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0 \}$$

and we call it the (*open*) *lightcone* at the origin. Then the subset

$$LC^*_+ = \{ \boldsymbol{x} \in LC^* \mid x_0 > 0 \}$$

of LC^* is called the *future lightcone.* If $\boldsymbol{x} = (x_0, x_1, x_2)$ is a non-zero lightlike vector, then $x_0 \neq 0$. Therefore we have

$$\widetilde{\boldsymbol{x}} = \left(1, \frac{x_1}{x_0}, \frac{x_2}{x_0}\right) \in S^1_+ = \{\boldsymbol{x} = (x_0, x_1, x_2) \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0,\ x_0 = 1\}.$$

We call S^1_+ the *lightcone* (or, *spacelike*) *unit circle.* Here we define

$$\boldsymbol{a} \wedge \boldsymbol{b} = \begin{vmatrix} -\boldsymbol{e}_0 & \boldsymbol{e}_1 & \boldsymbol{e}_2 \\ a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \end{vmatrix},$$

where $\boldsymbol{a} = (a_0, a_1, a_2), \boldsymbol{b} = (b_0, b_1, b_2)$ and $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2\}$ is the canonical basis of $\mathbb{R}^3$.

3 Curves on spacelike surface and lightcone Gauss maps

We consider a spacelilke embedding $\boldsymbol{X} : U \longrightarrow \mathbb{R}^3_1$ from an open subset $U \subset \mathbb{R}^2$. We write $M = \boldsymbol{X}(U)$ and identify M and U through the embedding $\boldsymbol{X}$. Here, we say that $\boldsymbol{X}$ is a *spacelike embedding* if the tangent space T_pM consists of spacelike vectors at any $p = \boldsymbol{X}(u)$. Let $\bar{\boldsymbol{\gamma}} : I \longrightarrow U$ be a regular curve and we have a curve $\boldsymbol{\gamma} : I \longrightarrow M \subset \mathbb{R}^3_1$ defined by $\boldsymbol{\gamma}(s) = \boldsymbol{X}(\bar{\boldsymbol{\gamma}}(s))$. We say that $\boldsymbol{\gamma}$ is a *curve on the spacelike surface M.* Since $\boldsymbol{\gamma}$ is a spacelike curve, we can reparameterize it by the arc-length s. So we have the unit tangent vector $\boldsymbol{t}(s) = \boldsymbol{\gamma}'(s)$ of $\boldsymbol{\gamma}(s)$. Since $\boldsymbol{X}$ is a spacelike embedding, we have a unit timelike normal vector field $\boldsymbol{n}$ along $M = \boldsymbol{X}(U)$ defined by

$$\boldsymbol{n}(p) = \frac{\boldsymbol{X}_{u_1}(u) \wedge \boldsymbol{X}_{u_2}(u)}{\|\boldsymbol{X}_{u_1}(u) \wedge \boldsymbol{X}_{u_2}(u)\|},$$

for $p = \boldsymbol{X}(u)$.

We say that $\boldsymbol{n}$ is *future directed* if $\langle \boldsymbol{n}, \boldsymbol{e}_0 \rangle < 0$. We choose the orientation of M such that $\boldsymbol{n}$ is future directed. We define $\boldsymbol{n}_{\boldsymbol{\gamma}}(s) = \boldsymbol{n} \circ \boldsymbol{\gamma}(s)$, so that we have a unit timelike normal vector field $\boldsymbol{n}_{\boldsymbol{\gamma}}$ along $\boldsymbol{\gamma}$.

Therefore we can construct a spacelike unit normal section $\boldsymbol{b}(s) \in N_p(M)$ by $\boldsymbol{b}(s) = \boldsymbol{t}(s) \wedge \boldsymbol{n}_\gamma(s)$. It follows that we have $\langle \boldsymbol{n}_\gamma, \boldsymbol{n}_\gamma \rangle = -1,\ \langle \boldsymbol{n}_\gamma, \boldsymbol{b} \rangle = 0,\ \langle \boldsymbol{b}, \boldsymbol{b} \rangle = 1$. Then we have a pseudo-orthonormal frame $\{\boldsymbol{t}(s), \boldsymbol{n}_\gamma(s), \boldsymbol{b}(s)\}$, which is called the *Lorentzian Darboux frame* along $\boldsymbol{\gamma}$. By standard arguments, we have the following *Frenet-Serret type formulae*:

$$\begin{cases} \boldsymbol{t}'(s) = \kappa_n(s)\boldsymbol{n}_\gamma(s) + \kappa_g(s)\boldsymbol{b}(s), \\ \boldsymbol{n}_\gamma'(s) = \kappa_n(s)\boldsymbol{t}(s) + \tau_g(s)\boldsymbol{b}(s), \\ \boldsymbol{b}'(s) = -\kappa_g(s)\boldsymbol{t}(s) + \tau_g(s)\boldsymbol{n}_\gamma(s), \end{cases}$$

where $\kappa_n(s) = -\langle \boldsymbol{t}'(s), \boldsymbol{n}_\gamma(s) \rangle$, $\kappa_g(s) = \langle \boldsymbol{t}'(s), \boldsymbol{b}(s) \rangle$ and $\tau_g(s) = -\langle \boldsymbol{b}'(s), \boldsymbol{n}_\gamma(s) \rangle$.
Here, we have the following properties of $\boldsymbol{\gamma}$ characterized by the conditions of $\kappa_g, \kappa_n, \tau_g$.

$$\boldsymbol{\gamma} \text{ is } \begin{cases} \text{a geodesic curve if and only if } \kappa_g \equiv 0 \\ \text{an acymptotic curve if and only if } \kappa_n \equiv 0 \\ \text{a principal curve if and only if } \tau_g \equiv 0 \end{cases}$$

We now consider a smooth mapping provided by the lightcone normal vector field $\boldsymbol{n}_\gamma \pm \boldsymbol{b}$ at each point s $\in I$. And we write $\mathbb{L}^\pm(s) = \boldsymbol{n}_\gamma(s) \pm \boldsymbol{b}(s)$. By the above Frenet-Serret type formulae, we can calculate the following derivative of $\mathbb{L}^\pm(s)$:

$$(\mathbb{L}^\pm)'(s) = (\boldsymbol{n}'_\gamma(s) \pm \boldsymbol{b}'(s)) = \pm\tau_g(s)\mathbb{L}^\pm(s) + (\kappa_n(s) \mp \kappa_g(s))\boldsymbol{t}(s).$$

Now, we consider a future directed unit timelike normal vector filed $\boldsymbol{n}_{\boldsymbol{\gamma}}(s) \in N_pM$ and the corresponding spacelike unit normal vector field $\boldsymbol{b}(s) \in N_pM$ along $\boldsymbol{\gamma}$ constructed in the previous paragraph, where $p = \boldsymbol{X}(u)$. Here we define the *lightcone Gauss map of* $\boldsymbol{\gamma}$ *relative to* M by

$$\widetilde{\mathbb{L}}^\pm : \ U \ \longrightarrow \ S^1_+; \quad s \longmapsto \widetilde{(\boldsymbol{n}_\gamma \pm \boldsymbol{b})}(s).$$

By definition, we have $\ell_0^\pm \widetilde{\mathbb{L}}^\pm = \mathbb{L}^\pm$ where $\mathbb{L}^\pm(s) = (\ell_0^\pm(s), \ell_1^\pm(s), \ell_2^\pm(s))$. It follows that $\ell_0^\pm(\widetilde{\mathbb{L}}^\pm)' = (\mathbb{L}^\pm)' - \ell_0^{\pm\prime}\widetilde{\mathbb{L}}^\pm$. Consider the orthogonal projection $\pi^t : T_pM \bigoplus N_pM \longrightarrow T_pM$, since $\widetilde{\mathbb{L}}^\pm(s) \in N_pM$ and $\pi^t \circ (\mathbb{L}^\pm)'(s) \in T_pM$, we obtain

$$\pi^t \circ (\widetilde{\mathbb{L}}^\pm)'(s) = \frac{1}{\ell_0^\pm(s)} \pi^t \circ (\mathbb{L}^\pm)'(s) = \frac{1}{\ell_0^\pm(s)}(\kappa_n(s) \mp \kappa_g(s))\boldsymbol{t}(s).$$

According to the above calculation, we define new invariants $\kappa_l^\pm$ by $\kappa_l^\pm(s) = \kappa_n(s) \mp \kappa_g(s)$, which are called *lightcone curvatures of* $\boldsymbol{\gamma}$ *relative to* M. Therefore $\pi^t \circ (\widetilde{\mathbb{L}}^\pm)'(s) = \frac{1}{\ell_0^\pm(s)}\kappa_l^\pm(s)\boldsymbol{t}(s)$. We also define $\widetilde{\kappa}_l^\pm$ by $\widetilde{\kappa}_l^\pm(s) = \pm\frac{1}{\ell_0^\pm(s)}\kappa_l^\pm(s)$. We call $\widetilde{\kappa}_l^\pm$ *normalized lightcone curvatures of* $\boldsymbol{\gamma}$ *relative to* M. Let σ be $+$ or $-$. Then we have the following proposition :

Proposition 3.1 *Under the above notation, the following conditions are equivalent*:
(1) $\kappa_l^\sigma = 0$.
(2) $\widetilde{\mathbb{L}}^\sigma$ *is constant.*
(3) *There exists a lightlike vector* $\boldsymbol{v} \in S^1_+ \subset LC^*$ *such that* $Im\boldsymbol{\gamma} = P(\boldsymbol{v}, c) \cap M$.

