ABOUT THE LOCAL AND GLOBAL WELLPOSEDENSS OF THE NAVIER-STOKES-MAXWELL COUPLED SYSTEM

SLIM IBRAHIM

Department of Mathematics and Statistics, University of Victoria
PO Box 3060 STN CSC, Victoria, BC, Canada, V8W 3R4

1. Introduction

This is the summary of joint works with S. Keraani (University of Lille 1, France) [8], and more recently with N. Masmoudi (Courant Institute, NYU, USA) [9].

We investigate the wellposedness of solutions of a full Magneto-Hydro-Dynamic system (MHD) in the space dimension two and three. The full MHD system is a coupling of a forced Navier-Stokes equations with Maxwell equations. It reads as follows

$$(1.1)\qquad \begin{cases} \partial_t v + v\cdot\nabla v - \nu\Delta v + \nabla p = j\times B \\ \partial_t E - \nabla\times B = -j \\ \partial_t B + \nabla\times E = 0 \\ \mathrm{div} v = \mathrm{div} B = 0 \\ \sigma(E + v\times B) = j \end{cases}$$

with the initial data

$$v|_{t=0} = v_0, \quad B|_{t=0} = B_0,\ E|_{t=0} = E_0.$$

Here, $v, E, B : \mathbb{R}_t^+ \times \mathbb{R}_x^d \longrightarrow \mathbb{R}^3$ are vector fields defined on $\mathbb{R}^d$ ($d = 2$ or 3). The vector field $v = (v_1, ..., v_d)$ represents the velocity of the fluid, and the positive constants ν and σ are its viscosity and resistivity, respectively. The scalar function p stands for the pressure. The vector fields E and B are the electric and magnetic fields of the fluid, respectively. The last equation in the system expresses Ohm's law for the electric current j. The force term $j\times B$ in the Navier-Stokes equations comes from Lorentz force under a quasi-neutrality assumption of the net charge carried by the fluid. Note that the pressure p can be recovered from v and $j\times B$ via an explicit Caldéron-Zygmund type operator (for example, see [4]). The second equation in (1.1) is the Ampère-Maxwell equation for an electric field E. The third equation is nothing but Farady's law. For a detailed introduction to the MHD, we refer to Davidson [6] and Biskamp [1].

Our main goal is to solve the system of equation (1.1). Before going any further, we first recall a few fundamental known results for the standard Navier-Stokes equations.

The incompressible Navier-Stokes equations are

$$(1.2)\qquad \frac{\partial v}{\partial t} + v\cdot\nabla v - \nu\Delta v + \nabla p = 0, \quad \nabla\cdot v = 0.$$

From the one hand, multiplying (1.2) by v and integrating in space formally gives the energy identity

$$(1.3)\qquad \frac{1}{2}\frac{d}{dt}\|v\|_{L^2}^2 + \nu\|\nabla v\|_{L^2}^2 = 0,$$

which shows that the viscosity dissipates the energy. Given an L^2 initial data, J. Leray [11] constructed a global weak solution $v \in L^\infty((0,\infty), L^2(\mathbb{R}^d)) \cap L^2((0,\infty), \dot H^1(\mathbb{R}^d))$ satisfying the energy inequality.

On the other hand, applying Leray's projection $\mathcal{P}$ to (1.2) the pressure disappears and the solution of (1.2) can be written in the integral form (mild solution)

$$v(t) = e^{\nu t\Delta} v_0 - \int_0^t e^{\nu(t-t')\Delta} \mathcal{P}\nabla(v \otimes v)(t')\, dt'.$$

With $\dot{H}^{\frac{d}{2}-1}$ initial data, Fujita and Kato [7] constructed a unique (in $\mathcal{C}_t(\dot{H}^{\frac{d}{2}-1}) \cap L^2_t(\dot{H}^{\frac{d}{2}})$) local mild solution which is global if the data is small. Moreover, $u \in L^4_t(\dot{H}^{\frac{d-1}{2}})$. When the space dimension is two, the two above solutions coincide and therefore we have both uniqueness and regularity of the solution. However, when the space dimension is three, the questions of the uniqueness of Leray's solutions and the global regularity of mild solutions remain outstanding open problems in contemporary Mathematics.

Similarly, for the full MHD system (1.1), one can formally get the following energy identity

$$\frac{1}{2}\frac{d}{dt}\left[\|v\|^2_{L^2} + \|B\|^2_{L^2} + \|E\|^2_{L^2}\right] + \|j\|^2_{L^2} + \nu\|\nabla v\|^2_{L^2} = 0.$$

showing that both the viscosity and the resistivity effects dissipate the energy. Therefore, one may wonder to extend Leray's result of the existence of global weak solutions to (1.1). Unfortunately, a such result remain an interesting open problem in both space dimension two and three. Indeed, one cannot have compactness for the term $E \times B$ in the Lorentz force. Recently, Masmoudi [12] constructed, in dimension two, a unique global strong solutions to (1.1) starting from initial data $(v^0, E^0, B^0) \in L^2(\mathbb{R}^2) \times \big(H^s(\mathbb{R}^2)\big)^2$ with $s > 0$. This extra regularity on the electromagnetic field was needed to get an $L^1_t(L^\infty_x)$ bound of the velocity field through a standard bi-dimensional logarithmic estimate. One of our main goals in this work is to reduce as much as possible the regularity required on the electromagnetic field.

