

Analyticity of the Stokes semigroup in spaces of bounded functions

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1 Introduction

1.1 Analyticity of the Stokes semigroup

We consider the initial-boundary problem for the Stokes equations

$$v_t - \Delta v + \nabla q = 0 \quad \text{in } \Omega \times (0, T) \quad (1.1)$$

$$\operatorname{div} v = 0 \quad \text{in } \Omega \times (0, T) \quad (1.2)$$

$$v = 0 \quad \text{on } \partial\Omega \times (0, T) \quad (1.3)$$

$$v(x, 0) = v_0 \quad \text{on } \Omega \times \{t = 0\} \quad (1.4)$$

in a domain Ω in \mathbf{R}^n with $n \geq 2$. It is well-known that the solution operator $S(t) : v_0 \mapsto v(t) = v(\cdot, t)$ forms an analytic semigroup in the solenoidal L^r space, $L^r_\sigma(\Omega)$ for $r \in (1, \infty)$ for various kind of domains Ω including a smoothly bounded domain [52], [26]. However, it has been a long-standing open problem whether or not the Stokes semigroup $\{S(t)\}_{t \geq 0}$ is analytic in L^∞ -type space even if Ω is bounded. When Ω is a half space it is known that the Stokes semigroup $\{S(t)\}_{t \geq 0}$ is analytic in L^∞ -type space since explicit solution formulas are available [12], [42], [56].

The goal of this paper is to give an affirmative answer to this open problem at least when Ω is bounded as a typical example. For a precise statement let $C_{0,\sigma}(\Omega)$ denote the L^∞ -closure of $C_{c,\sigma}^\infty(\Omega)$, the space of all smooth solenoidal vector fields with compact support in Ω . When Ω is bounded, $C_{0,\sigma}(\Omega)$ agrees with the space of all solenoidal vector fields continuous in $\bar{\Omega}$ vanishing on $\partial\Omega$ [41]. One of our main results is

Theorem 1.1 (Analyticity in $C_{0,\sigma}$). *Let Ω be a bounded domain in \mathbf{R}^n with C^3 boundary. Then the solution operator (the Stokes semigroup) $S(t) : v_0 \mapsto v(t)$ ($t \geq 0$) is a C_0 -analytic semigroup in $C_{0,\sigma}(\Omega)$.*

For the Laplace operator or general elliptic operators it is well-known that the corresponding semigroup is analytic in L^∞ -type space. The first pioneering work goes back to K. Yosida [64] for second order operators in \mathbf{R} . Unfortunately, it seems difficult to extend his method to multi-dimensional elliptic operators. K. Masuda [43], [44] (see [45]) first proved the analyticity of the semigroup generated by a general elliptic operator (including higher order operators) in $C_0(\mathbf{R}^n)$, the space of continuous functions vanishing at the space infinity. A key idea is to derive a corresponding resolvent estimate by a localization method together with L^p -estimates and interpolation inequalities. It is extended by H. B. Stewart for Dirichlet problems [59] and for more general boundary conditions [60]. (A complete proof is given by [4, Appendix].) The reader is referred to a book by A. Lunardi [40, Chapter 3] for this Masuda-Stewart method which applies to many other situations. By now, analyticity results in L^∞ spaces are established in various settings [4], [6], [61], [35], [40]. However, it seems that their localization argument does not apply to the Stokes equations and this may be a reason why this problem had been open for a long time.

1.2 Blow-up arguments

Our approach to prove the analyticity is completely different from conventional approaches. We appeal to a blow-up argument which is often used in a study of non-linear elliptic and parabolic equations. Let us give a heuristic idea of our argument. A key step (to prove analyticity in Theorem 1.1) is to establish a bound for

$$N(v, q)(x, t) = |v(x, t)| + t^{\frac{1}{2}} |\nabla v(x, t)| + t |\nabla^2 v(x, t)| + t |\partial_t v(x, t)| + t |\nabla q(x, t)| \quad (1.5)$$

of the form

$$\sup_{0 < t < T_0} \|N(v, q)\|_\infty(t) \leq C \|v_0\|_\infty \quad (1.6)$$

for some $T_0 > 0$ and C depending only on the domain Ω , where $\|v_0\|_\infty = \|v_0\|_{L^\infty(\Omega)}$ denotes the sup-norm of $|v_0|$ in Ω .

We argue by contradiction. Suppose that (1.6) were false for any choice of T_0 and C . Then there would exist a sequence $\{(v_m, q_m)\}_{m=1}^\infty$ of solutions of (1.1)-(1.4) with $v_0 = v_{0m}$ and a sequence $\tau_m \downarrow 0$ such that $\|N(v_m, q_m)\|_\infty(\tau_m) > m \|v_{0m}\|_\infty$. There is $t_m \in (0, \tau_m)$ such that

$$\|N(v_m, q_m)\|_\infty(t_m) \geq \frac{1}{2} M_m, \quad M_m = \sup_{0 < t < \tau_m} \|N(v_m, q_m)\|_\infty(t).$$

We normalize v_m, q_m by dividing M_m to observe that

$$\sup_{0 < t < t_m} \|N(\tilde{v}_m, \tilde{q}_m)\|_\infty(t) \leq 1, \quad (1.7)$$

$$\|N(\tilde{v}_m, \tilde{q}_m)\|_\infty(t_m) \geq 1/2, \quad (1.8)$$

$$\|\tilde{v}_{0m}\|_\infty < 1/m \quad (1.9)$$

with $\tilde{v}_m = v_m/M_m$, $\tilde{q}_m = q_m/M_m$. We rescale $(\tilde{v}_m, \tilde{q}_m)$ around a point $x_m \in \Omega$ satisfying

$$|N(\tilde{v}_m, \tilde{q}_m)(x_m, t_m)| \geq 1/4 \quad (1.10)$$

to get a blow-up sequence of (v_m, q_m) of the form

$$u_m(x, t) = \tilde{v}_m(x_m + t_m^{\frac{1}{2}}x, t_m t), p_m(x, t) = t_m^{\frac{1}{2}}\tilde{q}_m(x_m + t_m^{\frac{1}{2}}x, t_m t).$$

(Such an x_m exists because of (1.8).) Because of the scaling invariance of the equations (1.1) and (1.2), the rescaled function (u_m, p_m) solves (1.1) and (1.2) in a rescaled domain $\Omega_m \times (0, 1)$. Note that the time interval is normalized to a unit interval and Ω_m tends to either a half space or the whole space \mathbf{R}^n as $m \rightarrow \infty$.

The basic strategy is to prove that the blow-up sequence $\{(u_m, p_m)\}_{m=1}^{\infty}$ (subsequently) converges to a solution (u, p) of (1.1)-(1.4) with zero initial data. If the convergence is strong enough, (1.10) implies that $N(u, p)(0, 0) \geq 1/4$. If the limit (u, p) is unique, it is natural to expect $u \equiv 0$, $\nabla p \equiv 0$. This evidently yields a contradiction to $N(u, p)(0, 0) \geq 1/4$. The first part corresponds to "compactness" of a blow-up sequence and the second part corresponds to "uniqueness" of a blow-up limit. (Similar rescaling argument is explained in detail in a recent textbook [25].) When the problem is the heat equation, this strategy is easy to realize. However, for the Stokes equations it turns out that this procedure is highly nontrivial because of the presence of the pressure.

A blow-up argument was first introduced by E. De Giorgi [11] to study regularity of a minimal surface. B. Gidas and J. Spruck [23] adjusted a blow-up argument to derive a priori bound for solutions of a semilinear elliptic problem. It seems that the first application to (semilinear) parabolic problems to get a priori bound goes back to [27] (see also [30]). The method has been further developed in recent years to obtain several a priori bounds; see e.g. [48] and [47]. However, it is quite recent to apply to the Navier-Stokes equations. For example, a blow-up argument was used to conclude non-existence of type I blow-up for axisymmetric solutions [36], [49] and solutions having continuously varying vorticity directions [33].

1.3 Pressure gradient estimates and admissible domains

To derive both compactness of the blow-up sequence $\{(u_m, p_m)\}_{m=1}^{\infty}$ and uniqueness of its limit we invoke the fact that the pressure is determined by the velocity through the Helmholtz decomposition. Take the divergence of (1.1) to observe that q is harmonic in Ω (for each time). If one takes the normal component of (1.1), it turns out that q solves the Neumann problem

$$-\Delta q = 0 \quad \text{in } \Omega, \quad \partial q / \partial n_{\Omega} = \Delta v \cdot n_{\Omega} \quad \text{on } \partial\Omega \quad (1.11)$$

where n_Ω is the outward unit normal vector field of $\partial\Omega$. The correspondence $\Delta v \mapsto \nabla q$ is nothing but the projection operator $Q = I - P$ where P is the Helmholtz projection at least formally.

A key observation is that this harmonic pressure gradient is estimated by the velocity gradient of the form

$$\sup_{x \in \Omega} d_\Omega(x) |\nabla q(x, t)| \leq C \|\nabla v\|_{L^\infty(\partial\Omega)}(t) \quad (1.12)$$

with C depending only on Ω at least when Ω is bounded. Here $d_\Omega(x)$ denotes the distance from $\partial\Omega$ to $x \in \Omega$. This follows from a following regularizing type estimate for the Neumann problem (1.11) which depends only on Ω :

$$\sup_{x \in \Omega} d_\Omega(x) |Q[\nabla \cdot f]| \leq C \|f\|_{L^\infty(\partial\Omega)} \quad (1.13)$$

for all matrix-valued function $f \in C^1(\bar{\Omega})$ satisfying $\nabla \cdot f = (\sum_{j=1}^n \partial_j f_{ij}) \in L^2 \cap L^r(\Omega)$ for some $r \geq n$ such that

$$\text{tr } f = 0 \quad \text{and} \quad \partial_l f_{ij} = \partial_j f_{il}, \quad (1.14)$$

where $\text{tr } f = \sum_{i=1}^n f_{ii}$ and $\partial_l = \partial/\partial x_l$. If (1.13) is valid, then (1.12) follows by setting $f_{ij} = \partial_j v^i$ in (1.13). Since (1.13) may not be true for a general domain, we say that Ω is *admissible* if (1.13) holds for f satisfying (1.14). It is easy to prove that a half space \mathbf{R}_+^n is admissible by using an explicit representation formula of the solution of (1.11); see Remark 2.4 (iv). In this paper we shall prove that a bounded C^3 -domain is admissible by a blow-up argument as explained later in the introduction.

1.4 Compactness and uniqueness

We now study compactness of the blow-up sequence $\{(u_m, p_m)\}_{m=1}^\infty$. The situation is divided into two cases depending on whether the limit of Ω_m is a half space or the whole space \mathbf{R}^n . Let us consider the case when the limit is the whole space \mathbf{R}^n because it is easier than the half space case. We would like to prove that $N(u_m, p_m)$ converges to $N(u, p)$ near $(0, 1) \in \mathbf{R}^n \times (0, 1]$ uniformly by taking a subsequence. For this purpose it is enough to prove that the local space-time Hölder norm in $\mathbf{R}^n \times (0, 1]$ near $(0, 1)$ for $u_m, \nabla u_m, \nabla^2 u_m, \nabla p_m$ is bounded as $m \rightarrow \infty$. We are tempted to derive such an interior regularity estimate from (1.7) by localizing the problem. This idea works for the heat equation but for the Stokes equations it does not work (Remark 3.3(i)). In fact, if we consider a solution of (1.1)-(1.2) of the form $v = g(t), q = -g'(t) \cdot x$ for $g \in C^1[0, 1]$, we do not expect the (local) Hölder continuity in time for ∇q and v_t although $N(v, q)$ is bounded in $\mathbf{R}^n \times (0, 1]$. We invoke the admissibility of Ω and derive a uniform time Hölder estimate for $d_{\Omega_m}(x) \nabla p_m$ in $\Omega_m \times (\delta, 1]$ ($\delta > 0$) from (1.12). Then one can use usual parabolic interior regularity

theory [39] to derive necessary interior regularity estimate. Note that the constant in (1.12) is independent of the rescaling procedure so our Hölder estimate is uniform in m .