Proof. We only give the proof for $\sigma = +$. In this case we write $\mathbb{L} = \mathbb{L}^+$, $\kappa_l = \kappa_l^+$ and $\ell_0 = \ell_0^+$.

Assume that $\kappa_l = 0$. Then $\pi^t \circ \widetilde{\mathbb{L}}' = 0$. This means that

$$\widetilde{\mathbb{L}}'(s) = -\frac{\ell_0{}'(s)}{\ell_0(s)^2}\mathbb{L}(s) + \frac{1}{\ell_0(s)}\tau_g(s)\mathbb{L}(s) = (-\frac{\ell_0{}'(s)}{\ell_0(s)^2} + \frac{1}{\ell_0(s)}\tau_g(s))\mathbb{L}(s).$$

Here we define λ by $\lambda(s) = -\frac{\ell_0(s)'}{\ell_0(s)^2} + \frac{1}{\ell_0(s)}\tau_g(s)$, then

$$\widetilde{\mathbb{L}}'(s) = \lambda(s)\mathbb{L}(s) = (\lambda(s)\ell_0(s), \lambda(s)\ell_1(s), \lambda\ell_2(s)).$$

Since we have $\widetilde{\mathbb{L}}(s) = (1, \frac{\ell_1}{\ell_0}(s), \frac{\ell_2}{\ell_0}(s)), \widetilde{\mathbb{L}}'(s) = (0, (\frac{\ell_1}{\ell_0})'(s), (\frac{\ell_2}{\ell_0})'(s))$. Therefore $\lambda(s)\ell_0(s) \equiv 0$. By the above calculation and the fact $\ell_0(s) \neq 0$, we have $\lambda(s) \equiv 0$. Thus, $\widetilde{\mathbb{L}}'(s) \equiv 0$, so that $\widetilde{\mathbb{L}}$ is constant.

On the other hand, assume that $\widetilde{\mathbb{L}}$ is constant. By the definition of $\widetilde{\mathbb{L}}$, we have $\kappa_l = 0$. This completes the proof of the equivalence between (1) and (2).

Assume that $\widetilde{\mathbb{L}}$ is constant. This means that there exists a lightlike vector $\boldsymbol{v} \in S^1_+ \subset LC^*$ such that $\widetilde{\mathbb{L}} = \boldsymbol{v}$. Here we define $F : I \longrightarrow \mathbb{R}$ by $F(s) = \langle \boldsymbol{\gamma}(s), \boldsymbol{v} \rangle$. Since $F'(s) = \langle \boldsymbol{\gamma}'(s), \boldsymbol{v} \rangle = 0$, there exists a scalar $c \in \mathbb{R}$ such that $F(s) = \langle \boldsymbol{\gamma}(s), \boldsymbol{v} \rangle = c$. Therefore $Im\boldsymbol{\gamma} = P(\boldsymbol{v}, c) \cap M$.

For the converse, assume that $Im\boldsymbol{\gamma} = P(\boldsymbol{v}, c) \cap M$. The tangent space of $P(\boldsymbol{v}, c)$ can be identified with $P(\boldsymbol{v}, 0)$. Since $Im\boldsymbol{\gamma} \subset P(\boldsymbol{v}, c)$, we have $T_p Im\boldsymbol{\gamma} \subset P(\boldsymbol{v}, 0)$, so that $N_p Im\boldsymbol{\gamma} \cap P(\boldsymbol{v}, 0)$ is the line generated by $\boldsymbol{v}$. For the future directed timelike unit normal vector field $\boldsymbol{n}_\gamma$ along $\boldsymbol{\gamma}$, there exists a lightlike vector $\bar{\boldsymbol{v}}$ such that $\bar{\boldsymbol{v}}$ is parallel to $\boldsymbol{v}$ and $\bar{\boldsymbol{v}} - \boldsymbol{n}_\gamma$ is a spacelike unit normal vector field along $\boldsymbol{\gamma}$. We write $\boldsymbol{b} = \bar{\boldsymbol{v}} - \boldsymbol{n}_\gamma$, so that we have the Lorentzian Darboux frame $\{\boldsymbol{t}, \boldsymbol{n}_\gamma, \boldsymbol{b}\}$ along $\boldsymbol{\gamma}$ with $\widetilde{\boldsymbol{n}_\gamma + \boldsymbol{b}(s)} = \widetilde{\boldsymbol{v}}$. This means that the corresponding lightcone Gauss map $\widetilde{\mathbb{L}}$ is constant. This completes the proof of the equivalence between (2) and (3). This completes the proof. □

The above proposition suggests that curves of the form $P(\boldsymbol{v}, c) \cap M$ ($\boldsymbol{v} \in S^1_+$) are the candidates of model curves on M. These might play a similar role to lines in Euclidean plane. We call it a *lightlike-slice* (or, an *L-slice*) of M.

4 Lightlike height functions

In order to investigate the geometric properties of curves on spacelike surfaces, we introduce two families of functions and apply the theory of unfoldings of functions. Let $\boldsymbol{\gamma} : I \longrightarrow M$ be a curve on a spacelike surface M. Then we define two families of functions as follows:

$$H : I \times S^1_+ \longrightarrow \mathbb{R}; \quad (s, \boldsymbol{v}) \longmapsto \langle \boldsymbol{\gamma}(s), \boldsymbol{v} \rangle,$$

$$\widetilde{H} : I \times LC^* \longrightarrow \mathbb{R}; \quad (s, \boldsymbol{v}) \longmapsto \langle \boldsymbol{\gamma}(s), \boldsymbol{v} \rangle - v_0,$$

where $\boldsymbol{v} = (v_0, v_1, v_2)$. We call H the *lightcone height function* of $\boldsymbol{\gamma}$ on M and $\widetilde{H}$ the *extended lightcone height function* of $\boldsymbol{\gamma}$ on M. We denote $h_{\boldsymbol{v}}(s) = H(s, \boldsymbol{v})$ for any fixed $\boldsymbol{v} \in S^1_+$ and $\widetilde{h}_{\boldsymbol{v}}(s) = \widetilde{H}(s, \boldsymbol{v})$ for any fixed $\boldsymbol{v} \in LC^*$. Then we have the following proposition:

Proposition 4.1 *Under the above notations,we have the following:*
(1) $h_{\boldsymbol{v}}'(s) = 0$ *if and only if* $\boldsymbol{v} = \widetilde{\mathbb{L}}^\sigma(s)$.
(2) $h_{\boldsymbol{v}}'(s) = h_{\boldsymbol{v}}''(s) = 0$ *if and only if* $\boldsymbol{v} = \widetilde{\mathbb{L}}^\sigma(s)$ *and* $\kappa_l{}^\sigma(s) = 0$.
(3) $h_{\boldsymbol{v}}'(s) = h_{\boldsymbol{v}}''(s) = h_{\boldsymbol{v}}'''(s) = 0$ *if and only if* $\boldsymbol{v} = \widetilde{\mathbb{L}}^\sigma(s)$, $\kappa_l{}^\sigma(s) = 0$ *and* $(\kappa_l{}^\sigma)'(s) = 0$.