Notice that in dimension two, if the electromagnetic field is just an L^2 function, then the term $E \times B$ in the Lorentz force is just integrable, and therefore one cannot gain regularity using the parabolic regularization of the Stokes operator. In addition, to define (in the distributional sense) the trilinear term $(v \times B) \times B$, the vector field should (heuristically) be a bounded function. Our first result then requires less regularity on the electromagnetic field than Masmoudi's one. The regularity we impose is the minimal that enables us to have parabolic regularization. However, our condition on the velocity vector field is a little bit more restrictive in order for us to define the trilinear term. To be more precise, let us first set the functional spaces we work in.

Definition 1.1.
- *For $s \in \mathbb{R}$, define the space $\dot{H}^s_{\log}$* as the set of tempered distributions that satisfy

$$\|\psi\|^2_{\dot{H}^s_{\log}} := \sum_{q\le 0} 2^{2qs}\|\Delta_q\psi\|^2_{L^2} + \sum_{q>0} q 2^{2qs}\|\Delta_q\psi\|^2_{L^2} < \infty.$$

  Here, Δ_q stands for the dyadic localization operator in the frequency space.
- Recall the Besov space $\dot{B}^s_{p,q}$ defined by

$$\|u\|^q_{\dot{B}^s_{p,q}} = \Sigma_{j\in\mathbb{Z}} 2^{qjs}\|\Delta_j u\|^q_{L^p}$$

- For every $r \in [1, +\infty]$ the space $\tilde{L}^r_T L^2_{\log}$ is endowed with the norm

$$\|\phi\|^2_{\tilde{L}^r_T L^2_{\log}} := \sum_{q\le 0} \|\Delta_q\phi\|^2_{L^r_T L^2} + \sum_{q>0} q\|\Delta_q\phi\|^2_{L^r_T L^2}.$$

Our main Theorem is

Theorem 1.2. *Let $d = 2$, and set*

$$\mathcal{X}_2 := \dot{B}^0_{2,1}(\mathbb{R}^2) \times L^2_{\log}(\mathbb{R}^2) \times L^2_{\log}(\mathbb{R}^2).$$

There exists a small constant $\delta > 0$ such that to any initial data $(v^0, E^0, B^0) \in \mathcal{X}_2$ satisfying

$$\|(v^0, E^0, B^0)\|_{\mathcal{X}_2} \le \delta,$$

corresponds a unique global solution (v, E, B) of (1.1)

$$v \in \mathcal{C}(\mathbb{R}^+; \dot{B}^0_{2,1}) \cap \tilde{L}^2(\mathbb{R}^+; \dot{B}^1_{2,1}), \quad E,\ B \in \tilde{L}^\infty(\mathbb{R}^+; L^2_{\log}).$$

The proof of the above result goes by compactness. It basically relies on two major steps. The first one is a microlocal refinement of the parabolic regularization given by the following Lemma

Lemma 1.3. *Let u be a smooth divergence free vectors fields solving*

$$\partial_t u - \Delta u + \nabla p = F \times B, \qquad u_{|t=0} = 0,$$

on some time interval $[0, T]$. Then,

$$\|u\|_{\tilde{L}^2_T \dot{B}^1_{2,1}} \lesssim \|F\|_{L^2_T L^2_{\log}} \Big(\|B\|_{\tilde{L}^\infty_T L^2_{\log}} + \|B\|_{L^2_T L^2_H} \Big). \tag{1.4}$$

The norm in L^2_H is given by

$$\|\psi\|^2_{L^2_H} := \sum_{q \le 0} 2^{2q} \|\Delta_q \psi\|^2_{L^2} + \sum_{q \ge 0} \|\Delta_q \psi\|^2_{L^2} < \infty.$$

To use this Lemma, one has to prove a decay (in time) of the magnetic field. This is given by the following result

Lemma 1.4. *Let (E, B) be a smooth solution of*

$$\begin{aligned} \partial_t E - curl\, B + E &= f \times g, \\ \partial_t B + curl\, E &= 0 \end{aligned}$$

on some interval $[0, T]$. Then, we have

$$\|E\|_{\tilde{L}^\infty_T L^2_{\log}} + \|E\|_{L^2_T L^2_{\log}} + \|B\|_{\tilde{L}^\infty_T L^2_{\log}} \lesssim \|(E_0, B_0)\|_{L^2_{\log}} + \|f \times g\|_{L^2_T L^2_{\log}},$$

and

(1.5)

$$\|B\|_{L^2_T L^2_H} \lesssim \|(E_0, B_0)\|_{L^2} + \|g\|_{L^\infty_T L^2} \|f\|_{\tilde{L}^2_T \dot{B}^1_{2,1}} + \Big(\|\nabla f\|_{L^2_T L^2} + \|f\|_{L^\infty_T L^2} \Big) \|g\|_{L^2_T L^2_H}.$$

Observe that the factor in front of $\|g\|_{L^2_T L^2_H}$ in (1.5) can be made arbitrary small when the data is small. Thus, the term in (1.5) containing $\|g\|_{L^2_T L^2_H}$ can be absorbed in the left hand side giving an *à priori* bound on $\|g\|_{L^2_T L^2_H}$.
The proof of this lemma uses the representation of solutions of the damped wave equation satisfied by B. For high frequencies, such an equation behaves like the damped wave and therefore solutions decay exponentially in time. However, for low frequencies, it behaves like the heat equation, and the decay rate is rather weak.