The case when Ω_m tends to a half space is more involved. We still use the admissibility of Ω to derive necessary Hölder estimates for p_m . Then instead of using conventional parabolic local Hölder estimate, we are forced to use Schauder estimates for the Stokes equations and Helmholtz decomposition for Hölder spaces developed by V. A. Solonnikov [58] since the boundary value problem for the Stokes equations cannot be reduced to usual parabolic theory [39].

We also invoke admissibility of Ω to derive uniqueness of the blow-up limit (u, p) . If Ω_m tends to the whole space, by (1.12) we observe that ∇p_m tends to zero locally uniformly in $\mathbf{R}^n \times (0, 1]$. This reduces the problem to the uniqueness result for the heat equation. If Ω_m tends to a half space, we use a uniqueness result for spatially non-decaying velocity in the half space $\mathbf{R}_+^n = \{(x', x_n) \mid x_n > 0, x' \in \mathbf{R}^{n-1}\}$ which is essentially due to V. A. Solonnikov [56]. Note that to assert the uniqueness of solutions (u, p) of the Stokes equations (1.1)-(1.4) with zero initial data and a bound for $\|N(u, p)\|_\infty(t)$, we need to assume some decay for ∇p , otherwise there is a counterexample (Remark 4.2). In fact, it suffices to assume that $\nabla p \rightarrow 0$ for $x_n \rightarrow \infty$. In our setting since (1.12) is a scaling invariant, this estimate is inherited to (u_m, p_m) . Since $x_n = d_{\mathbf{R}_+^n}(x)$, we are able to conclude that $t^{\frac{1}{2}}x_n|\nabla p(x, t)|$ is bounded in $\mathbf{R}_+^n \times (0, 1)$, which is enough to apply this available uniqueness result. Note that in the above uniqueness result we do not assume any spatial decay condition for velocity fields at the space infinity.

1.5 A priori L^∞ estimates for \tilde{L}^r -solutions

As we have seen above a blow-up argument yields a key estimate (1.6) for a solution of the Stokes equations (1.1)-(1.4) provided that $\|N(v, q)\|_\infty(t)$ (see (1.5)) is finite for $t > 0$ as far as Ω is admissible not necessarily bounded. A question is whether or not such a solution actually exists. It is by now well-known [22] that if a uniformly C^3 -domain admits the Helmholtz decomposition in L^r , there exists an L^r -solution and the Stokes semigroup $S(t)$ is analytic in L_σ^r , the closure of $C_{c,\sigma}^\infty(\Omega)$ in L^r . However, in general, it is also known that the Helmholtz decomposition in L^r space may not hold (see [9], [46]), unless $r = 2$. Fortunately, R. Farwig, H. Kozono and H. Sohr [14], [15], [16] established an \tilde{L}^r -theory with $\tilde{L}_\sigma^r = L_\sigma^r \cap L_\sigma^2$ for $r \geq 2$ for any uniformly C^2 -domain for (1.1)-(1.4); in particular, they showed that the Stokes semigroup is analytic in \tilde{L}_σ^r space. For an application to the Navier-Stokes equations see [19]. It turns out that their solution (called an \tilde{L}^r -solution) has a property

$$\sup_{0 < t < T} \|N(v, q)\|_\infty(t) < \infty$$

provided that $r > n$ and v_0 is sufficiently regular. So one can claim a priori L^∞ -estimates (1.6) for an \tilde{L}^r -solution which is very useful to study a domain not necessarily bounded. Here is our main result.

Theorem 1.2 (A priori L^∞ -estimates). *Let Ω be an admissible, uniformly C^3 -domain in \mathbf{R}^n with $r > n$. Then there exists positive constants C and T_0 depending only on Ω such that (1.6), i.e.*

$$\sup_{0 < t < T_0} \|N(v, q)\|_\infty(t) \leq C \|v_0\|_\infty \quad (1.15)$$

holds for all \tilde{L}^r -solution (v, q) of (1.1)-(1.4) with $v_0 \in C_{c,\sigma}^\infty(\Omega)$.

1.6 General analyticity result

By a density argument with (1.15) we are able to construct a solution semigroup $S(t)$ for (1.1)-(1.4) in $C_{0,\sigma}(\Omega)$. In particular, the estimate

$$\sup_{0 < t < T_0} t \|v_t\|_\infty(t) \leq C \|v_0\|_\infty$$

from (1.15) shows that this semigroup is analytic in $C_{0,\sigma}(\Omega)$. Let us give a precise form of our result which includes Theorem 1.1 as a particular example.

Theorem 1.3 (Analyticity for a general domain). *Let Ω be an admissible, uniformly C^3 -domain in \mathbf{R}^n . Then the Stokes semigroup $S(t)$ is uniquely extendable to a C_0 -analytic semigroup in $C_{0,\sigma}(\Omega)$. Moreover, the estimate (1.15) holds with some $C > 0$ and $T_0 > 0$ for $v = S(t)v_0, v_0 \in C_{0,\sigma}(\Omega)$ with a suitable choice of pressure q .*

Although there are several results on analyticity of $S(t)$ in L_σ^r for various domains such as a half space, a bounded domain [26], [52], an exterior domain [10], [34], an aperture domain [18], a layer domain [1], a perturbed half space [17] (even for variable viscosity coefficients) [3], [2], the result corresponding to Theorem 1.3 is available only for a half space [12], [42], [56] (and the whole space, where the Stokes semigroup agrees with the heat semigroup.)

We do not touch the problem for the large time behavior of the Stokes semigroup. In particular, we do not know in general whether or not the Stokes semigroup is bounded in time. This is known for a half space [12], [42], [56]. For a bounded domain it is not difficult to derive even exponential decay as $t \rightarrow \infty$. In fact, for a bounded domain we prove that $S(t)$ is a bounded analytic semigroup in $C_{0,\sigma}$ (Remark 5.4 (i)). Moreover, the operator norm of $\|S(t)\|$ is bounded in t when Ω is bounded. Such a type of results is called a maximum modulus result and studied in the literature [63], [54], [55] (Remark 5.4 (ii)).

1.7 Admissible domains

We also use a blow-up argument to prove that a bounded C^3 -domain is indeed an admissible domain. Suppose that (1.13) does not hold for f satisfying (1.14). There would exist a sequence of functions $\{\Phi_m\}_{m=1}^\infty$ with $\nabla\Phi_m = Q[\nabla \cdot f_m]$ and a sequence of points $\{x_m\}_{m=1}^\infty \subset \Omega$ such that

$$\frac{1}{2} \leq d_\Omega(x_m) |\nabla\Phi_m(x_m)| \leq \sup_{x \in \Omega} d_\Omega(x) |\nabla\Phi_m(x)| = 1 \quad (1.16)$$

and f_m tends to zero uniformly on $\partial\Omega$. If a subsequence of $\{x_m\}_{m=1}^\infty$ converges to an interior point, the limit Φ solves the homogeneous Neumann problem (for the Laplace equation) with a bound

$$\sup_{x \in \Omega} d_\Omega(x) |\nabla\Phi(x)| < \infty. \quad (1.17)$$

So if the solution of this problem is unique (i.e. $\nabla\Phi \equiv 0$), then one gets a contradiction. Note that Φ_m is harmonic so compactness part is easy. If $\{x_m\}_{m=1}^\infty$ converges to a boundary point (by taking a subsequence), we rescale Φ_m around x_m and set

$$\Psi_m(x) = \Phi_m(x_m + d_m x) \quad \text{with} \quad d_m = d_\Omega(x_m).$$

Then the rescaled domains Ω_m expands to a half space and the limit Ψ solves the homogeneous Neumann problem in a half space with an estimate inherited by (1.16). We prove its uniqueness by reducing the problem to the whole space via a reflection argument. The compactness part is easy since the distance between the origin for Ψ_m and the boundary $\partial\Omega_m$ is always one.

It is possible to prove that an exterior domain or a perturbed half space is admissible but we do not discuss these problems in the present paper. We expect that a layer domain $\Omega = \{a < x_n < b\}$ is not admissible since the uniqueness under (1.17) is not valid. For example $\Phi(x) = x_1$ is a nontrivial solution satisfying (1.17) for the homogeneous Neumann problem in Ω . We conjecture that an unbounded domain (with smooth boundary) is admissible if and only if Ω is *not quasicylindrical* (see [5, 6.32]), i.e. $\overline{\lim}_{|x| \rightarrow \infty} d_\Omega(x) = \infty$.

1.8 Extension to L^∞_σ space

It is natural to extend the Stokes semigroup in L^∞_σ , the solenoidal L^∞ space defined by

$$L^\infty_\sigma(\Omega) = \left\{ f \in L^\infty(\Omega) \mid \int_\Omega f \cdot \nabla\varphi \, dx = 0 \quad \text{for all} \quad \varphi \in \hat{W}^{1,1}(\Omega) \right\},$$

where $\hat{W}^{1,1}(\Omega)$ is the homogeneous Sobolev space of the form

$$\hat{W}^{1,1}(\Omega) = \left\{ \varphi \in L^1_{loc}(\Omega) \mid \nabla\varphi \in L^1(\Omega) \right\}.$$

When Ω is bounded, the Stokes semigroup $S(t)$ is defined in $L^r_\sigma(\Omega)$ ($1 < r < \infty$) and $L^\infty_\sigma \subset L^r_\sigma$ one is able to extend the estimate (1.15) when initial data v_0 is merely in L^∞_σ by an approximation argument. Note that $C^\infty_{c,\sigma}$ (or $C_{0,\sigma}$) is not dense in L^∞_σ so one cannot approximate v_0 by elements of $C^\infty_{c,\sigma}$ in a uniform topology. However, by a mollifying procedure keeping the divergence free condition there is a sequence $\{v_{0m}\}_{m=1}^\infty \subset C^\infty_{c,\sigma}$ converges to v_0 a.e. and $\|v_{0m}\|_\infty \leq C\|v_0\|_\infty$ with C independent of v_0 . This is very easy to prove when Ω is star-shaped while in general it is nontrivial. We localize the problem to reduce it to star-shaped case. Since Ω is bounded, $v_{0m} \rightarrow v$ in L^r_σ so we extend the estimate (1.15) to $v = S(t)v_0$ with the associated pressure q when $v_0 \in L^\infty_\sigma$. Thus we have

Theorem 1.4 (Analyticity in L^∞_σ for a bounded domain). *Let Ω be a bounded C^3 -domain in \mathbf{R}^n . Then the Stokes semigroup $S(t)$ is a (non C_0 -) analytic semigroup in $L^\infty_\sigma(\Omega)$.*

Since smooth functions are not dense in $L^\infty_\sigma(\Omega)$ and $S(t)v_0$ is smooth for $t > 0$, $S(t)v_0 \rightarrow v_0$ as $t \downarrow 0$ in L^∞_σ does not hold for some $v_0 \in L^\infty_\sigma(\Omega)$. This means $S(t)$ is a non C_0 -semigroup.