Proof. (1) Since $\{\boldsymbol{n}_\gamma, \boldsymbol{b}, \boldsymbol{t}\}$ is a basis of the vector space $T_p\mathbb{R}^3_1$ where $p = \boldsymbol{X}(u)$, there exist real numbers λ, μ, ξ such that $\boldsymbol{v} = \lambda \boldsymbol{n}_\gamma + \mu \boldsymbol{b} + \xi \boldsymbol{t}$. Then we have

$$h'_{\boldsymbol{v}}(s) = \langle \boldsymbol{\gamma}'(s), \boldsymbol{v} \rangle = \langle \boldsymbol{t}(s), \lambda \boldsymbol{n}_\gamma(s) + \mu \boldsymbol{b}(s) + \xi \boldsymbol{t}(s) \rangle = \xi.$$

Since $h_{\boldsymbol{v}}'(s) = 0$, we have $\xi = 0$, and the fact that $\boldsymbol{v} \in S^1_+$ implies that $\lambda = \pm\mu = \pm 1$. This completes the proof of (1).
We only give the proof (2) and (3) for $\sigma = +$. In this case we write $\mathbb{L} = \mathbb{L}^+$ $\kappa_l = \kappa_l{}^+$ and $\ell_0 = \ell_0^+$.
(2) Since $h_{\boldsymbol{v}}''(s) = \langle \boldsymbol{\gamma}''(s), \boldsymbol{v} \rangle$, $h_{\boldsymbol{v}}'(s) = h_{\boldsymbol{v}}''(s) = 0$ if and only if $h_{\boldsymbol{v}}''(s) = \langle \boldsymbol{\gamma}''(s), \widetilde{\mathbb{L}}(s) \rangle = 0$. Here, we have $\widetilde{\mathbb{L}}(s) = \frac{1}{\ell_0(s)}\mathbb{L}$ by definition. It follows form the *Frenet-Serret type formulae* that

$$\langle \boldsymbol{\gamma}''(s), \widetilde{\mathbb{L}}(s) \rangle = \frac{1}{\ell_0(s)} \langle \kappa_n(s) \boldsymbol{n}_\gamma + \kappa_g(s) \boldsymbol{b}(s), \boldsymbol{n}_\gamma(s) + \boldsymbol{b}(s) \rangle = \frac{1}{\ell_0(s)}(-\kappa_n(s) + \kappa_g(s)) = -\frac{1}{\ell_0(s)}\kappa_l.$$

Since $\langle \boldsymbol{\gamma}''(s), \widetilde{\mathbb{L}}(s)\rangle = 0$ and $\ell_0(s) \neq 0$, we have $\kappa_l = 0$. This completes the proof of (2).
(3) By the assertions (1) and (2), $h_{\boldsymbol{v}}'(s) = h_{\boldsymbol{v}}''(s) = h_{\boldsymbol{v}}'''(s) = 0$ if and only if $h_{\boldsymbol{v}}'''(s) = \langle \boldsymbol{t}''(s), \widetilde{\mathbb{L}}(s)\rangle = 0$ and $\kappa_l = 0$. Since $\boldsymbol{t}''(s) = \kappa_n'(s)\boldsymbol{n}_\gamma(s) + \kappa_n(s)\boldsymbol{n}_\gamma'(s) + \kappa_g'(s)\boldsymbol{b}(s) + \kappa_g(s)\boldsymbol{b}'(s)$, we have

$$\langle \boldsymbol{t}''(s), \widetilde{\mathbb{L}}(s)\rangle = \frac{1}{\ell_0(s)}(-\kappa_n'(s) + \kappa_g'(s) - \kappa_g(s)\tau_g(s) + \kappa_n(s)\tau_g(s)) = -\frac{1}{\ell_0(s)}(\kappa_n'(s) - \kappa_g'(s)) = 0.$$

Then we have $\kappa_l' = 0$. This completes the proof. □

By definition, we have $\partial\widetilde{H}/\partial s = \partial H/\partial s,\ \partial^2\widetilde{H}/\partial s^2 = \partial^2 H/\partial s^2, \partial^3\widetilde{H}/\partial s^3 = \partial^3 H/\partial s^3$. Then we have the following proposition:

Proposition 4.2 *Under the notations as the above, we have the following:*

$$\widetilde{H}(s, \boldsymbol{v}) = \frac{\partial\widetilde{H}}{\partial s}(s, \boldsymbol{v}) = 0 \textit{ if and only if } \boldsymbol{v} = \langle \boldsymbol{\gamma}(s), \widetilde{\mathbb{L}}^\sigma(s)\rangle\widetilde{\mathbb{L}}^\sigma(s).$$

The above proposition induces the following notion: We define the *lightcone pedal of* $\boldsymbol{\gamma}$ *relative to* M as a smooth mapping

$$\mathbb{LP}^\sigma_{(\gamma,M)} : \ I \longrightarrow LC^*; \ \ s \longmapsto \langle \boldsymbol{\gamma}(s), \widetilde{\mathbb{L}}^\sigma(s)\rangle\widetilde{\mathbb{L}}^\sigma(s).$$

The image $\mathbb{LP}^\sigma_{(\gamma,M)}(I)$ is called the *lightcone pedal curve of* $\boldsymbol{\gamma}$ *relative to* M.

On the other hand, we consider the following another family of function:

$$\mathcal{H} : \mathbb{R}^3_1 \times S^1_+ \longrightarrow \mathbb{R}; \ \ (\boldsymbol{x}, \boldsymbol{v}) \longmapsto \langle \boldsymbol{x}, \boldsymbol{v}\rangle.$$

We denote $\mathfrak{h}_{\boldsymbol{v}}(\boldsymbol{x}) = \mathcal{H}(\boldsymbol{x}, \boldsymbol{v})$ for any fixed $\boldsymbol{v} \in S^1_+$, then, we have

$$h_{\boldsymbol{v}_0}(s) = \langle \boldsymbol{\gamma}(s), \boldsymbol{v}_0\rangle = \mathcal{H}(\boldsymbol{\gamma}(s), \boldsymbol{v}_0) = \mathfrak{h}_{\boldsymbol{v}_0}(\boldsymbol{\gamma}(s))$$

Moreover, for any $s \in \mathbb{R}$ and $\boldsymbol{v}_0 = \widetilde{\mathbb{L}}^\sigma(s_0)$, $\mathfrak{h}^{-1}_{\boldsymbol{v}_0}(s)$ is an *L-slice* when we consider $\mathcal{H} = \mathcal{H}|_{M\times S^1_+}$. By Proposition 4.1, (1), $\mathfrak{h}^{-1}_{\boldsymbol{v}_0}(c_0)$ is an L-slice of M tangent to $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}(s_0)$, where $c_0 = h_{\boldsymbol{v}_0}(s_0)$. We call $\mathfrak{h}^{-1}_{\boldsymbol{v}_0}(c_0)$ a *tangent L-slice* of $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}(s_0)$. We have two tangent L-slices at each point depending on $\sigma = \pm$. We denote it by $TL(M, \boldsymbol{\gamma})^\sigma_{s_0}$. Now let $F : M \longrightarrow \mathbb{R}$ be a submersion and $\boldsymbol{\gamma}(s_0) \subset F^{-1}(0)$. We say that $\boldsymbol{\gamma}$ and $F^{-1}(0)$ have *contact of order* k if the function $f = F \circ \boldsymbol{\gamma}(s)$ satisfies $f(s_0) = f'(s_0) = \cdots = f^{(k)}(s_0) = 0$ and $f^{(k+1)}(s_0) \neq 0$.
Then we have the following proposition:

Proposition 4.3 *A curve* $\boldsymbol{\gamma}$ *and the tangent L-slice* $TL(M, \boldsymbol{\gamma})^\sigma_{s_0}$ *have a contact of order two at point* s_0 *if and only if* $\kappa_l{}^\sigma(s_0) = 0$ *and* $(\kappa_l{}^\sigma)'(s_0) \neq 0$.

Proof. Here we have a vector $\boldsymbol{v}_0 \in S^1_+$ and the tangent L-slice is given as $\mathfrak{h}^{-1}_{\boldsymbol{v}_0}(s)$ for any $s \in \mathbb{R}$. By Proposition 4.1, the conditions $\widetilde{\mathbb{L}}^\sigma(s_0) = \boldsymbol{v}_0, \kappa_l{}^\sigma(s_0) = 0$ and $(\kappa_l{}^\sigma)'(s_0) \neq 0$ are equivalent to the conditions $h_{\boldsymbol{v}_0}'(s) = h_{\boldsymbol{v}_0}''(s) = 0, h_{\boldsymbol{v}_0}'''(s) \neq 0$. Since $h_{\boldsymbol{v}_0}(s) = \langle \boldsymbol{\gamma}(s), \boldsymbol{v}_0\rangle = \mathcal{H}(\boldsymbol{\gamma}(s), \boldsymbol{v}_0) = \mathfrak{h}_{\boldsymbol{v}_0}(\boldsymbol{\gamma}(s))$, we can calculate that $h_{\boldsymbol{v}_0}(s) = \mathfrak{h}_{\boldsymbol{v}_0}(\boldsymbol{\gamma}(s))$ satisfies $h_{\boldsymbol{v}_0}'(s_0) = h_{\boldsymbol{v}_0}''(s_0) = 0, h_{\boldsymbol{v}_0}'''(s_0) \neq 0$. This means $\boldsymbol{\gamma}$ and the tangent L-slice have a contact of order two at the point s_0. This completes the proof. □

5 The Lightcone Gauss map and the lightcone pedal

In this section we apply the theory of unfoldings of functions and give a proof of the main theorem.