In the three dimensional case, we construct mild solution using a fixed point argument based on the following nonlinear estimates

Lemma 1.5. *There exists a constant $C > 0$ such that*

$$\begin{aligned} \|FB\|_{\tilde{L}^2_T \dot{B}^{-\frac{1}{2}}_{2,1}} &\le C \|F\|_{L^2 \dot{H}^{\frac{1}{2}}} \|B\|_{\tilde{L}^\infty_T \dot{H}^{\frac{1}{2}}}, \\ \|vB\|_{L^2_T \dot{H}^{\frac{1}{2}}} &\le C \|v\|_{\tilde{L}^2_T \dot{B}^{\frac{3}{2}}_{2,1}} \|B\|_{\tilde{L}^\infty_T \dot{H}^{\frac{1}{2}}}, \\ \|uv\|_{L^1_T \dot{B}^{\frac{3}{2}}_{2,1}} &\le C \|u\|_{\tilde{L}^2_T \dot{B}^{\frac{3}{2}}_{2,1}} \|v\|_{\tilde{L}^2_T \dot{B}^{\frac{3}{2}}_{2,1}}, \end{aligned}$$

for all smooth functions F, B, v and u defined on some interval $[0, T]$.

With a more careful study of the full MHD system, we recently obtained with N. Masmoudi [9] the following result about the local and global wellposedness of (1.1).

Theorem 1.6. *Let $d = 2, 3$, and set*

$$\mathcal{X}_d := \dot{H}^{\frac{d}{2}-1} \times \dot{H}_{\log}^{\frac{d}{2}-1}(\mathbb{R}^d) \times \dot{H}_{\log}^{\frac{d}{2}-1}(\mathbb{R}^d).$$

For any $\Gamma^0 := (v^0, E^0, B^0) \in \mathcal{X}_d$, there exists $T > 0$ and a unique mild solution Γ of (1.1) with initial data Γ^0 and

$$v \in \mathcal{C}((0,T); \dot{H}^{\frac{d}{2}-1}) \cap \tilde{L}^2((0,T); \dot{H}^{\frac{d}{2}}) + \tilde{L}^2((0,T); \dot{B}^{\frac{d}{2}}_{2,1}), \quad E,\ B \in \tilde{L}^\infty(\mathbb{R}^+; \dot{H}_{\log}^{\frac{d}{2}-1}).$$

Moreover, the solution is global (i.e. $T = \infty$) if the initial data is sufficiently small.

References

[1] D. Biskamp. *Nonlinear magnetohydrodynamics*, volume 1 of *Cambridge Monographs on Plasma Physics*. Cambridge University Press, Cambridge, 1993.
[2] J.-M. Bony. Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires. *Ann. Sci. École Norm. Sup. (4)*, 14(2):209–246, 1981.
[3] M. Cannone, Q. Chen, and C. Miao. A losing estimate for the ideal MHD equations with application to blow-up criterion. *SIAM J. Math. Anal.*, 38(6):1847–1859 (electronic), 2007.
[4] J. -Y. Chemin, *Perfect incompressible Fluids*, Clarendon press, Oxford, 1998.
[5] J.-Y. Chemin and N. Lerner. Flot de champs de vecteurs non lipschitziens et équations de Navier-Stokes. *J. Differential Equations*, 121(2):314–328, 1995.
[6] P. A. Davidson. *An introduction to magnetohydrodynamics*. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 2001.
[7] H. Fujita and T. Kato. On the Navier-Stokes initial value problem I, *Archiv for Rationnal Mechanic Analysis*, 16, (1964), pages 269315.
[8] S. Ibrahim and S. Keraani Global small solutions of the Navier-Stokes Maxwell system, to appear in *SIAM Journal on Math. Analysis.*
[9] S. Ibrahim and N. Masmoudi On the wellposedness of the Navier-Stokes Maxwell system, preprint *SIAM Journal on Math. Analysis.*
[10] P. G. Lemarié-Rieusset. Recent developments in the Navier-stokes problem. *CRC Press, (2002).*
[11] J. Leray. Essai sur le mouvement dun liquide visqueux emplissant lespace, *Acta Mathematica*, 63, (1933), pages 193248.
[12] N. Masmoudi. Global well posedness for the Maxwell-Navier-Stokes system in $2D$. *Journal de Math. Pures et Appl.* 93 (2010) 559571.

E-mail address: S. Ibrahim: ibrahim@math.uvic.ca
URL: http://www.math.uvic.ca/~ibrahim/