To extend analyticity in L^∞_σ in a general admissible domain we have to construct $S(t)$ in L^∞_σ in a unique way since \tilde{L}^r_σ does not contain L^∞_σ . This attempt is so far carried out for a half space in [12], where an explicit solution formula is available. Moreover, it is also shown in [12] that $S(t)$ is a C_0 -analytic semigroup in

$$BUC_\sigma(\Omega) = \{f \in BUC(\Omega) \mid \operatorname{div} f = 0 \text{ in } \Omega, f = 0 \text{ on } \partial\Omega\},$$

when Ω is a half space; see also [56]. Here $BUC(\Omega)$ denotes the space of all bounded, uniformly continuous functions. We shall discuss these problems for a general unbounded admissible domain in forthcoming papers. (Note that $BUC_\sigma(\Omega) = C_{0,\sigma}(\Omega)$ when Ω is bounded.) The analyticity as well as (1.15) is fundamental to study the Navier-Stokes equations. So far L^∞ -type theory is only established when $\Omega = \mathbf{R}^n$ [29], [31] and \mathbf{R}^n_+ [56], [7]. We shall also discuss the nonlinear problem in forthcoming papers.

This paper is organized as follows. In Section 2 we define an admissible domain and prove that a bounded C^3 -domain is admissible by a blow-up argument. In Section 3 we derive local Hölder estimates both interior and up to boundary which are key to derive necessary compactness for a blow-up sequence. In Section 4 we review a uniqueness result for spatially non-decaying solutions for the Stokes equations as well as the heat equation. In Section 5 we prove key a priori estimates (Theorem 1.2) by a blow-up argument. As an application we prove Theorem 1.3 (and Theorem 1.1 as a particular example.) In Section 6 we prove Theorem 1.4.

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2 Admissible domains

In this section we introduce the notion of an admissible domain and prove that a bounded domain is admissible by a blow-up argument. We also give a short proof that a half space is admissible. We first recall the Helmholtz decomposition.

2.1 Helmholtz decomposition

Let Ω be an arbitrary domain in $\mathbf{R}^n (n \geq 2)$. Let $L_\sigma^r(\Omega) (1 < r < \infty)$ denote the L^r -closure of $C_{c,\sigma}^\infty(\Omega)$, the space of all smooth solenoidal vector fields with compact support in Ω . The Helmholtz decomposition is a topological direct sum decomposition of the form

$$L^r(\Omega) = L_\sigma^r(\Omega) \oplus G^r(\Omega), G^r(\Omega) = \{\nabla p \in L^r(\Omega) \mid p \in L_{loc}^r(\Omega)\}.$$

Although this decomposition is known to hold (see e.g. [20, III.1]) for various domains like a bounded or exterior domain with smooth boundary, in general there is a domain with (uniformly) smooth boundary such that the L^r -Helmholtz decomposition does not hold (cf. [9], [46]). Note that this decomposition is an orthogonal decomposition if $r = 2$ and that the case $r = 2$ is valid for any domain Ω .

In [14] Farwig, Kozono and Sohr introduced an \tilde{L}^r space and proved that Helmholtz decomposition is valid for any uniformly C^2 -domain for $n = 3$. Later, it is generalized for arbitrary uniformly C^1 -domain for $n \geq 2$ [15]. Let us recall their results. We set

$$\tilde{L}^r(\Omega) = \begin{cases} L^2(\Omega) \cap L^r(\Omega), & 2 \leq r < \infty \\ L^2(\Omega) + L^r(\Omega), & 1 < r < 2. \end{cases}$$

Note that $\tilde{L}^{r_1} \subset \tilde{L}^r$ for $r_1 > r$. We define \tilde{L}_σ^r and \tilde{G}^r in a similar way. We then recall a definition of uniformly C^k -domain for $k \geq 1$; see e.g. [51, I.3.2].

Definition 2.1 (Uniformly C^k -domain). Let Ω be a domain in \mathbf{R}^n with $n \geq 2$. Assume that there exists $\alpha, \beta, K > 0$ such that for each $x_0 \in \partial\Omega$, there exists C^k -function h of $n - 1$ variable y' such that

$$\sup_{|l| \leq k, |y'| < \alpha} |\partial_{y'}^l h(y')| \leq K, \quad \nabla' h(0) = 0, \quad h(0) = 0$$

and denote a neighborhood of x_0 by

$$U_{\alpha,\beta,h}(x_0) = \{(y', y_n) \in \mathbf{R}^n \mid h(y') - \beta < y_n < h(y') + \beta, |y'| < \alpha\}.$$

Assume that up to rotation and translation we have

$$U_{\alpha,\beta,h}(x_0) \cap \Omega = \{(y', y_n) \mid h(y') < y_n < h(y') + \beta, |y'| < \alpha\}$$

and

$$U_{\alpha,\beta,h}(x_0) \cap \partial\Omega = \{(y', y_n) \mid y_n = h(y'), |y'| < \alpha\}.$$

Then we call Ω a uniformly C^k -domain of type α, β, K . Here $\partial_x^l = \partial_{x_1}^{l_1} \cdots \partial_{x_n}^{l_n}$ with multi-index $l = (l_1, \dots, l_n)$ and $\partial_{x_j} = \partial/\partial x_j$ as usual and ∇' denotes the gradient in $y' \in \mathbf{R}^{n-1}$.

Proposition 2.2 ([14], [15]). *Let Ω be a uniformly C^1 -domain of type $\alpha, \beta, K > 0$ and $1 < r < \infty$. Then $\tilde{L}^r(\Omega)$ has a topological direct sum decomposition $\tilde{L}^r(\Omega) = \tilde{L}_\sigma^r(\Omega) \oplus \tilde{G}^r(\Omega)$. Let $P(= P_r)$ be the projection to $\tilde{L}_\sigma^r(\Omega)$ associated to this decomposition. Then there is a constant $C = C(r, \alpha, \beta, K) > 0$ such that the operator norm of P is bounded by C .*

The operator P is often called the Helmholtz projection. In this paper we shall use \tilde{L}^r space for $r \geq 2$ so \tilde{L}^r norm is given as

$$\|f\|_{\tilde{L}^r} = \max(\|f\|_{L^r}, \|f\|_{L^2}).$$

2.2 Definition of an admissible domain

We give a rigorous definition of an admissible domain. Let $d_\Omega(x)$ denote the distance function from $\partial\Omega$, i.e.,

$$d_\Omega(x) = \inf\{|x - y| \mid y \in \partial\Omega\}.$$

Let $Q_r = I - P_r$ be the projection to $\tilde{G}^r(\Omega)$ associated to the Helmholtz decomposition. We shall suppress a subscript r of Q_r .

Definition 2.3 (Admissible domain). Let Ω be a uniformly C^1 -domain in \mathbf{R}^n ($n \geq 2$) with $\partial\Omega \neq \emptyset$. We call Ω *admissible* if there exists $r \geq n$ and a constant $C = C_\Omega$ such that

$$\sup_{x \in \Omega} d_\Omega(x) |Q[\nabla \cdot f](x)| \leq C_\Omega \|f\|_{L^\infty(\partial\Omega)}$$

holds for all matrix-valued function $f = (f_{ij})_{1 \leq i, j \leq n} \in C^1(\bar{\Omega})$ which satisfies $\nabla \cdot f (= \sum_{j=1}^n \partial_j f_{ij}) \in \tilde{L}^r(\Omega)$,

$$\operatorname{tr} f = 0 \quad \text{and} \quad \partial_l f_{ij} = \partial_j f_{il} \tag{2.1}$$

for all $i, j, l \in \{1, \dots, n\}$.

Remark 2.4. (i) We note that $\nabla q = Q[\nabla \cdot f]$ is formally obtained by solving the Neumann problem

$$\begin{cases} -\Delta q = \operatorname{div}(\nabla \cdot f) & \text{in } \Omega, \\ \partial q / \partial n_\Omega = n_\Omega \cdot (\nabla \cdot f) & \text{on } \partial\Omega, \end{cases}$$

where n_Ω is the exterior unit normal of $\partial\Omega$. In particular q (and also ∇q) is harmonic in Ω since

$$\operatorname{div}(\nabla \cdot f) = \sum_{1 \leq i, j \leq n} \partial_i \partial_j f_{ij} = \sum_{1 \leq i, j \leq n} \partial_j \partial_i f_{ij} = 0$$

(ii) The left hand side of the inequality in Definition 2.3 is always finite. Indeed, since ∇q is harmonic, the mean value theorem (see e.g. [13, 2.2.2]) implies that

$$\nabla q(x) = \frac{1}{|B_\rho(x)|} \int_{B_\rho(x)} \nabla q(y) dy \quad \text{for } \rho < d_\Omega(x),$$

where $B_\rho(x)$ is the closed ball of radius ρ centered at x and $|B_\rho(x)|$ denotes its volume. Applying the Hölder inequality yields

$$\begin{aligned} |\nabla q(x)| &\leq |B_\rho(x)|^{-1/p} \|\nabla q\|_p, \quad 1/p + 1/p' = 1, \\ &\leq C\rho^{-n/p} \|\nabla \cdot f\|_{\tilde{L}^r} \quad \text{for } 2 \leq p \leq r \end{aligned}$$

by Proposition 2.2. If $d_\Omega(x) < 1$, we take $p = n$. If $d_\Omega(x) \geq 1$, we take $p = 2$. Since $n \geq 2$, this choice implies that $|\nabla q(x)|d_\Omega(x)$ is bounded in Ω . Although $|\nabla q(x)|d_\Omega(x)$ is continuous in Ω , this quantity may not be continuous up to the boundary.

(iii) Although the constant $C = C_\Omega$ in Definition 2.3 depends on a domain, it is independent of dilation and translation. In other words, $C_{\lambda\Omega+x_0} = C_\Omega$ for $x_0 \in \mathbf{R}^n$, $\lambda > 0$.

(iv) It is easy to see that the half space $\mathbf{R}_+^n = \{(x', x_n) \mid x_n > 0\}$ is admissible. In this case

$$Q[\nabla \cdot f] = \nabla q, \quad q(x', x_n) = \int_{x_n}^{\infty} \mathcal{P}_s[-n_\Omega \cdot (\nabla \cdot f)] ds,$$

where \mathcal{P}_s denotes the Poisson semigroup, i.e.

$$\mathcal{P}_s[h] = P_s * h \quad \text{with } P_s(x') = as/(|x'|^2 + s^2)^{n/2}, \quad x' \in \mathbf{R}^{n-1},$$

where $2/a$ is the surface area of the $n - 1$ dimensional unit sphere. Since

$$-n_\Omega \cdot (\nabla \cdot f) = \sum_j \partial_j f_{nj} = \sum_{1 \leq j \leq n-1} \partial_j f_{nj} - \sum_{1 \leq i \leq n-1} \partial_n f_{ii} = \sum_{1 \leq j \leq n-1} \partial_j (f_{nj} - f_{jn})$$

by (2.1), we end up with

$$\nabla q(x) = \sum_{1 \leq j \leq n-1} \nabla \partial_j \int_{x_n}^{\infty} \mathcal{P}_s[f_{nj} - f_{jn}] ds.$$

By an explicit form of the Poisson semigroup it is easy to see that

$$\|\partial_j \mathcal{P}_s[h]\|_{L^\infty(\mathbf{R}^{n-1})}(s) \leq c\|h\|_{L^\infty(\mathbf{R}^{n-1})}/s \quad \text{for } s > 0, \quad 1 \leq j \leq n-1$$

with $c > 0$ independent of s and h . Thus

$$\begin{aligned} \|\partial_k q\|_{L^\infty(\mathbf{R}^{n-1})}(x_n) &\leq \sum_{j=1}^{n-1} \int_{x_n}^{\infty} \|\partial_k \partial_j \mathcal{P}_s[h_j]\|_{L^\infty(\mathbf{R}^{n-1})} ds, \quad h_j = f_{nj} - f_{jn} \\ &\leq c^2(n-1) \int_{x_n}^{\infty} \frac{1}{s^2} ds \max_{1 \leq j \leq n-1} \|h_j\|_{L^\infty(\mathbf{R}^{n-1})} \leq C' \|f\|_{L^\infty} / x_n \end{aligned}$$

for $k \leq n-1$. For $k = n$ it is easier to obtain a similar estimate so we observe that the half space is admissible since $x_n = d_\Omega(x)$.