First we give a quick review on the theory of unfoldings of functions of one variable. Detailed descriptions are found in the book[1]. Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, x_0)) \to \mathbb{R}$ be a function germ. We call F an *r-parameter unfolding* of f, where $f(s) = F_{x_0}(s)$. We say that f has *type* A_k at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \leqq p \leqq k$, and $f^{(k+1)}(s_0) \neq 0$. We also say that f has *type* $A_{\geq k}$ at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \leqq p \leqq k$. Let F be an unfolding of f and $f(s)$ has type A_k $(k \geqq 1)$ at s_0. We denote the $(k-1)$-jet of the partial derivative $\frac{\partial F}{\partial x_i}$ at s_0 by $j^{(k-1)}(\frac{\partial F}{\partial x_i}(s, x_0))(s_0) = \sum_{j=1}^{k-1} \alpha_{ji} s^j$ for $i = 1, \dots, r$. Then F is called an $\mathcal{R}^+$*-versal unfolding* if the $(k-1) \times r$ matrix of coefficients (α_{ji}) has rank $k-1$ $(k-1 \leqq r)$. Under the same condition as the above, F is called an $\mathcal{R}$*-versal unfolding* if the $k \times r$ matrix of coefficients $(\alpha_{0i}, \alpha_{ji})$ has rank k $(k \leqq r)$, where $\alpha_{0i} = \frac{\partial F}{\partial x_i}(s_0, x_0)$.

We now introduce important sets concerning the unfoldings relative to the above notions. The *catastrophe set* of F is the set

$$C_F = \{(s, x) | \frac{\partial F}{\partial s}(s, x) = 0\}.$$

The *bifurcation set* B_F of F is the critical value set of the restriction to C_F of the canonical projection $\pi : \mathbb{R} \times \mathbb{R}^r \longrightarrow \mathbb{R}^r$:

$$B_F = \{x \in \mathbb{R}^r | \exists s; \text{ with } \frac{\partial F}{\partial s}(s, x) = \frac{\partial^2 F}{\partial s^2}(s, x) = 0 \}$$

By Proposition 4.1, we have $C_H = \{(s, \boldsymbol{v}) \mid \boldsymbol{v} = \widetilde{\mathbb{L}}^\sigma(s) \}$, so that $\pi | C_H(s, \boldsymbol{v}) = \widetilde{\mathbb{L}}^\sigma(s)$. The *discriminant set* of F is the set

$$D_F = \{x \in \mathbb{R}^r | \exists s; F(s, x) = \frac{\partial F}{\partial s}(s, x) = 0 \}.$$

By Proposition4.2, we have the discriminant set $D_{\widetilde{H}}$ of $\widetilde{H}$:

$$D_{\widetilde{H}} = \{\boldsymbol{v} \in LC^* | \boldsymbol{v} = \langle \boldsymbol{\gamma}(s), \widetilde{\mathbb{L}}(s) \rangle \widetilde{\mathbb{L}}(s)\}$$

On the other hand, we consider the special case when $r = 1$. Let $F : (\mathbb{R} \times \mathbb{R}, (s_0, x_0)) \to (\mathbb{R}, 0)$ be a one-parameter unfolding of $f(s)$ which has type A_k at s_0. Suppose that $(\partial^2 F / \partial x \partial s)(s_0, x_0) \neq 0$. Then, by the implicit function theorem, there exists a smooth function germ $h : (\mathbb{R}, s_0) \longrightarrow (\mathbb{R}, x_0)$ such that

$$\frac{\partial F}{\partial s}(s, h(s)) = 0.$$

We define a two-parameter unfolding $\overline{F} : (\mathbb{R} \times \mathbb{R}^2, (s_0, (x_0, r_0))) \longrightarrow (\mathbb{R}, 0)$ by $\overline{F}(s, (x, r)) = F(s, x) - r$. We call $\overline{F}$ an *extended unfolding* of F. Let $\boldsymbol{\delta}_F : I \longrightarrow \mathbb{R}^2$ be a curve defined by $\boldsymbol{\delta}_F(s) = (h(s), F(s, h(s)))$. Then $\boldsymbol{\delta}_F(I) = D_{\overline{F}}$ as set germs at (s_0, x_0). Let $\boldsymbol{\gamma} : (\mathbb{R}, s_0) \longrightarrow \mathbb{R}^2$ be a smooth map germ. We say that $\boldsymbol{\gamma}$ has the *ordinary cusp* at $s_0 \in \mathbb{R}$ if $\boldsymbol{\gamma}'(s_0) = \mathbf{0}$ and $\boldsymbol{\gamma}''(s_0), \boldsymbol{\gamma}'''(s_0)$ are linearly independent. It is known that the image of $\boldsymbol{\gamma}$ at the ordinary cusp is diffeomorphic to $C = \{(x_1, x_2) | {x_1}^2 = {x_2}^3\}$ as set germs[1, Page 154]. We also say that a smooth function germ $g : (\mathbb{R}, s_0) \longrightarrow \mathbb{R}$ has the *fold* singularity at $s_0 \in R$ if g has type A_1 at s_0. In this case, it is easy to show that there exist diffeomorphism germs $\phi : (\mathbb{R}, s_0) \longrightarrow (\mathbb{R}, 0)$ and $\psi : (\mathbb{R}, g(s_0)) \longrightarrow (\mathbb{R}, 0)$ such that $\psi \circ g \circ \phi^{-1}(x) = x^2$. Then we have the following result.

Proposition 5.1 *Under the above notations, suppose that f has type $A_{\geq 2}$ at s_0. Then the following conditions are equivalent:*
(1) f has type A_2 at s_0,
(2) h has type A_1 (i.e., the fold) at s_0,
(3) $\boldsymbol{\delta}_F : I \longrightarrow \mathbb{R}^2$ has the ordinary cusp at s_0.
(4) F is an $\mathcal{R}^+$-versal unfolding of f.
(5) $\overline{F}$ is an $\mathcal{R}$-versal unfolding of f.

Proof. If we calculate the derivative of the equation $\partial F/\partial s(s, h(s)) = 0$, then we have

$$\begin{aligned} 0 &= \frac{d}{ds}\left(\frac{\partial F}{\partial s}(s,h(s))\right) = \frac{\partial^2 F}{\partial s^2}(s,h(s)) + \frac{\partial^2 F}{\partial x \partial s}(s,h(s))h'(s), \\ 0 &= \frac{d^2}{ds^2}\left(\frac{\partial F}{\partial s}(s,h(s))\right) = \frac{\partial^3 F}{\partial s^3}(s,h(s)) + \frac{\partial^3 F}{\partial x \partial s^2}(s,h(s))h'(s) \\ &\quad + \frac{\partial^3 F}{\partial s^2 \partial x}(s,h(s))h'(s) + \frac{\partial^3 F}{\partial x^2 \partial s}(s,h(s))(h'(s))^2 + \frac{\partial^2 F}{\partial x \partial s}(s,h(s))h''(s). \end{aligned}$$

Therefore, we have

$$0 = \frac{\partial^2 F}{\partial s^2}(s_0,x_0) + \frac{\partial^2 F}{\partial x \partial s}(s_0,x_0)h'(s_0) = f''(s_0) + \frac{\partial^2 F}{\partial x \partial s}(s_0,x_0)h'(s_0).$$

Since $(\partial^2 F/\partial x \partial s)(s_0, x_0) \neq 0$, $f''(s_0) = 0$ if and only if $h'(s_0) = 0$. Under the condition that $h'(s_0) = 0$, we have

$$0 = \frac{\partial^3 F}{\partial s^3}(s_0,x_0) + \frac{\partial^2 F}{\partial x \partial s}(s_0,x_0)h''(s_0) = f'''(s_0) + \frac{\partial^2 F}{\partial x \partial s}(s_0,x_0)h''(s_0),$$

so that $f'''(s_0) = 0$ if and only if $h''(s_0) = 0$. Therefore, the conditions (1) and (2) are equivalent.

By the relations $f(s_0) = F_{x_0}(s_0), h'(s_0) = 0$, $\partial F/\partial s(s_0, x_0) = 0$ and the straightforward calculatios, we have

$$\begin{aligned} \boldsymbol{\delta}'_F(s_0) &= (0,0), \\ \boldsymbol{\delta}''_F(s_0) &= \left(h''(s_0), \frac{\partial F}{\partial x}(s_0,x_0)h''(s_0)\right), \\ \boldsymbol{\delta}'''_F(s_0) &= \left(h'''(s_0), f'''(s_0) + 2\frac{\partial^2 F}{\partial x \partial s}(s_0,x_0)h''(s_0) + \frac{\partial F}{\partial x}(s_0,x_0)h'''(s_0)\right). \end{aligned}$$

The curve $\boldsymbol{\delta}_F(s)$ has the ordinary cusp at s_0 if and only if $\boldsymbol{\delta}'_F(s_0) \neq \mathbf{0}$ and the rank of the following matrx is two:

$$A = \begin{pmatrix} h''(s_0) & \frac{\partial F}{\partial x}(s_0,x_0)h''(s_0) \\ h'''(s_0) & f'''(s_0) + 2\frac{\partial^2 F}{\partial x \partial s}(s_0,x_0)h''(s_0) + \frac{\partial F}{\partial x}(s_0,x_0)h'''(s_0) \end{pmatrix}.$$

Here we have

$$\operatorname{rank} A = \operatorname{rank} \begin{pmatrix} h''(s_0) & 0 \\ h'''(s_0) & f'''(s_0) + 2\frac{\partial^2 F}{\partial x \partial s}(s_0,x_0)h''(s_0) \end{pmatrix}.$$

Since $f'''(s_0) + (\partial^2 F/\partial x \partial s(s_0, x_0))h''(s_0) = 0$, we have

$$\operatorname{rank} A = \operatorname{rank} \begin{pmatrix} h''(s_0) & 0 \\ h'''(s_0) & -f'''(s_0) \end{pmatrix}.$$

Therefore rank $A = 2$ if and only if $h''(s_0), f'''(s_0) \neq 0$. This condition is equivalent to the conditions (1) and (2). Thus, the conditions (1), (2) and (3) are equivalent.

Since F is a one-parameter unfolding, F is $\mathcal{R}^+$-versal if and only if the rank of the 1×1-matrix $\alpha_{11} = (\partial^2 F/\partial x \partial s)(s_0, x_0)$ is 1 and f has type A_2 at s_0. However, by the assumption, we have $(\partial^2 F/\partial x \partial s)(s_0, x_0) \neq 0$. Therefore the conditions (1) and (4) are equivalent. Moreover, the two-parameter unfolding $\overline{F}$ is $\mathcal{R}$-versal if and only if f has type A_2 at s_0 and the rank of the following matrix is two:

$$\begin{pmatrix} \frac{\partial F}{\partial x}(s_0, x_0) & \frac{\partial^2 F}{\partial s \partial x}(s_0, x_0) \\ -1 & 0 \end{pmatrix}.$$

By the assumotion, the rank of the above matrix is two. So, the conditions (1) and (5) are equivalent. This completes the proof. □

Remark 5.2 In the proof of the above proposition, we calculate that

$$\boldsymbol{\delta}'_F(s) = \left(h'(s), \frac{\partial F}{\partial s}(s, h(s)) + \frac{\partial F}{\partial x}(s, h(s))h'(s)\right) = h'(s)\left(1, \frac{\partial F}{\partial x}(s, h(s))\right).$$

Therefore, $h'(s_0) = 0$ if and only if $\boldsymbol{\delta}'_F(s) = 0$.

In order to apply the above proposition to our situattion, we denote $\boldsymbol{\gamma}(s) = (x_0(s), x_1(s), x_2(s))$, and fix the parameterization of a vector $\boldsymbol{v} \in S^1_+$ as $\boldsymbol{v} = (1, \cos\theta, \sin\theta)$. Then we have

$$H(s, \boldsymbol{v}) = H(s, \theta) = \langle \boldsymbol{\gamma}(s), \boldsymbol{v} \rangle = -x_0(s) + \cos\theta x_1(s) + \sin\theta x_2(s_0).$$

We have the following lemma.

Lemma 5.3 *Under the above notation, we have*

$$\frac{\partial^2 H}{\partial \theta \partial s}(s_0, \boldsymbol{v}_0) \neq 0,$$

for $\boldsymbol{v}_0 = \widetilde{\mathbb{L}}^\sigma(s_0)$.

Proof. By straightforward calculations, we have

$$\frac{\partial H}{\partial s}(s, \theta) = -x'_0(s) + \cos\theta x'_1(s) + \sin\theta x'_2(s), \ \frac{\partial^2 H}{\partial \theta \partial s}(s, \theta) = -\sin\theta x'_1(s) + \cos\theta x'_2(s).$$

We suppose that

$$-\sin\theta {x_1}'(s_0) + \cos\theta {x_2}'(s_0) = 0 \tag{i}$$

Here we denote $\boldsymbol{v}_0 = \widetilde{\mathbb{L}}^\sigma(s_0) = (1, \cos\theta_0, \sin\theta_0)$ and $\boldsymbol{t}(s_0) = ({x_0}'(s_0), {x_1}'(s_0), {x_2}'(s_0))$, then we have

$$0 = \langle \boldsymbol{t}(s_0), \widetilde{\mathbb{L}}(s_0) \rangle = -{x_0}'(s_0) + \cos\theta_0 {x_1}'(s_0) + \sin\theta_0 {x_2}'(s_0) \tag{ii}$$

By the equation (i) and (ii),we have $-\sin\theta_0 \cos\theta_0 {x_1}'(s_0) + \cos^2\theta_0 {x_2}'(s_0) = 0$ and $-\sin\theta_0 {x_0}'(s_0) + \sin\theta_0 \cos\theta_0 {x_1}'(s_0) + \sin^2\theta_0 {x_2}'(s_0) = 0$. Then $-\sin\theta_0 {x_0}'(s_0) + \cos^2\theta_0 {x_2}'(s_0) + \sin^2\theta_0 {x_2}'(s_0) = 0$, that is

$$-\sin\theta_0 {x_0}'(s_0) + {x_2}'(s_0) = 0 \tag{iii}$$

By the same calculation as the above, the equations (i) and (ii), we have $-\sin^2\theta_0 x_1{}'(s_0) + \sin\theta_0\cos\theta_0 x_2{}'(s_0) = 0$ and $-\cos\theta_0 x_0{}'(s_0) + \cos^2\theta_0 x_1{}'(s_0) + \cos\theta_0\sin\theta_0 x_2{}'(s_0) = 0$. Then $-\cos\theta_0 x_0{}'(s_0) + \cos^2\theta_0 x_1{}'(s_0) + \sin^2\theta_0 x_1{}'(s_0) = 0$, that is

$$-\cos\theta_0 x_0{}'(s_0) + x_1{}'(s_0) = 0 \tag{iv}$$

By the equations (iii) and (iv), we have

$$\begin{aligned}\boldsymbol{t}(s) &= (x_0{}'(s), x_1{}'(s), x_2{}'(s)) = (x_0{}'(s), \cos\theta_0 x_0{}'(s), \sin\theta_0 x_0{}'(s)) \\ &= x_0{}'(s_0)(1, \cos\theta_0, \sin\theta_0) = x_0{}'(s_0)\boldsymbol{v}_0.\end{aligned}$$

Since $\boldsymbol{t}$ is a spacelike vector and $\boldsymbol{v}_0$ is a lightlike vector, we have a contradiction. This completes the proof. □

We denote that $\boldsymbol{\gamma}(s) = (x_0(s), x_1(s), x_2(s))$. Let $\phi : S^1_+ \times (\mathbb{R}\setminus\{0\}) \longrightarrow LC^*$ be a diffeomorphism defined by $\phi((1,\cos\theta,\sin\theta), r) = (r, r\cos\theta, r\sin\theta)$. We define a family of functions

$$\overline{H} : I \times S^1 \times (\mathbb{R}\setminus\{0\}) \longrightarrow \mathbb{R}$$

defined by $\overline{H} = \widetilde{H}\circ(1_I \times \phi)$. By a straightforward calculation, $\widetilde{H}$ is an $\mathcal{R}$-versal unfolding of f if and only $\overline{H}$ is an $\mathcal{R}$-versal unfolding of f. Therefore, we consider $\overline{H}$ instead of $\widetilde{H}$. We remark that $\overline{H}$ is the extended unfolding of H.
As a consequence, we have the following theorem:

Theorem 5.4 *For $\boldsymbol{v}_0 = \widetilde{\mathbb{L}}^\sigma(s_0)$, we have the following:*
(A) *The following conditions are equivalent:*
(1) $\kappa_l{}^\sigma(s_0) \neq 0$,
(2) $\boldsymbol{\gamma}$ *and the tangent L-slice* $TL(M,\boldsymbol{\gamma})^\sigma_{s_0}$ *have a contact of order one at point* s_0,
(3) $h_{\boldsymbol{v}_0}$ *has type* A_1 *at* s_0,
(4) $\widetilde{\mathbb{L}}^\sigma : I \longrightarrow S^1_+$ *is non-singular at* s_0,
(5) $\mathbb{LP}^\sigma_{(\boldsymbol{\gamma},M)} : I \longrightarrow LC^*$ *is an immersion at* s_0.
(B) *The following conditions are equivalent:*
(1) $\kappa_l{}^\sigma(s_0) = 0$ *and* $(\kappa_l{}^\sigma)'(s_0) \neq 0$,
(2) $\boldsymbol{\gamma}$ *and the tangent L-slice* $TL(M,\boldsymbol{\gamma})^\sigma_{s_0}$ *have a contact of order two at point* s_0,
(3) $h_{\boldsymbol{v}_0}$ *has type* A_2 *at* s_0,
(4) $\widetilde{\mathbb{L}}^\sigma : I \longrightarrow S^1_+$ *is the fold point at* s_0,
(5) $\mathbb{LP}^\sigma_{(\boldsymbol{\gamma},M)} : I \longrightarrow LC^*$ *is the ordinary cusp at* s_0,
(6) $h''_{\boldsymbol{v}_0}(s_0) = 0$ *and* H *is an* $\mathcal{R}^+$*-versal unfolding of* $h_{\boldsymbol{v}_0}$,
(7) $\widetilde{h}''_{\boldsymbol{v}_0}(s_0) = 0$ *and* $\widetilde{H}$ *is an* $\mathcal{R}$*-versal unfolding of* $h_{\boldsymbol{v}_0}$.