2.3 Blow-up arguments

Our goal in this subsection is to prove

Theorem 2.5. *A bounded domain with C^3 boundary is admissible.*

We shall prove this theorem by an indirect method - a blow-up argument although it might be possible to prove directly. For this purpose we first derive a weak formulation for $\nabla \Phi = Q[\nabla \cdot f]$.

Lemma 2.6. *Let Ω be a C^1 -domain. Assume that $f = (f_{ij}) \in C^1(\bar{\Omega})$ satisfies (2.1) with $\nabla \cdot f \in L^2(\Omega)$ so that $\nabla \Phi = Q[\nabla \cdot f] \in G^2(\Omega)$. Then*

$$-\int_{\Omega} \Phi \Delta \varphi dx = \sum_{i,j=1}^n \int_{\partial\Omega} f_{ij}(x) (n_\Omega^j(x) \partial_i \varphi(x) - n_\Omega^i(x) \partial_j \varphi(x)) d\mathcal{H}^{n-1} \quad (2.2)$$

for all $\varphi \in C_c^2(\bar{\Omega})$ satisfying $\partial\varphi/\partial n_\Omega = 0$ on $\partial\Omega$, where $d\mathcal{H}^{n-1}$ is the surface element of $\partial\Omega$, and $n_\Omega(x) = (n_\Omega^1(x), \dots, n_\Omega^n(x))$.

Proof. The L^2 -Helmholtz decomposition says that for $h = \nabla \cdot f$ there is a unique $h_0 \in L_\sigma^2(\Omega)$ and $Q[h] \in G^2(\Omega)$ such that $h = h_0 + Q[h]$ with $Q[h] = \nabla \Phi$. Multiply $\nabla \varphi$ with h and use the orthogonality to get

$$\int_{\Omega} h \cdot \nabla \varphi dx = \int_{\Omega} \nabla \varphi \cdot \nabla \Phi dx. \quad (2.3)$$

Since $\partial\varphi/\partial n_\Omega = 0$ on $\partial\Omega$, we have

$$\int_{\Omega} \nabla \varphi \cdot \nabla \Phi dx = - \int_{\Omega} \Phi \Delta \varphi dx \quad (2.4)$$

by integration by parts. (Note that $\Phi \in L_{loc}^2(\bar{\Omega})$ by the Poincaré inequality e.g. [13].) We now calculate the left hand side of (2.3). We observe that

$$\begin{aligned} (\partial_j f_{ij})(\partial_i \varphi) &= \partial_j (f_{ij} \partial_i \varphi) - f_{ij} \partial_i \partial_j \varphi, \\ f_{ij} \partial_i \partial_j \varphi &= \partial_i (f_{ij} \partial_j \varphi) - (\partial_i f_{ij}) \partial_j \varphi, \quad 1 \leq i, j \leq n. \end{aligned}$$

Since

$$\sum_{i=1}^n \partial_i f_{ij} = \sum_{i=1}^n \partial_j f_{ii} = 0$$

by (2.1), we now obtain an identity

$$\int_{\Omega} h \cdot \nabla \varphi dx = \sum_{i,j=1}^n \int_{\partial\Omega} f_{ij} (n_{\Omega}^i \partial_i \varphi - n_{\Omega}^j \partial_j \varphi) d\mathcal{H}^{n-1}. \quad (2.5)$$

Identities (2.3)-(2.5) yield (2.2). \square

Proof of Theorem 2.5. We argue by contradiction. Suppose that the condition were false. Then there would exist a sequence $\{\tilde{f}_m\}_{m=1}^{\infty} \subset C^1(\bar{\Omega})$ satisfying (2.1) such that

$$\infty > M_m = \sup_{x \in \Omega} d_{\Omega}(x) |\nabla \tilde{\Phi}_m(x)| > m \|\tilde{f}_m\|_{L^{\infty}(\partial\Omega)}$$

with $\nabla \tilde{\Phi}_m = Q[\nabla \cdot \tilde{f}_m]$. (Note that M_m is always finite by Remark 2.4 (ii)). We normalize by $\Phi_m = \tilde{\Phi}_m/M_m$ and $f_m = \tilde{f}_m/M_m$. There is a sequence of points $\{x_m\}_{m=1}^{\infty} \subset \Omega$ such that

$$\sup_{x \in \Omega} d_{\Omega}(x) |\nabla \Phi_m(x)| = 1, \quad (2.6)$$

$$d_{\Omega}(x_m) |\nabla \Phi_m(x_m)| \geq 1/2, \quad (2.7)$$

$$\|f_m\|_{L^{\infty}(\partial\Omega)} < 1/m. \quad (2.8)$$

Since $\bar{\Omega}$ is compact, x_m subsequently converges to some $x_{\infty} \in \bar{\Omega}$ as $m \rightarrow \infty$.

Case 1. $x_{\infty} \in \Omega$. We may assume $\Phi_m(x_{\infty}) = 0$. Since $\nabla \Phi_m$ is harmonic, (2.6) implies that $\{\Phi_m\}_{m=1}^{\infty}$ subsequently converges to some function $\Phi \in C^{\infty}(\Omega)$ locally uniformly in Ω with its all derivatives. By (2.6) the sequence $\{\Phi_m\}$ is bounded in $L^r(\Omega)$ for any $r \in [1, \infty)$ so Φ_m subsequently converges to Φ weakly in $L^r(1 < r < \infty)$. We apply Lemma 2.6 with $\Phi = \Phi_m$ and $f = f_m$ and send $m \rightarrow \infty$ to observe that $\Phi \in L^1(\Omega) \cap C^{\infty}(\Omega)$ fulfills

$$\int_{\Omega} \Phi(x) \Delta \varphi(x) dx = 0$$

for all $\varphi \in C_c^2(\bar{\Omega}) (= C^2(\bar{\Omega}))$ satisfying $\partial\varphi/\partial n_{\Omega} = 0$ on $\partial\Omega$ since the right hand side of (2.2) converges to zero by (2.8). Thus Φ formally solves the homogeneous Neumann problem so that $\nabla \Phi \equiv 0$. (In fact we apply Lemma 2.8 in the next subsection for a rigorous proof.)

Since $\nabla \Phi_m$ subsequently converges to $\nabla \Phi$ locally uniformly in Ω , (2.7) implies that $d_{\Omega}(x_{\infty}) |\nabla \Phi(x_{\infty})| \geq 1/2$. This contradicts the fact $\nabla \Phi \equiv 0$ so we get a contradiction for the case 1.

Case 2. $x_\infty \in \partial\Omega$. By taking a subsequence we may assume that $x_m \rightarrow x_\infty$. We rescale Φ_m and f_m around x_m so that the distance from the origin to the boundary equals 1. More precisely, we set

$$\Psi_m(x) = \Phi_m(x_m + d_m x), \quad g_m(x) = f_m(x_m + d_m x)$$

with $d_m = d_\Omega(x_m)$. It follows from (2.6)-(2.8) that

$$\sup_{x \in \Omega_m} d_{\Omega_m}(x) |\nabla \Psi_m(x)| = 1, \quad (2.9)$$

$$|\nabla \Psi_m(0)| \geq 1/2, \quad (2.10)$$

$$\|g_m\|_{L^\infty(\partial\Omega_m)} < 1/m. \quad (2.11)$$

Here Ω_m is the rescaled domain of the form

$$\Omega_m = \left\{ x \in \mathbf{R}^n \mid x = \frac{y - x_m}{d_m}, y \in \Omega \right\}.$$

We apply (2.2) for Ψ_m , g_m and Ω_m and send $m \rightarrow \infty$. Since the domain is moving, we have to take φ_m satisfying $\partial\varphi_m/\partial n_{\Omega_m} = 0$ so that it converges to some function φ . If $\partial\Omega$ is C^k ($k \geq 2$), there exists $\mu > 0$ such that $d_\Omega(x) \in C^k(\Gamma_{\Omega,\mu})$ with a tubular neighborhood $\Gamma_{\Omega,\mu} = \{x \in \bar{\Omega} \mid d_\Omega(x) < \mu\}$ and that, for any $z \in \Gamma_{\Omega,\mu}$ there is a unique projection $z^p \in \partial\Omega$ to $\partial\Omega$ i.e. $|z - z^p| = d_\Omega(z)$; cf. Proposition 3.6 (i). Let $x_m^p \in \partial\Omega$ be the projection of x_m to $\partial\Omega$ for sufficiently large m . The sequence of unit vector $(x_m - x_m^p)/d_m$ converges to a unit vector e . By translation and rotation we may assume that $e = (0, \dots, 0, 1)$. Then Ω_m converges to a half space $\mathbf{R}_{+,-1}^n$, where

$$\mathbf{R}_{+,c}^n = \{(x', x_n)\} \in \mathbf{R}^n \mid x_n > c\}.$$

More precisely, for any $R > 0$ there is m_0 such that for $m \geq m_0$ there is $h_m \in C^2(B_R^{n-1}(0))$ converging to -1 up to third derivatives with the property

$$\Omega_m \cap B_R^{n-1}(0) \times [-R, R] = \{(x', x_n) \in \mathbf{R}^n \mid R > x_n > h_m(x'), x' \in B_R^{n-1}(0)\},$$

where $B_R^{n-1}(0)$ denotes the closed ball in \mathbf{R}^{n-1} with radius R centered at the origin. Let $\varphi \in C_c^2(\mathbf{R}_{+,-1}^n)$ satisfy $\partial\varphi/\partial x_n = 0$ on $\{x_n = -1\}$. We may assume $\varphi \in C_c^2(\mathbf{R}^n)$ by a suitable extension. Take $R > 0$ large so that the support of φ is included in the interior of $B_R^{n-1}(0) \times [-R, R]$. We take a normal coordinate associated with Ω_m . Let F_m be the mapping defined by

$$x = (x', x_n) \longmapsto X = z + d_{\Omega_m}(x) \nabla d_{\Omega_m}(z) \quad \text{with} \quad z = (x', h_m(x')).$$

We set $\varphi_m(X) = \varphi(F_m^{-1}(X))$. This is well-defined for sufficiently large m . We further observe that $\partial\varphi_m/\partial n_{\Omega_m} = 0$ on $\partial\Omega_m$ since $n_{\Omega_m} = -\nabla d_{\Omega_m}$. If $\partial\Omega$ is C^3 , then F_m^{-1} is still C^2 . Thus $\varphi_m \in C_c^2(\bar{\Omega}_m)$ for sufficiently large m . Here we invoke C^3 regularity.