Proof. (A) By the proof of proposition 3.1, we have $(\widetilde{\mathbb{L}}^\sigma)'(s_0) \neq 0$ if and only if $\kappa^\sigma_l(s_0) \neq 0$. This means that the conditions (1) and (4) are equivalent. By the proof of Proposition 4.1, $h_{\boldsymbol{v}_0}{}'(s_0) = 0$ and $h_{\boldsymbol{v}_0}{}''(s_0) \neq 0$ if and only if $\boldsymbol{v}_0 = \widetilde{\mathbb{L}}^\sigma(s_0)$ and $\kappa^\sigma_l(s_0) \neq 0$. This means that the conditions (1) and (3) are equivalent. Moreover, by the relation $h_{\boldsymbol{v}_0}(s_0) = \mathfrak{h}_{\boldsymbol{v}_0}(\boldsymbol{\gamma}(s_0))$ and Proposition 4.1, $\boldsymbol{\gamma}$ and the tangent L-slice $TL(M,\boldsymbol{\gamma})^\sigma_{s_0}$ have a contact of order one at point s_0 if and only if $\kappa_l{}^\sigma(s_0) \neq 0$. Therefore, the conditions (2) and (3) are equivalent. By Remark 5.2,

${\mathbb{LP}^{\sigma}_{(\gamma,M)}}'(s_0) \neq 0$ if and only if $(\widetilde{\mathbb{L}}^{\sigma})'(s) \neq 0$. This means that the conditions (4) and (5) are equivalent.

(B) By Lemma 5.3, the assumption of Proposition 5.1 is satisfied for H. The assertions of Lemma 5.3 and Proposition 5.1 in the case $F = H$ mean that the conditions (3),(4),(5),(6) and (7) are equivalent to each other. By Proposition 4.1, the conditions (1) and (3) are equivalent. The assertion of Proposition 4.3 means that the conditions (1) and (2) are equivalent. □

6 A global property of the lightcone curvature

In this section we consider regular homotopy invariants of regular closed curves on a spacelike surface. Here it has been known that the regular homotopy classification among regular plane curves are classified by the winding number and that the winding number of the projection of a closed spacelike regular curve is a spacelike homotopy invariant [2]. Here we calculate the winding number of a regular closed curve on a spacelike surface by using the normalized lightcone curvature of $\boldsymbol{\gamma}$ relative to M. Let $\boldsymbol{\gamma} : S^1 \longrightarrow M$ be a unit speed closed regular curve on a spacelike surface. We now fix the parameterization of lightcone circle :

$$S^1_+ = \{(1, \cos\theta, \sin\theta) \in LC^* | 0 \leq \theta \leq 2\pi\}$$

It follows that there exists a smooth function $\theta(s)$ such that

$$\widetilde{\mathbb{L}}^{\sigma}(s) = (1, \cos\theta(s), \sin\theta(s)).$$

Then we have the following proposition:

Proposition 6.1 *Under the same notations as the above, we have the following relation:*

$$\frac{d\theta(s)}{ds} = \sigma \frac{{\kappa_\ell}^{\sigma}(s)}{\ell_0^{\sigma}(s)}$$

at $s_0 \in S^1_+$ *with* ${\kappa_\ell}^{\sigma} \neq 0$. *If* ${\kappa_\ell}^{\sigma} = 0$ *then* $\frac{d\theta(s)}{ds}(s) = 0$

Proof. Firstly we assume that $\kappa_\ell^{\sigma} \neq 0$. By definition, we have

$$(\widetilde{\mathbb{L}}^{\sigma})'(s) = (0, -\sin\theta(s)\frac{d\theta(s)}{ds}, \cos\theta(s)\frac{d\theta(s)}{ds}).$$

If we write $\boldsymbol{n}_\gamma(s) = (n_0(s), n_1(s), n_2(s))$, then we calculate the following determinant:

$$\begin{aligned}
\left| \boldsymbol{n}_\gamma(s) \;\; \widetilde{\mathbb{L}}^{\sigma}(s) \;\; (\widetilde{\mathbb{L}}^{\sigma})'(s) \right| &= \begin{vmatrix} n_0(s) & n_1(s) & n_2(s) \\ 1 & \cos\theta(s) & \sin\theta(s) \\ 0 & -\sin\theta(s)\frac{d\theta(s)}{ds} & \cos\theta(s)\frac{d\theta(s)}{ds} \end{vmatrix} \\
&= \frac{d\theta(s)}{ds}(n_0(s) - n_1(s)\cos\theta(s) - n_2(s)\sin\theta(s)) \\
&= -\frac{d\theta(s)}{ds}\langle \boldsymbol{n}_\gamma(s), \widetilde{\mathbb{L}}^{\sigma}(s)\rangle \\
&= -\frac{d\theta(s)}{ds}\langle \boldsymbol{n}_\gamma(s), \frac{1}{\ell_0^{\sigma}(s)}(\boldsymbol{n}_\gamma + \sigma\boldsymbol{b}(s))\rangle \\
&= \frac{1}{\ell_0^{\sigma}(s)}\frac{d\theta(s)}{ds}
\end{aligned}$$

On the other hand, since $\ell_0^\sigma \widetilde{\mathbb{L}}^\sigma = \mathbb{L}^\sigma$ where $\mathbb{L}^\sigma(s) = (\ell_0^\sigma(s), \ell_1^\sigma(s), \ell_2^\sigma(s))$, we have $(\mathbb{L}^\sigma)' = \ell_0^{\sigma\prime}\widetilde{\mathbb{L}}^\sigma + \ell_0^\sigma(\widetilde{\mathbb{L}}^\sigma)'$. Moreover, by the *Frenet-Serret type formulae*, we have $(\mathbb{L}^\sigma)' = \sigma\tau_g(\boldsymbol{n}_\gamma + \sigma\boldsymbol{b}) + \kappa_\ell{}^\sigma\boldsymbol{t}$. It follows that

$$\begin{aligned}
\left|\boldsymbol{n}_\gamma(s) \quad \widetilde{\mathbb{L}}^\sigma(s) \quad (\widetilde{\mathbb{L}}^\sigma)'(s)\right| &= \left|\boldsymbol{n}_\gamma(s) \quad \tfrac{1}{\ell_0^\sigma(s)}\mathbb{L}^\sigma(s) \quad \tfrac{1}{\ell_0^\sigma(s)}(\mathbb{L}^\sigma)'(s) - \tfrac{\ell_0^\sigma(s)'}{\ell_0^\sigma(s)^2}\mathbb{L}^\sigma(s)\right| \\
&= \left|\boldsymbol{n}_\gamma(s) \quad \tfrac{1}{\ell_0^\sigma(s)}(\boldsymbol{n}_\gamma(s) + \sigma\boldsymbol{b}(s)) \quad \tfrac{1}{\ell_0^\sigma(s)}\sigma\tau_g(s)(\boldsymbol{n}_\gamma + \sigma\boldsymbol{b}(s)) + \tfrac{1}{\ell_0^\sigma(s)}\kappa_\ell{}^\sigma(s)\boldsymbol{t}(s)\right| \\
&= \frac{\sigma}{\ell_0^\sigma(s)^2}\kappa_\ell{}^\sigma(s)\left|\boldsymbol{n}_\gamma(s) \quad \boldsymbol{b}(s) \quad \boldsymbol{t}(s)\right| \\
&= \sigma\frac{\kappa_\ell{}^\sigma(s)}{\ell_0^\sigma(s)^2}
\end{aligned}$$

Therefore we have the desired relation. By Proposition 3.1, $\kappa_\ell{}^\sigma(s) = 0$ if and only if s is a singular point of the lightcone Gauss map. This is equivalent to the condition $\frac{d\theta(s)}{ds}(s) = 0$. This completes the proof. □

By the above proposition and the definition of $\widetilde{\kappa}_l^\sigma$, we have the following proposition:

Proposition 6.2 *For any unit speed closed regular spacelike immersion* $\boldsymbol{\gamma} : S^1 \longrightarrow M$, *we have*

$$\frac{1}{2\pi}\int_{S^1} \widetilde{\kappa}_l^\sigma ds = \deg(\widetilde{\mathbb{L}}^\sigma),$$

where $\deg(\widetilde{\mathbb{L}}^\sigma)$ *is the mapping degree of* $\widetilde{\mathbb{L}}^\sigma : I \longrightarrow S^1_+$.

Proof. By Proposition 6.1, we have

$$\int_{S^1} \widetilde{\kappa}_l^\sigma ds = \int_0^{2\pi} \frac{d\theta}{ds}(s)ds = 2\pi \deg(\widetilde{\mathbb{L}}^\sigma).$$

□

By using the canonical projection $\pi : \mathbb{R}^3_1 \longrightarrow \mathbb{R}^2_0$, we have an orientation preserving diffeomorphism $\pi|S^1_+ \longrightarrow S^1$. We now consider the (Euclidean) Gauss map $\mathbb{N} : S^1 \longrightarrow S^1$ on $\pi \circ \boldsymbol{\gamma}$. Since $\boldsymbol{\gamma} : S^1 \longrightarrow M \subset \mathbb{R}^3_1$ is a spacelike curve in $\mathbb{R}^3_1$, we have the following lemma as a special case of [2, Lemma 3.6].