Since we may assume that $\Psi_m(0) = 0$, by (2.9) the sequence $\{\Psi_m\}$ is bounded in $L^r(\Omega_m \cap B_R(0) \times [-R, R])$, $r \in (1, \infty)$ for any $R > 1$. Since $\{\nabla\Psi_m\}$ is harmonic in Ω_m , Ψ_m subsequently converges to some function $\Psi \in C^\infty(\mathbf{R}_{+,-1}^n)$ locally uniformly with its all derivatives and weakly in $L_{loc}^r(\bar{\mathbf{R}}_{+,-1}^n)$ ($1 < r < \infty$). Since (2.11) implies that $g_m \rightarrow 0$ uniformly, we apply (2.2) with Ψ_m , φ_m and g_m and send $m \rightarrow \infty$ to get

$$\int_{\mathbf{R}_{+,-1}^n} \Psi \Delta \varphi dx = 0 \quad (2.12)$$

since F_m^{-1} converges to the identity in C^2 so that $\varphi_m \rightarrow \varphi$ in C^2 in a neighborhood of the support $\text{spt } \varphi$. We thus observe that (2.12) is valid for all $\varphi \in C_c^3(\mathbf{R}_{+,-1}^n)$ with $\partial\varphi/\partial x_n = 0$ on $\{x_n = -1\}$. We apply a uniqueness result for the Neumann problem with an estimate $\sup x_n |\nabla\Psi|(x', x_n) \leq 1$ obtained from (2.9) to get $\nabla\Psi \equiv 0$. (One should apply Lemma 2.9 in the next subsection for a rigorous proof.)

Since $\nabla\Psi_m$ subsequently converges to $\nabla\Psi$ locally uniformly in $\mathbf{R}_{+,-1}^n$, (2.10) implies $|\nabla\Psi(0)| \geq 1/2$. This contradicts the fact $\nabla\Phi \equiv 0$ so the proof is now complete. \square

Remark 2.7. (i) Even in Case 1 the estimate (2.6) does not imply that $\{\nabla\Psi_m\}$ is uniformly bounded in any Lebesgue spaces on Ω . Thus it is not clear that

$$\int_{\Omega_m} \nabla\Phi_m \cdot \nabla\varphi dx \rightarrow \int_{\Omega} \nabla\Phi \cdot \nabla\varphi dx$$

though we know

$$- \int_{\Omega_m} \Phi_m \Delta\varphi dx \rightarrow - \int_{\Omega} \Phi \Delta\varphi dx$$

since Φ_m converges weakly in all L^r ($1 < r < \infty$) spaces as $m \rightarrow \infty$ by taking a subsequence. This is a reason we need to assume that φ is at least C^2 and $\partial\varphi/\partial n_\Omega = 0$ on the boundary.

(ii) The proof of Theorem 2.5 actually yields an estimate

$$\sup_{x \in \Omega} d_\Omega(x) |Q[\nabla \cdot f](x)| \leq C_\Omega \|n_\Omega \cdot (f - {}^t f)\|_{L^\infty(\partial\Omega)}$$

which is stronger than (1.13). Here, $n_\Omega \cdot f = \sum_{j=1}^n n_\Omega^j f_{ij}$ and ${}^t f_{ij} = f_{ji}$.

If $f_{ij} = \partial_j v^i$ with $\text{div } v = 0$, the quantity $n_\Omega \cdot (f - {}^t f)$ is nothing but the tangential trace of the vorticity, i.e. $\omega \times n_\Omega$ when $n = 3$. Moreover, the right hand side of (2.2) equals

$$\int_{\partial\Omega} (\omega \times n_\Omega) \cdot \nabla\varphi d\mathcal{H}^{n-1}.$$

Since $\partial\varphi/\partial n_\Omega = 0$ so that $\nabla\varphi = \nabla_{\text{tan}}\varphi$ and since $\omega \times n_\Omega$ is a tangent vector field on $\partial\Omega$, the above quantity equals

$$- \int_{\partial\Omega} (\text{div}_{\partial\Omega}(\omega \times n_\Omega)) \varphi d\mathcal{H}^{n-1}.$$

This implies formally that Φ with $f = \partial_j v^i$ solves

$$-\Delta\Phi = 0 \text{ in } \Omega, \quad \partial\Phi/\partial n_\Omega = -\operatorname{div}_{\partial\Omega}(\omega \times n_\Omega) \text{ on } \partial\Omega,$$

where $\operatorname{div}_{\partial\Omega}$ denotes the surface divergence see e.g. [28], [50]. In general, since $n_\Omega \cdot (f - {}^t f)$ is tangential, we have

$$\partial\Phi/\partial n_\Omega = -\operatorname{div}_{\partial\Omega}(n_\Omega \cdot (f - {}^t f)) \text{ on } \partial\Omega.$$

2.4 Uniqueness of the Neumann problem

We shall give uniqueness results which are used in the proof of Theorem 2.5.

Lemma 2.8 (Uniqueness for bounded domains). *Let Ω be a bounded domain with C^3 boundary. Assume that $\Phi \in L^1(\Omega) \cap C(\Omega)$ satisfies*

$$\int_{\Omega} \Phi(x) \Delta\varphi(x) dx = 0 \tag{2.13}$$

for all $\varphi \in C^2(\bar{\Omega})$ satisfying $\partial\varphi/\partial n_\Omega = 0$ on $\partial\Omega$. Then Φ is a constant.

Proof. We consider a dual problem

$$-\Delta\varphi = \operatorname{div} \psi \text{ in } \Omega, \quad \partial\varphi/\partial n_\Omega = 0 \text{ on } \partial\Omega.$$

For arbitrary $\psi \in C_c^\infty(\Omega)$, there exists a solution $\varphi \in W^{3,r}(\Omega)$ for all $r > 1$ (see e.g. [34, Lemma 2.3].) By the Sobolev embedding we conclude that $\varphi \in C^2(\bar{\Omega})$. From (2.13) it follows that

$$\int_{\Omega} \Phi \operatorname{div} \psi \, dx = 0$$

for all $\psi \in C_c^\infty(\Omega)$. This implies $\nabla\Phi = 0$, so Φ is a constant. \square

Lemma 2.9 (Uniqueness for the half space). *Let $\Phi \in L^1_{loc}(\bar{\mathbf{R}}_+^n)$ satisfy*

$$\int_{\mathbf{R}_+^n} \Phi(x) \Delta\varphi(x) dx = 0$$

for all $\varphi \in C_c^\infty(\bar{\mathbf{R}}_+^n)$ satisfies $\partial\varphi/\partial x_n = 0$ on $\{x_n = 0\}$. Assume that Φ satisfies

$$\sup_{x \in \mathbf{R}_+^n} x_n |\nabla\Phi(x)| < \infty \tag{2.14}$$

Then Φ is a constant.

Proof. The problem can be reduced to the whole space. Let $\tilde{\Phi}$ be an even extension of Φ to the whole space i.e. $\tilde{\Phi}(x', x_n) = \Phi(x', -x_n)$ for $x_n < 0$. For arbitrary $\varphi \in C_c^\infty(\mathbf{R}^n)$ let φ_{even} and φ_{odd} are even and odd part of φ , i.e.,

$$\varphi_{\text{even}}(x) = \frac{\varphi(x', x_n) + \varphi(x', -x_n)}{2}, \quad \varphi_{\text{odd}}(x) = \frac{\varphi(x', x_n) - \varphi(x', -x_n)}{2}.$$

The integration by parts yields

$$\begin{aligned} \int_{\mathbf{R}^n} \tilde{\Phi}(x) \Delta \varphi(x) dx &= \int_{\mathbf{R}^n} \tilde{\Phi}(x) \Delta (\varphi_{\text{even}}(x) + \varphi_{\text{odd}}(x)) dx \\ &= \int_{\mathbf{R}^n} \tilde{\Phi}(x) \Delta \varphi_{\text{even}}(x) dx \\ &= 2 \int_{\mathbf{R}_+^n} \Phi(x) \Delta \varphi_{\text{even}}(x) dx. \end{aligned}$$

Since φ_{even} satisfies $\partial \varphi_{\text{even}} / \partial x_n = 0$ on $\{x_n = 0\}$, we conclude that

$$\int_{\mathbf{R}^n} \tilde{\Phi}(x) \Delta \varphi(x) dx = 0. \quad (2.15)$$

By (2.14) we know $\tilde{\Phi}$ is locally integrable in \mathbf{R}^n . Since (2.15) says that $\tilde{\Phi}$ is weakly harmonic, $\tilde{\Phi} = \eta_\epsilon * \tilde{\Phi}$ by the mean value theorem if η_ϵ is a symmetric mollifier i.e. η_ϵ is radially symmetric (see e.g. [13, 2.2.3]). Moreover, by integrating $\tilde{\Phi}$ from $x_0 = (0, (x_0)_n) \in \mathbf{R}^n$, $(x_0)_n \neq 0$ to x , we observe that (2.14) yields

$$|\tilde{\Phi}(x)| \leq C \left(1 + |\log |x_n|| + |x| |\log |x_n|| \right)$$

for $x' \in \mathbf{R}^n$, $|x_n| < 1/2$ with some constant C independent of x . This implies that $\nabla \tilde{\Phi} = \nabla \eta_\epsilon * \tilde{\Phi}$ enjoys an estimate

$$|\nabla \tilde{\Phi}(x)| \leq C_\epsilon (1 + |x|) \quad (2.16)$$

for $x' \in \mathbf{R}^{n-1}$, $|x_n| < 2\epsilon$ with C_ϵ independent of x . By (2.14) we conclude that $\nabla \tilde{\Phi}$ satisfies (2.16) for all $x \in \mathbf{R}^n$. Since $\tilde{\Phi}$ is weakly harmonic, (2.16) implies that $\nabla \tilde{\Phi}$ is harmonic in \mathbf{R}^n . By (2.16) the classical Liouville theorem implies that $\nabla \tilde{\Phi}$ is a polynomial of degree one. However, by the decay estimate (2.14) for $|x_n| \rightarrow \infty$ this polynomial must be zero. Thus $\nabla \tilde{\Phi} = 0$ i.e. $\tilde{\Phi}$ is a constant. \square

Remark 2.10. We actually need only C^2 -regularity of the boundary $\partial\Omega$ in the Case 1 of the proof of Theorem 2.5. Note that the identity (2.2) is still valid for $\varphi \in W^{2,2}(\Omega)$ having compact support in $\bar{\Omega}$. (In this paper $W^{m,r}(\Omega)$ denotes the L^r -Sobolev space of order m .) When $\partial\Omega$ is C^2 , a slightly modified version of Lemma 2.8 is valid. In fact, for $\Phi \in L^2(\Omega)$ we still assert $\nabla \Phi \equiv 0$ if (2.13) is fulfilled for all $\varphi \in W^{2,2}(\Omega)$ with $\partial\varphi/\partial n_\Omega = 0$ on $\partial\Omega$. (The constructed φ in the proof is now in $W^{2,2}(\Omega)$ not necessarily in $W^{3,r}(\Omega)$.) Based on these assertions the proof of Case 1 goes through with trivial modifications.

3 Uniform Hölder estimates for pressure gradients

The goal of this section is to establish local Hölder estimates for second spatial derivatives and the time derivative of the velocity solving the Stokes equations both interior and up to boundary. This procedure is a key to derive necessary compactness for blow-up sequences. Unlike the heat equation the result is not completely local even interior case since we need a uniform Hölder estimates in time for pressure gradients. For this purpose we invoke admissibility of domains.