Lemma 6.3 *Under the choice of a suitable direction of* $\mathbb{N}$, $\pi \circ \widetilde{\mathbb{L}}^\sigma$ *and* $\mathbb{N}$ *are homotopic.*

Since the mapping degree is a homotopy invariant and an invariant under orientation preserving diffeomorphisms, we have the following corollary.

Corollary 6.4 *Under the same assumptions as those in Proposition 6.2, we have*

$$\deg(\widetilde{\mathbb{L}}^\sigma) = W(\pi \circ \boldsymbol{\gamma})$$

where $W(\pi \circ \boldsymbol{\gamma})$ *denotes the winding number of* $\pi \circ \boldsymbol{\gamma}$.

This proves the following theorem:

Theorem 6.5 *For any unit speed closed spacelike immersion* $\boldsymbol{\gamma} : S^1 \longrightarrow M$ *with* $\boldsymbol{\gamma}'' \neq 0$, *we have*

$$\frac{1}{2\pi}\int_{S^1} \widetilde{\kappa}_l^\sigma ds = W(\pi\circ\boldsymbol{\gamma}).$$

On the other hand, we consider the total absolute normalized lightcone curvature of $\boldsymbol{\gamma}$. We define two subsets of S^1 as follows:

$$\begin{aligned} S_+ &= \{s \in S^1 \mid \sigma\kappa_l^\sigma(s) > 0\} \\ S_- &= \{s \in S^1 \mid \sigma\kappa_l^\sigma(s) < 0\}. \end{aligned}$$

Then we have

$$\begin{aligned} \int_{S^1} |\widetilde{\kappa}_l^\sigma| ds &= \int_{S_+} \widetilde{\kappa}_l^\sigma ds - \int_{S_-} \widetilde{\kappa}_l^\sigma ds \\ \int_{S^1} \widetilde{\kappa}_l^\sigma ds &= \int_{S_+} \widetilde{\kappa}_l^\sigma ds + \int_{S_-} \widetilde{\kappa}_l^\sigma ds. \end{aligned}$$

By Theorem 6.5 and the fact that $\int_{S_-} \widetilde{\kappa}_l^\sigma ds \leq 0$, we have

$$\int_{S^1} |\widetilde{\kappa}_l^\sigma| ds \geq 2\pi W(\pi\circ\boldsymbol{\gamma}).$$

Moreover, we have

$$-2\pi W(\pi\circ\boldsymbol{\gamma}) = -\int_{S^1} \widetilde{\kappa}_l^\sigma ds = -\int_{S_+} \widetilde{\kappa}_l^\sigma ds - \int_{S_-} \widetilde{\kappa}_l^\sigma ds,$$

so that we have

$$-\int_{S_-} \widetilde{\kappa}_l^\sigma ds \geq -2\pi W(\pi\circ\boldsymbol{\gamma}).$$

Thus, we have

$$\int_{S^1} |\widetilde{\kappa}_l^\sigma| ds \geq -2\pi W(\pi\circ\boldsymbol{\gamma}).$$

Therefore we have

$$\int_{S^1} |\widetilde{\kappa}_l^\sigma| ds \geq 2\pi |W(\pi\circ\boldsymbol{\gamma})|.$$

On the other hand, we have the following lemma:

Lemma 6.6 *We have the following inequality:*

$$\int_{S^1} |\widetilde{\kappa}_l^\sigma| ds \geq 2\pi.$$

Proof. We consider the lightcone Gauss map $\widetilde{\mathbb{L}}^\sigma : S^1 \longrightarrow S^1_+$. Let $C(\widetilde{\mathbb{L}}^\sigma)$ be the critical value set of $\widetilde{\mathbb{L}}^\sigma$. By the Sard theorem, $D = S^1_+ \setminus C(\widetilde{\mathbb{L}}^\sigma)$ is an open dense subset of S^1_+. For any $\boldsymbol{v} \in D$, the lightcone height function $h_{\boldsymbol{v}}$ has at least two critical points (i.e., one if the maximum another is the minimum). Suppose that s_0 is one of such points. By Theorem 5.4, $\kappa_l^\sigma(s_0) \neq 0$ if and only if $h_{\boldsymbol{v}}$ has type A_1 at s_0. This means that $h''_{\boldsymbol{v}}(s_0) \neq 0$. By the calculation of the proof of

Proposition 4.1, (2), $h''_{\boldsymbol{v}}(s_0) = -\kappa_l^\sigma(s_0)$. This means that $\widetilde{\mathbb{L}}^\sigma|(S_+ \cup S_-) : S_+ \cup S_- \longrightarrow D$ is surjective. Thereofore,

$$\int_{S^1} |\widetilde{\kappa}_l^\sigma| ds = \int_{S^1} \left| \frac{d\theta}{ds}(s) \right| ds = \int_{S_+ \cup S_-} \left| \frac{d\theta}{ds}(s) \right| ds = \int_{S_+} \frac{d\theta}{ds}(s) ds - \int_{S_-} \frac{d\theta}{ds}(s) ds \geq 2\pi.$$

□

Then we have the following theorem.

Theorem 6.7 *We have the following inequality:*

$$\frac{1}{2\pi} \int_{S^1} |\widetilde{\kappa}_l^\sigma| ds \geq \max\left(|W(\pi \circ \boldsymbol{\gamma})|, 1\right).$$

If $\pi \circ \boldsymbol{\gamma} : S^1 \longrightarrow \mathbb{R}^2_0$ is an embedding, then $W(\pi \circ \boldsymbol{\gamma}) = \pm 1$. Therefore, we have the following corollary:

Corollary 6.8 *Let $\boldsymbol{\gamma} : S^1 \longrightarrow M$ be a regular curve on M. Suppose that $\pi \circ \boldsymbol{\gamma} : S^1 \longrightarrow \mathbb{R}^2_0$ is an embedding. Then we have the following inequality:*

$$\frac{1}{2\pi} \int_{S^1} |\widetilde{\kappa}_l^\sigma| ds \geq 1.$$

In order to characterize the curve with the equality in the above corollary, we introduce the following notion: Let L be a spacelike line in $\mathbb{R}^3_1$. We define $\overline{L}$ as a line through the origin which is parallel to L. Since $L^\perp$ is a Lorentz plane, there exists a pseduo-orthonromal basis $\{\boldsymbol{v}^T, \boldsymbol{v}^S\}$ of $L^\perp$. Here, $\boldsymbol{v}^T$ is timelike and $\boldsymbol{v}^S$ is spacelike. Then we have lightlike vectors $\boldsymbol{v}^\pm = \boldsymbol{v}^T \pm \boldsymbol{v}^S$. By definition, there exists $p \in \mathbb{R}^3_1$ such that $L = p + \overline{L}$. For any $\boldsymbol{x} \in \overline{L}$, $\langle p + \boldsymbol{x}, \boldsymbol{v}^\pm \rangle = \langle p, \boldsymbol{v} \rangle = c^\pm$ are constant numbers. Thus we have lightlike planes $P(\boldsymbol{v}^\pm, c^\pm)$. Then we have

$$L = P(\boldsymbol{v}^+, c^+) \cap P(\boldsymbol{v}^-, c^-).$$

For a regular curve $\boldsymbol{\gamma} : I \longrightarrow M$, we consider the tangent line L_p of $\boldsymbol{\gamma}$ at $p = \boldsymbol{\gamma}(s_0)$. Then we say that the corresponding lightlike planes are *tangent lightlike planes* of $\boldsymbol{\gamma}$ at $p = \boldsymbol{\gamma}(s_0)$. Let K be a subset of $M \subset \mathbb{R}^3_1$. A plane Π through a point $\boldsymbol{x} \in K$ is called a *support plane* if K lies entirely in one of the closed half-space determined by Π. Let $\boldsymbol{\gamma} : S^1 \longrightarrow M$ be a spacelike embedding. We say that $\boldsymbol{\gamma}$ is *lightlike-convex* (or, *L-convex*) relative to M if the tangent lightlike planes at each point $\boldsymbol{\gamma}(s)$ are support planes of $\boldsymbol{\gamma}(S^1)$. Then we have the following proposition.

Proposition 6.9 *Let $\boldsymbol{\gamma} : S^1 \longrightarrow M$ be a regular curve on M. Suppose that $\pi \circ \boldsymbol{\gamma} : S^1 \longrightarrow \mathbb{R}^2_0$ is an embedding. Then*

$$\frac{1}{2\pi} \int_{S^1} |\widetilde{\kappa}_l^\sigma| ds = 1 \qquad (*)$$

if and only if $\boldsymbol{\gamma}$ is L-convex relative to M.