3.1 Interior Hölder estimates for pressure gradients

We use conventional notation [39] for Hölder (semi)norms for space-time functions. Let $f = f(x, t)$ be a real-valued or an \mathbf{R}^n -valued function defined in $Q = \Omega \times (0, T]$, where Ω is a domain in \mathbf{R}^n . For $\mu \in (0, 1)$ we set several Hölder semi-norms

$$\begin{aligned} [f]_{(0,T]}^{(\mu)}(x) &= \sup \left\{ |f(x, t) - f(x, s)| / |t - s|^\mu \mid t, s \in (0, T], t \neq s \right\} \\ [f]_{\Omega}^{(\mu)}(t) &= \sup \left\{ |f(x, t) - f(y, t)| / |x - y|^\mu \mid x, y \in \Omega, x \neq y \right\} \end{aligned}$$

and

$$[f]_{t,Q}^{(\mu)} = \sup_{x \in \Omega} [f]_{(0,T]}^{(\mu)}(x), \quad [f]_{x,Q}^{(\mu)} = \sup_t [f]_{\Omega}^{(\mu)}(t).$$

In the parabolic scale for $\gamma \in (0, 1)$ we set

$$[f]_Q^{(\gamma, \gamma/2)} = [f]_{t,Q}^{(\gamma/2)} + [f]_{x,Q}^{(\gamma)}.$$

For later convenience we also define the case $\gamma = 1$ so that

$$[f]_Q^{(1, 1/2)} = \|\nabla f\|_{L^\infty(Q)} + [f]_{t,Q}^{(1/2)}.$$

If $l = [l] + \gamma$ where $[l]$ is a nonnegative integer and $\gamma \in (0, 1)$, we set

$$[f]_Q^{(l, l/2)} = \sum_{|\alpha|+2\beta=[l]} [\partial_x^\alpha \partial_t^\beta f]_Q^{(\gamma, \gamma/2)}$$

and the parabolic Hölder norm

$$|f|_Q^{(l, l/2)} = \sum_{|\alpha|+2\beta \leq [l]} \|\partial_x^\alpha \partial_t^\beta f\|_{L^\infty(Q)} + [f]_Q^{(l, l/2)}.$$

When f is time-independent, we simply write $[f]_{x,Q}^{(\mu)}$ by $[f]_{\Omega}^{(\mu)}$.

Let Ω be a uniformly C^2 -domain in \mathbf{R}^n . For a given $v_0 \in \tilde{L}_\sigma^r(\Omega)$, $1 < r < \infty$ it is proved in [14], [16] that there exists a unique solution (v, q) of the Stokes equations (1.1)-(1.4) satisfying $v_t, \nabla q, \nabla^2 v, \nabla v, v \in \tilde{L}^r(\Omega)$ at each $t \in (0, T)$ such that the solution operator $S(t) : v_0 \mapsto v(\cdot, t)$ is an analytic semigroup in $\tilde{L}_\sigma^r(\Omega)$. Here $T > 0$ is taken arbitrary large. In this paper we simply say that (v, q) is an \tilde{L}^r -solution of (1.1)-(1.4). Note that $\nabla q = Q[\Delta v]$ for an \tilde{L}^r -solution.

Lemma 3.1. *Let Ω be an admissible, uniformly C^2 -domain in \mathbf{R}^n (with $r \geq n$). Then there exists a constant $M(\Omega) > 0$ such that*

$$[d_\Omega(x)\nabla q]_{t,Q_\delta}^{(1/2)} \leq \frac{M}{\delta} \sup\left\{ (\|v_t\|_\infty(t) + \|\nabla^2 v\|_\infty(t))t \mid \delta \leq t \leq T \right\}$$

holds for all \tilde{L}^r -solution (v, q) of (1.1)-(1.4) and all $\delta \in (0, T)$, where $Q_\delta = \Omega \times (\delta, T)$. The constant M can be taken uniform with respect to translation and dilation i.e., $M(\lambda\Omega + x_0) = M(\Omega)$ for all $\lambda > 0$ and $x_0 \in \Omega$.

Proof. By an interpolation inequality (e.g. [62], [38, 3.2]) there is a dilation invariant constant C such that for any $\varepsilon > 0$ the estimate

$$\|\nabla v\|_\infty(t) \leq \varepsilon \|\nabla^2 v\|_\infty(t) + (C/\varepsilon)\|v\|_\infty(t)$$

holds. Since our solution is an \tilde{L}^r -solution, we have

$$\nabla q = Q[\nabla \cdot f], \quad f = (f_{ij}) = \partial_j v^i$$

and moreover

$$\nabla q(x, t) - \nabla q(x, s) = Q[\Delta v(x, t) - \Delta v(x, s)].$$

Since Ω is admissible, we have

$$\begin{aligned} d_\Omega(x) |\nabla q(x, t) - \nabla q(x, s)| &\leq C(\Omega) \|\nabla(v(\cdot, t) - v(\cdot, s))\|_\infty \\ &\leq C(\Omega) [\varepsilon \max(\|\nabla^2 v\|_\infty(t), \|\nabla^2 v\|_\infty(s)) + (C/\varepsilon) \|v(\cdot, t) - v(\cdot, s)\|_\infty]. \end{aligned}$$

Since

$$\begin{aligned} \|v(\cdot, t) - v(\cdot, s)\|_\infty &\leq |t - s| \sup\{\|v_t\|_\infty(\tau) \mid \tau \text{ is between } t \text{ and } s\}, \\ &\leq |t - s| \frac{1}{\delta} \sup\{\tau \|v_t\|_\infty(\tau) \mid \delta \leq \tau \leq T\} \end{aligned}$$

for $t, s \geq \delta$, the desired inequality follows by taking $\varepsilon = |t - s|^{1/2}$. Since C_Ω is also dilation and translation invariant by Remark 2.4 (iii), so is $M(\Omega)$. \square

Proposition 3.2 (Interior Hölder estimates). *Let Ω be an admissible, uniformly C^2 -domain in \mathbf{R}^n (with $r \geq n$). Take $\gamma \in (0, 1)$, $\delta > 0$, $T > 0$, $R > 0$. Then there exists a constant $C = C(M(\Omega), \delta, R, d, \gamma, T)$ such that the estimate*

$$[\nabla^2 v]_{Q'}^{(\gamma, \gamma/2)} + [v_t]_{Q'}^{(\gamma, \gamma/2)} + [\nabla q]_{Q'}^{(\gamma, \gamma/2)} \leq CN_T \quad (3.1)$$

holds for all \tilde{L}^r -solution (v, q) of (1.1)-(1.4) provided that $B_R(x_0) \subset \Omega$ and $x_0 \in \Omega$, where $Q' = \text{int}B_R(x_0) \times (\delta, T]$ and d denotes the distance of $B_R(x)$ and $\partial\Omega$. Here

$$N_T = \sup_{0 < t < T} \|N(v, q)\|_\infty(t) < \infty \quad (3.2)$$

and $M(\Omega)$ is the constant in Lemma 3.1.

Proof. Since ∇q is harmonic in Ω , the Cauchy type estimate implies

$$\sup_{x \in B_{R+d/2}(x_0)} |\nabla^2 q(x, t)| \leq \frac{C_0}{d} \|\nabla q\|_{L^\infty(\Omega)}(t), \quad B_{R+d/2}(x_0) \subset \Omega,$$

where C_0 depends only on n . This together with Lemma 3.1 implies

$$[\nabla q]_{Q''}^{(1,1/2)} \leq \left(\frac{C_0 R'}{d} + M \right) \frac{1}{\delta} N_T, \quad R' = R + d/2$$

for any $x_0 \in \Omega$, $R > 0$, $\delta > 0$, where $Q'' = \text{int} B_{R+d/2}(x_0) \times (\delta/2, T]$. By the standard local Hölder estimate for the heat equation

$$v_t - \Delta v = -\nabla q \quad \text{in} \quad Q''$$

this pressure gradient estimate implies estimates for $\nabla^2 v$, v_t for Q' [39, Chapter IV, Theorem 10.1]. \square

Remark 3.3. (i) We are tempted to claim that if (v, q) solves the Stokes system (1.1)-(1.2) without boundary and initial condition, then the desired interior Hölder estimate would be valid. Such a type estimate is in fact true for the heat equation [39, Chapter IV, Theorem 10.1]. However, for the Stokes equations this is no longer true. In fact, if we take $v(x, t) = g(t)$ and $p(x, t) = -g'(t) \cdot x$ with $g \in C^1[0, \infty)$, this is always a solution of (1.1)-(1.2) satisfying $N_{T_1} < \infty$ for any $T_1 > 0$. However, evidently v_t may not be Hölder continuous in time unless ∇p is Hölder continuous in time. This is why we use a global setting with admissibility of the domain.

(ii) In the constant C the dependence of Ω is through $M(\Omega)$ so it is invariant under a dilation provided that d and R are taken independent of a dilation.

3.2 Local Hölder estimates up to the boundary

The regularity up to boundary is more involved. We begin with the statement and give a proof in subsequent sections.

Theorem 3.4 (Estimates near the boundary). *Let Ω be an admissible, uniformly C^3 -domain of type (α, β, K) in \mathbf{R}^n (with $r \geq n$). Then there exists $R_0 = R_0(\alpha, \beta, K) > 0$ such that for any $\gamma \in (0, 1)$, $\delta \in (0, T)$ and $R \leq R_0/2$ there exists a constant*

$$C = C(M(\Omega), \delta, \gamma, T, R, \alpha, \beta, K)$$

such that (3.1) is valid for all \tilde{L}^r -solution (v, q) of (1.1)-(1.4) with

$$Q' = Q'_{x_0, R, \delta} = \Omega_{x_0, R} \times (\delta, T], \quad \Omega_{x_0, R} = \text{int} B_R(x_0) \cap \Omega$$

provided that $x_0 \in \partial\Omega$.

The proof is more involved. We first localize the Stokes equations near the boundary by using cut-off technique and the Bogovskii operator [20, III.3] to recover divergence free property. Then we apply a global Schauder estimate for the Stokes equations in a localized domain. As in the interior case we use the admissibility of the domain to obtain the Hölder estimate for the pressure in time.

We begin with Hölder estimates for q in time since we are not able to control the Hölder norm of ∇q up to the boundary.

Lemma 3.5. *Assume the same hypotheses of Lemma 3.1. Then there exists $R_0 = R_0(\alpha, \beta, K) > 0$ such that for $\nu \in (0, 1)$ and $R \in (0, R_0]$ there exists a constant $C_0 = C_0(M(\Omega), \nu, \alpha, R, \beta, K)$ such that*

$$[q]_{Q'}^{(\nu, \nu/2)} \leq C_0 N_T / \delta. \quad (3.3)$$

is valid for all \tilde{L}^r -solution (v, q) of (1.1)-(1.4) and $Q' = Q'_{x_0, R, \delta}$ for $x_0 \in \partial\Omega$.

For this purpose we prepare a basic fact for a distance function.

Proposition 3.6. *Let Ω be a uniformly C^2 -domain of type (α, β, K) .*

- (i) *There is a constant $R_* = R_*(\alpha, \beta, K) > 0$ such that $x \in \Gamma_{\Omega, R_*} = \{x \in \Omega \mid d_\Omega(x) < R_*\}$ has the unique projection $x_p \in \partial\Omega$ (i.e., $|x - x_p| = d_\Omega(x)$) and x is represented as $x = x_p - dn_\Omega(x_p)$ with $d = d_\Omega(x)$. The mapping $x \mapsto (x_p, d)$ is C^1 in Γ_{Ω, R_*} .*
- (ii) *There is a positive constant $R_1 = R_1(\alpha, \beta, K) \leq R_*$ such that $\Omega_{x_0, R_1} \subset U_{\alpha, \beta, h}(x_0)$ and the projection x_p of $x \in \Omega_{x_0, R_1}$ is on $x_0 + \text{graph } h$.*
- (iii) *For each $R \in (0, R_1)$ and $\nu \in [0, 1)$ there is a constant $C = C(\alpha, \beta, K, R, \nu)$ such that*

$$|\tilde{q}(x) - \tilde{q}(y)| \leq C \|d_\Omega^\nu \nabla \tilde{q}\|_{L^\infty(\Omega)} \left\{ |d_\Omega(y)^{1-\nu} - d_\Omega(x)^{1-\nu}| + |x_p - y_p| / \max(d_\Omega(x)^\nu, d_\Omega(y)^\nu) \right\}$$

for $x, y \in \Omega_{x_0, R}$

for all $\tilde{q} \in C^1(\Omega)$ and $x_0 \in \partial\Omega$.

Proof of Proposition 3.6. (i) This is nontrivial but well-known. See e.g. [24] or [37, 4.4].

(ii) This is easy by taking R smaller. The smallness depends on a bound for the second fundamental form of $\partial\Omega$.

(iii) For $x \in \Omega_{x_0, R}$ ($R \leq R_1$) we consider its normal coordinate (x_p, d) . Since $\Omega_{x_0, R} \subset U_{\alpha, \beta, h}(x_0)$, there is unique $x'_p \in \mathbf{R}^{n-1}$ such that $x_p = (x'_p, h(x'_p))$. Moreover, we are able to use (x'_p, d) as a coordinate system. For $x, y \in \Omega_{x_0, R}$ with $x = (x'_p, d_\Omega(x))$, $y = (y'_p, d_\Omega(y))$ with $d_\Omega(y) > d_\Omega(x)$ we estimate

$$|\tilde{q}(x) - \tilde{q}(y)| \leq |\tilde{q}(x) - \tilde{q}(z)| + |\tilde{q}(z) - \tilde{q}(y)|$$

with $z = (x'_p, d_\Omega(y))$. Thus we connect x and z by a straight line which parallels to $n_\Omega(x_p)$ and observe that

$$\begin{aligned} |\tilde{q}(x) - \tilde{q}(z)| &\leq |z - x| \int_0^1 \frac{1}{d_\Omega^\nu(x_\tau)} |d_\Omega^\nu \nabla \tilde{q}(x_\tau)| \, d\tau, \quad x_\tau = x(1 - \tau) + \tau z \quad (0 \leq \tau \leq 1) \\ &\leq \int_{d_\Omega(x)}^{d_\Omega(y)} \frac{1}{s^\nu} \, ds \|d_\Omega^\nu \nabla \tilde{q}\|_{L^\infty(\Omega)} \\ &\leq (d_\Omega(z)^{1-\nu} - d_\Omega(x)^{1-\nu}) \|d_\Omega^\nu \nabla \tilde{q}\|_{L^\infty(\Omega)} (1 - \nu)^{-1}. \end{aligned}$$

It remains to estimate $|\tilde{q}(z) - \tilde{q}(y)|$. We connect z and y by a curve $C_{z,y}$ of the form

$$C_{z,y} = \left\{ x(\tau) \mid 0 \leq \tau \leq 1, \quad x'_p(\tau) = x'_p(1 - \tau) + \tau y'_p, \quad d_\Omega(x(\tau)) = d_\Omega(y) \right\}$$

so that the projection in \mathbf{R}^{n-1} is a straight line connecting x'_p and y'_p . We now estimate

$$\begin{aligned} |\tilde{q}(z) - \tilde{q}(y)| &\leq \int_{C_{z,y}} \frac{1}{d_\Omega^\nu(y)} d_\Omega^\nu(y) |\nabla \tilde{q}|(x) \, d\mathcal{H}^1(x) \\ &= \frac{1}{d_\Omega(y)^\nu} \mathcal{H}^1(C_{z,y}) \|d_\Omega^\nu \nabla \tilde{q}\|_{L^\infty(\Omega)}. \end{aligned}$$

Since $\mathcal{H}^1(C_{z,y}) \leq C|x_p - y_p|$, the proof is now complete. \square

Proof of Lemma 3.5. We take $R_1 > 0$ as in Proposition 3.6. For $x_0 \in \partial\Omega$ we take $\tilde{x}_0 = x_0 - \frac{R_1}{2}n_\Omega(x_0)$. We may assume that $q(\tilde{x}_0, t) = 0$ for all $t \in (0, T)$. Since

$$[d_\Omega(x)^\nu \nabla q]_{t, Q_\delta}^{(1/2)} \leq \left([d_\Omega(x) \nabla q]_{t, Q_\delta}^{(1/2)} \right)^\nu (2\|\nabla q\|_{L^\infty(Q_\delta)})^{1-\nu},$$

Lemma 3.1 implies that

$$\|d_\Omega(x)^\nu \nabla \tilde{q}(x, \cdot)\|_{L^\infty(\Omega)}(t, s) \leq \frac{M^\nu N_T 2^{1-\nu}}{\delta} |t - s|^{\nu/2} \quad \text{for } t, s \in (\delta, T]$$

with $\tilde{q}(x, t, s) = q(x, t) - q(x, s)$. We now apply Proposition 3.6 (iii) with $y = \tilde{x}_0$ to get

$$|q(x, t) - q(x, s)| \leq C(d_\Omega(\tilde{x}_0)^{1-\nu} + |x_p - x_0|d_\Omega(\tilde{x}_0)^{-\nu}) \frac{M^\nu N_T 2^{1-\nu}}{\delta} |t - s|^{\nu/2}$$

for $t, s \in (\delta, T]$ and all $x \in \Omega_{x_0, R}$, $R \leq R_0 = R_1/4$. Since $d_\Omega(\tilde{x}_0) = 2R_0$ and $|x_p - x_0| < R$, the above inequality implies

$$[q]_{t, Q'}^{(\nu/2)} \leq C_0 N_T / \delta, \quad C_0 = C((2R_0)^{1-\nu} + R(2R_0)^{-\nu}) M^\nu 2^{1-\nu}.$$

For the Hölder estimate in space we simply apply Proposition 3.6 (iii) with $\nu = 0$ to get

$$\begin{aligned} |q(x, t) - q(y, t)| &\leq C \|\nabla q\|_{L^\infty(\Omega)}(t) (|d_\Omega(y) - d_\Omega(x)| + |x_p - y_p|) \\ &\leq C \|\nabla q\|_{L^\infty(\Omega)}(t) |x - y|, \quad x, y \in \Omega_{x_0, R}, \quad R \leq R_0, \quad t \in (0, T). \end{aligned}$$

This implies

$$[q]_{x, Q'}^{(\nu)} \leq C_0 N_T / \delta$$

so the proof is now complete. \square

3.3 Helmholtz decomposition and the Stokes equations in Hölder spaces

To prove local Hölder estimates up to boundary (Theorem 3.4) we recall several known Hölder estimates for the Helmholtz decomposition and the Stokes equations established by [52], [58] via potential theoretic approach. We recall notions for the spaces of Hölder continuous functions. By $C^\gamma(\bar{\Omega})$ with $\gamma \in (0, 1)$ we mean the space of all continuous functions in $\bar{\Omega}$ with $[f]_\Omega^{(\gamma)} < \infty$. Similarly, we use $C^{\gamma, \gamma/2}(\bar{Q})$ for the space of all continuous functions in \bar{Q} with $[f]_Q^{(\gamma, \gamma/2)} < \infty$.

Proposition 3.7 (Helmholtz decomposition). *Let Ω be a bounded $C^{2+\gamma}$ -domain in \mathbf{R}^n with $\gamma \in (0, 1)$.*

(i) *For $f \in C^\gamma(\bar{\Omega})$ there is a (unique) decomposition $f = f_0 + \nabla\Phi$ with $f_0, \nabla\Phi \in C^\gamma(\bar{\Omega})$ such that*

$$\int_{\Omega} f_0 \cdot \nabla\varphi dx = 0 \quad \text{for all } \varphi \in C^\infty(\bar{\Omega}). \quad (3.4)$$

(ii) *There is a constant $C_H > 0$ depending only on γ and Ω only through its $C^{2+\gamma}$ regularity such that*

$$|f_0|_\Omega^{(\gamma)} + |\nabla\Phi|_\Omega^{(\gamma)} \leq C_H |f|_\Omega^{(\gamma)} \quad \text{for all } f \in C^\gamma(\bar{\Omega}). \quad (3.5)$$

(iii) *For each $\varepsilon \in (0, 1 - \gamma)$ there is a constant $C'_H > 0$ depending only on γ, ε and Ω only through its $C^{2+\gamma}$ regularity such that*

$$|f_0|_Q^{(\gamma, \gamma/2)} + |\nabla\Phi|_Q^{(\gamma, \gamma/2)} \leq C'_H |f|_Q^{(\gamma + \varepsilon, \frac{\gamma + \varepsilon}{2})} \quad \text{for all } f \in C^{\gamma, \gamma/2}(\bar{Q}). \quad (3.6)$$

Proof. The part (i) and (ii) are established in [52], [58]; the dependence of the constant is not explicit but it is observed from the proof.

In [58, Corollary on p.175] it is proved that the left hand side of (3.6) is dominated by a (similar type) constant multiple of

$$|f|_Q^{(\gamma, \gamma/2)} + \sup_{\substack{x, y \in \Omega \\ t, s \in (0, T)}} \frac{|(f(x, t) - f(x, s)) - (f(y, t) - f(y, s))|}{|x - y|^\mu \cdot |t - s|^{\frac{\gamma}{2}}} \quad (3.7)$$

for arbitrary $\mu \in (0, 1)$. By the Young inequality we observe to get

$$\frac{1}{|x - y|^\varepsilon |t - s|^{\gamma/2}} \leq \frac{\varepsilon}{\gamma + \varepsilon} \frac{1}{|x - y|^{\gamma + \varepsilon}} + \frac{\gamma}{\gamma + \varepsilon} \frac{1}{|t - s|^{\frac{\gamma + \varepsilon}{2}}}.$$

Thus we take $\mu = \varepsilon$ to see that the second term of (3.7) is dominated by

$$\frac{2\varepsilon}{\gamma + \varepsilon} \sup_{t \in (0, T]} [f]_\Omega^{(\gamma + \varepsilon)}(t) + \frac{2\gamma}{\gamma + \varepsilon} \sup_{x \in \Omega} [f]_{(0, T]}^{(\frac{\gamma + \varepsilon}{2})}(x).$$

Thus the estimate (3.6) follows and (iii) is proved. \square

Remark 3.8. The operator $f \mapsto f_0$ is essentially the Helmholtz projection P for Hölder vector fields since (3.4) implies that $\operatorname{div} f = 0$ in Ω and $f \cdot n_\Omega = 0$ on $\partial\Omega$. The estimate (3.5) shows the continuity of P in the Hölder space $C^\gamma(\bar{\Omega})$. However, it is mentioned in [58] (without a proof) that P is not continuous in $C^{\gamma, \gamma/2}(\bar{Q})$. In other words, one cannot take $\varepsilon = 0$ in the estimate (3.6).

We next recall Schauder type estimates for the Stokes system

$$v_t - \Delta v + \nabla q = f_0 \quad \text{in } \Omega \times (0, T) \quad (3.8)$$

$$\operatorname{div} v = 0 \quad \text{in } \Omega \times (0, T) \quad (3.9)$$

$$v = 0 \quad \text{on } \partial\Omega \times (0, T) \quad (3.10)$$

$$v = 0 \quad \text{on } \Omega \times \{t = 0\}. \quad (3.11)$$

Proposition 3.9. *Let Ω be a bounded $C^{2+\gamma}$ -domain in \mathbf{R}^n with $\gamma \in (0, 1)$ and $T > 0$. Then for each $f_0 \in C^{\gamma, \gamma/2}(\bar{Q})$ satisfying (3.4) there is a unique solution $(v, \nabla q) \in C^{2+\gamma, 1+\gamma/2}(\bar{Q}) \times C^{\gamma, \gamma/2}(\bar{Q})$ (up to an additive constant for q) of (3.8)-(3.11). Moreover, there is a constant C_S depending only on γ , T and Ω only through its $C^{2+\gamma}$ -regularity such that*

$$|v|_Q^{(2+\gamma, 1+\gamma/2)} + |\nabla q|_Q^{(\gamma, \gamma/2)} \leq C_S |f_0|_Q^{(\gamma, \gamma/2)} \quad (3.12)$$

Remark 3.10. (i) This result is a special case of a very general result [58, Theorem 1.1] where the viscosity constant in front of Δ in (3.8) depends on space and time and the boundary and initial data are inhomogeneous. Note that the divergence free condition (3.4) for f_0 is assumed to establish (3.12).