Proof. By Corollary 6.8, the condition (*) is equivalent to the condition

$$\frac{1}{2\pi} \int_{S^1} |\widetilde{\kappa}_l^+| ds + \frac{1}{2\pi} \int_{S^1} |\widetilde{\kappa}_l^-| ds = 2 \qquad (**)$$

Moreover, the condition (**) is equivalent to the following condition:

$$h_{\boldsymbol{v}}|(S_+ \cup S_-) \text{ has type } A_1 \text{ at exactly two points for each } \boldsymbol{v} \in D. \qquad (***)$$

Suppose that the condition (***) holds. If $\boldsymbol{\gamma}$ is not L-convex relative to M, then there exists $s \in S^1$ and $\boldsymbol{v} \in S^1_+$ such that one of the tangent lightlike planes at $\boldsymbol{\gamma}(s)$ separates $\boldsymbol{\gamma}(S^1)$ into two parts. Then we have $\boldsymbol{v} = \widetilde{\mathbb{L}}^\sigma(s)$. If $h_{\boldsymbol{v}}$ has type A_1 at s, it contradicts to the condition (***). If $h_{\boldsymbol{v}}$ has type $A_{\geq 2}$, under a small perturvation of $\boldsymbol{v} \in S^1_+$, there exists a point $s_0 \in S^1$ such that $h_{\boldsymbol{v}}$ has type A_1 at s_0. This also contradicts to the condition (***).

On the other hand, if the condition (***) does not hold, then there exists $\boldsymbol{v} \in S^1_+$ such that $h_{\boldsymbol{v}}$ has at least three critical points. If necessary, under a small perturvation of $\boldsymbol{v} \in S^1_+$, all critical values of $h_{\boldsymbol{v}}$ are different. It follows that there exists a critical point $s \in S^1$ of $h_{\boldsymbol{v}}$ such that neither the maximum nor the minimum point of $h_{\boldsymbol{v}}$. This means that one of the tangent lightlike planes of $\boldsymbol{\gamma}$ at s locally separates $\boldsymbol{\gamma}(S^1)$ into at least two parts. Therefore, $\boldsymbol{\gamma}$ is not L-convex relative to M. This completes the proof. □

7 Special cases

In this section we consider the case when M is a spacelike plane or the hyperbolic plane as special cases.

7.1 Curves on a spacelike plane

Suppose that $M = \mathbb{R}^2_0 = \{\boldsymbol{x} = (x_0, x_1, x_2) \in \mathbb{R}^3_1 \mid x_0 = 0\}$. We consider a plane curve $\boldsymbol{\gamma} : I \longrightarrow \mathbb{R}^2_0$. In this case we have $\boldsymbol{n}_{\boldsymbol{\gamma}} = \boldsymbol{e}_0$, $\boldsymbol{t}(s) = \boldsymbol{\gamma}'(s)$ and $\boldsymbol{b}(s) = \boldsymbol{t}(s) \wedge \boldsymbol{e}_0$. It follows that $\mathbb{L}^\pm(s) = \boldsymbol{e}_0 \pm \boldsymbol{b}(s) = \widetilde{\mathbb{L}}^\pm(s)$, $\kappa_n(s) \equiv \tau_g(s) \equiv 0$ and $\kappa_g(s) = \langle \boldsymbol{t}'(s), \boldsymbol{b}(s) \rangle = \kappa(s)$. Thus, $\kappa^\pm_l(s) = \mp\kappa(s)$. Then we have the following classical Frenet-Serret formulae on Euclidean plane:

$$\begin{cases} \boldsymbol{t}'(s) = \kappa(s)\boldsymbol{b}(s), \\ \boldsymbol{b}'(s) = -\kappa(s)\boldsymbol{t}(s). \end{cases}$$

The interesection of a lightlike plane with $\mathbb{R}^2_0$ is a line, so that an L-slice of $\mathbb{R}^2_0$ is a line. By Proposition 3.1, $\boldsymbol{\gamma}$ is an L-slice of $\mathbb{R}^2_0$ if and only if $\kappa \equiv 0$. All results in this paper correspond to classical results on plane curves.

We remark that if we consider a constant timelike unit vector $\boldsymbol{v}$ and a real number c, then we have a spacelike plane $P(\boldsymbol{v}, c)$. For a curve $\boldsymbol{\gamma}$ on $P(\boldsymbol{v}, c)$, we have $\boldsymbol{n}_{\boldsymbol{\gamma}}(s) = \boldsymbol{v}$. Then all the results for curves on $\mathbb{R}^2_0$ hold for curves on $P(\boldsymbol{v}, c)$.

7.2 Curves on the hyperbolic plane $H^2_+(-1)$

Suppose that $M = H^2_+(-1)$. In this case, by definition, an L-slice of $H^2_+(-1)$ is $P(\boldsymbol{v}, c) \cap H^2_+(-1)$ for some $\boldsymbol{v} \in S^1_+, c \in \mathbb{R}$ which is known to be a *horocycle* of the hyperbolic plane. We have $\boldsymbol{n}_{\boldsymbol{\gamma}}(s) = \boldsymbol{\gamma}(s)$, $\boldsymbol{t}(s) = \boldsymbol{\gamma}'(s)$ with $\|\boldsymbol{t}(s)\| = 1$ and $\boldsymbol{b}(s) = \boldsymbol{t}(s) \wedge \boldsymbol{n}_{\boldsymbol{\gamma}}(s)$. We call $\{\boldsymbol{t}, \boldsymbol{\gamma}, \boldsymbol{b}\}$ the *Lorentzian Saban frame*. Therefore, we have $\kappa_n(s) \equiv 1, \tau_g(s) \equiv 0$. By the Frenet-Serret type formulae in §3, we have the following[3]:

$$\begin{cases} \boldsymbol{t}'(s) = \boldsymbol{\gamma} + \kappa_g(s)\boldsymbol{b}(s), \\ \boldsymbol{\gamma}'(s) = \boldsymbol{t}(s) \\ \boldsymbol{b}'(s) = -\kappa_g(s)\boldsymbol{t}(s), \end{cases}$$

Here, by Proposition 3.1, $\boldsymbol{\gamma}$ is a *horocycle* if and only if $\kappa_l^\sigma = \kappa_n \mp \kappa_g = 0$ that is $\kappa_g(s) \equiv \pm 1$. Moreover, we have $\widetilde{\kappa}_l^\pm(s) = \sigma(\widetilde{1 \mp \kappa_g(s)})$. We now denote that $\widetilde{\kappa}_h^\pm(s) = \widetilde{1 \mp \kappa_g(s)}$ which are called *horocyclic curvatures* of $\boldsymbol{\gamma}$. By Theorem 5.4, $\boldsymbol{\gamma}$ and the tangent horocycle $TH(H_+^2(-1), \boldsymbol{\gamma})_{s_0}^\sigma$ have a contact of order one at point s_0 if and only if $\widetilde{\kappa}_h^\pm(s_0) \neq 0$ and a contact of order two at point s_0 if and only if $\widetilde{\kappa}_h^\pm(s_0) = 0$ and $(\widetilde{\kappa}_h^\pm)'(s_0) \neq 0$. Moreover, by Theorem 6.5, we have

$$W(\pi \circ \boldsymbol{\gamma}) = \frac{1}{2\pi}\int_{S^1} \widetilde{\kappa}_h^\pm ds = \frac{1}{2\pi}\int_{S^1} \sigma(\widetilde{1 \mp \kappa_g(s)})ds.$$

We also have

$$\frac{1}{2\pi}\int_{S^1} |\widetilde{\kappa}_h^\pm| ds = \frac{1}{2\pi}\int_{S^1} |\widetilde{1 \mp \kappa_g(s)}| ds \geq \max\left(|W(\pi \circ \boldsymbol{\gamma})|, 1\right).$$

by Theorem 6.7. By Proposition 6.9, $\frac{1}{2\pi}\int_{S^1}|\widetilde{\kappa}_h^\pm|ds = 1$ if and only if $\boldsymbol{\gamma}$ is L-convex relative to M. It means that $\boldsymbol{\gamma}$ is inside of one of the two horocycles at each point of $\boldsymbol{\gamma}(s)$ which are the intersections of tangent lightlike planes of $\boldsymbol{\gamma}$ and $H_+^2(-1)$, where they are support planes of $\boldsymbol{\gamma}$.

References

[1] J. W. Bruce and P. J. Giblin, *Curves and singularities* (*second edition*). Cambridge University press (1992)

[2] S. Izumiya, M. Kikuchi and M. Takahashi, *Global Properties of spacelike curves in Minkowski 3-space.* Journal of Knot Theory and Its Ramifications, **15**, No.7(2006) 869-881

[3] S. Izumiya, D.-H, Pei, T. Sano and E. Torii, *Evoutes of Hyperbolic Plane Curves.* Acta Math. Sinica, English Series, **20**, No. 3 (2004) 543–550

[4] S. Izumiya, M. C. Romero Fuster, *the lightlike flat geometry in spacelike submanifolds of codimension two in Minkowski space.* Sel. math.,New ser., **13** (2007),23-55

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