(ii) If the domain is a bounded C^3 -domain, clearly it is a uniformly C^3 -domain of type (α, β, K) with some (α, β, K) . The constants C_H , C'_H and C_S in Propositions 3.7 and 3.9 depends on Ω only through this (α, β, K) when Ω is a bounded C^3 -domain (which is of course a $C^{2+\gamma}$ -domain for all $\gamma \in (0, 1)$).

3.4 Localization procedure

We shall prove Theorem 3.4 by Lemma 3.5 and a localization procedure with necessary Hölder estimates (Propositions 3.7 and 3.9). We first recall the Bogovskiĭ operator B_E in [8]. Let E be a bounded subdomain in Ω with a Lipschitz boundary. The Bogovskiĭ operator B_E is a rather explicit operator but here we only need a few properties. This linear operator B_E is well-defined for average-zero function i.e. $\int_E g dx = 0$. Moreover, $\operatorname{div} B_E(g) = g$ in E and if the support $\operatorname{spt} g \subset E$, then $\operatorname{spt} B_E(g) \subset E$.

The operator B_E fulfills estimates

$$\|B_E(g)\|_{W^{1,p}(E)} \leq C_B \|g\|_{L^p(E)} \text{ for } g \in L^p(E) \text{ satisfying } \int_E g dx = 0 \quad (3.13)$$

$$\|B_E(g)\|_{L^p(E)} \leq C_B \|g\|_{W_0^{-1,p}(E)} \text{ for } h \in W_0^{-1,p}(E) = (W^{1,p'}(E))^* \quad (3.14)$$

with some constant C_B independent of g , where $1/p' + 1/p = 1$ with $1 < p < \infty$. In particular B_E is bounded from $L_{av}^p = \{g \in L^p(E) \mid \int_E g dx = 0\}$ to the Sobolev space $W^{1,p}(E)$. The result (3.14) is a special case of that of [21, Theorem 2.5] which asserts that B_E is bounded from $W_0^{s,p}(\Omega)$ to $W_0^{s+1,p}(\Omega)$ for $s > -2 + 1/p$. The bound C_B depends on p but its dependence on E is through Lipschitz regularity constant of ∂E .

Proof of Theorem 3.4. We take R_0 as in Lemma 3.5 and take $R \leq R_0/2$. For $x_0 \in \partial\Omega$ we take a bounded C^3 -domain Ω' such that $\Omega_{x_0,3R/2} \subset \Omega' \subset \Omega_{x_0,2R}$. Evidently $\partial\Omega_{x_0,R} \cap \partial\Omega$ is strictly included in $\partial\Omega' \cap \partial\Omega$. Moreover, one can arrange that Ω' is of type (α', β', K') such that (α', β', K) depends on (α, β, K) and R . Such Ω' is constructed for example by considering $\Omega'' = \Omega_{x_0,7R/4}$ and mollify near the set of intersection $\partial B_{7R/4}(x_0)$ and $\partial\Omega$ in a suitable way to get Ω' .

Let θ be a smooth cut-off function of $[0,1]$ supported in $[0, 3/2)$, i.e. $\theta \in C^\infty[0, \infty)$ such that $\theta \equiv 1$ on $[0, 1]$ and $0 \leq \theta \leq 1$ with $\operatorname{spt} \theta \subset [0, 3/2)$. We set $\theta_R(x) = \theta(|x - x_0|/R)$ which is a cut-off function of $\Omega_{x_0,R}$ supported in Ω' . Because of construction, its derivatives depend only on R . We also take a cut-off function ρ_δ in time variable. Let $\rho \in C^\infty[0, \infty)$ satisfies $\rho \equiv 1$ on $[1, \infty)$ and $\rho = 0$ on $[0, 1/2)$ with $0 \leq \rho \leq 1$. For $\delta > 0$ we set $\rho_\delta(t) = \rho(t/\delta)$. We set $\xi = \theta_R \rho_\delta$ and observe that $u = v\xi$ and $p = q\xi$ solves

$$u_t - \Delta u + \nabla p = f, \operatorname{div} u = g$$

in $U = \Omega' \times (0, T)$ with

$$f = v\xi_t - 2\nabla v \cdot \nabla \xi - v\Delta \xi + q\nabla \xi, \quad g = v\nabla \xi (= \operatorname{div}(v\xi)).$$

We next use the Bogovskiĭ operator $B_{\Omega'}$ so that the vector field is solenoidal. We set $u^* = B_{\Omega'}(g)$ and $\tilde{u} = u - u^*$. Then (\tilde{u}, p) solve

$$\tilde{u}_t - \Delta \tilde{u} + \nabla p = \tilde{f}, \operatorname{div} \tilde{u} = 0 \text{ in } U$$

with $\tilde{f} = f + u_t^* - \Delta u^*$. We shall fix Ω' so that C'_H in (3.6) and C_S in (3.12) depends on Ω' only through (α, β, K) and R . If we know $\tilde{f} \in C^{\gamma+\varepsilon, \frac{\gamma+\varepsilon}{2}}(\bar{U})$ with $\varepsilon \in (0, 1 - \gamma)$ then by the Helmholtz decomposition in Hölder spaces (Proposition 3.7), one finds $\tilde{f} = f_0 + \nabla \Phi$ with $f_0 \in C^{\gamma, \gamma/2}(\bar{U})$ satisfying (3.4) and

$$|f_0|_{(\gamma)} + |\nabla \Phi|_{(\gamma)} \leq C'_H |\tilde{f}|_{(\gamma+\varepsilon)}, \quad (3.15)$$

where we use a short hand notation $|f|_{(\gamma)} = |f|_U^{(\gamma, \gamma/2)}$. If we set $\tilde{p} = p - \Phi$, then (\tilde{u}, \tilde{p}) solves (3.8)-(3.11) with $\Omega = \Omega'$, where f_0 satisfies the solenoidal condition (3.4). Applying the Schauder estimate (3.12) yields

$$|\tilde{u}|_{(2+\gamma)} + |\nabla \tilde{p}|_{(\gamma)} \leq C_S |f_0|_{(\gamma)}. \quad (3.16)$$

By definition of \tilde{f} we observe that

$$\begin{aligned} |\tilde{f}|_{(\gamma+\varepsilon)} &\leq |f|_{(\gamma+\varepsilon)} + |u_t^*|_{(\gamma+\varepsilon)} + |\Delta u^*|_{(\gamma+\varepsilon)} \\ &\leq c_0 \left(|v|_{\Omega' \times (\frac{\delta}{2}, T]}^{(\gamma+\varepsilon, \frac{\gamma+\varepsilon}{2})} + |\nabla v|_{\Omega' \times (\frac{\delta}{2}, T]}^{(\gamma+\varepsilon, \frac{\gamma+\varepsilon}{2})} + |q|_{\Omega' \times (\frac{\delta}{2}, T]}^{(\gamma+\varepsilon, \frac{\gamma+\varepsilon}{2})} \right) + |u^*|_{(2+\gamma+\varepsilon)} \end{aligned}$$

with c_0 depends only on R, T, δ and $\gamma + \varepsilon$. Since N_T in (3.2) is finite, by an interpolation inequality as in the proof of Lemma 3.1 we have $|\nabla v|_{t, Q_\delta}^{(1/2)} \leq CN_T / \delta$ with C depending only on (α, β, K) . We now apply this estimate together with estimate (3.3) for q in Lemma 3.5 to get

$$|\tilde{f}|_{(\gamma+\varepsilon)} \leq CN_T + |u^*|_{(2+\gamma+\varepsilon)} \quad (3.17)$$

with a constant $C = C(M(\Omega), \gamma + \varepsilon, \alpha, \beta, K, R, \delta)$. Since

$$\begin{aligned} |v|_{Q'}^{(2+\gamma, 1+\gamma/2)} &\leq |u|_{(2+\gamma)} \leq |\tilde{u}|_{(2+\gamma)} + |u^*|_{(2+\gamma)} \\ |\nabla q|_{Q'}^{(\gamma, \gamma/2)} &\leq |\nabla \tilde{p}|_{(\gamma)} + |\nabla \Phi|_{(\gamma)}, \end{aligned}$$

the desired estimates follow from (3.15)-(3.17) once we have established that

$$|u^*|_{(2+\gamma+\varepsilon)} \leq CN_T.$$

with $C = C(M(\Omega), \gamma + \varepsilon, \alpha, \beta, K, R, \delta)$.

We shall present a proof for

$$[u_t^*]_{t, U}^{(\mu/2)} \leq CN_T \quad (3.18)$$

for $\mu \in (0, 1)$ since other quantities can be estimated in a similar way and even easier. By (3.13) and (3.14) we have

$$\|u_t^*\|_{L^p(\Omega')} \leq C_B \|\operatorname{div} u_t\|_{W_0^{-1, p}(\Omega')} \quad (3.19)$$

$$\|u_t^*\|_{W^{1, p}(\Omega')} \leq C_B \|\operatorname{div} u_t\|_{L^p(\Omega')}. \quad (3.20)$$

To estimate $\|\operatorname{div} u_t\|_{W_0^{-1,p}(\Omega')}$ we use the equations $v_t - \Delta v + \nabla q = 0$ and $\operatorname{div} v = 0$. For an arbitrary $\varphi \in W^{1,p'}(\Omega')$ we have

$$\begin{aligned} \int_{\Omega'} \varphi \operatorname{div} u_t \, dx &= \int_{\Omega'} (\varphi v_t \cdot \nabla \xi + \varphi \nabla \xi_t \cdot v) \, dx \\ &= \int_{\Omega'} (\varphi \nabla \xi \cdot (\Delta v - \nabla q) + \varphi \nabla \xi_t \cdot v) \, dx \\ &= \int_{\Omega'} \left\{ - \sum_{i=1}^n \partial_{x_i}(\varphi \nabla \xi) \cdot \partial_{x_i} v + q \operatorname{div}(\varphi \nabla \xi) + \varphi \nabla \xi_t \cdot v \right\} \, dx \\ &\quad + \int_{\partial\Omega'} \{ \varphi \nabla \xi \cdot \partial v / \partial n_{\Omega'} - q \varphi \partial \xi / \partial n_{\Omega'} \} \, d\mathcal{H}^{n-1}. \end{aligned}$$

This implies

$$\left| \int_{\Omega'} \varphi \operatorname{div} u_t \, dx \right| \leq C_\xi \left\{ \|\nabla v\|_\infty + \|q\|_\infty + \|v\|_\infty \right\} \left(\|\varphi\|_{W^{1,1}(\Omega')} + \|\varphi\|_{L^1(\partial\Omega')} \right) \quad (3.21)$$

with C_ξ depending only on R and δ (independent of t), where L^∞ -norm is taken on Ω' . By a trace theorem (e.g. [13, 5.5, Theorem 1]) there is a constant C (depending only on Lipschitz regularity of the domain) such that

$$\|\varphi\|_{L^1(\partial\Omega')} \leq C \|\varphi\|_{W^{1,1}(\Omega')}.$$

By the Hölder inequality $\|\varphi\|_{W^{1,1}(\Omega')} \leq C' \|\varphi\|_{W^{1,p}(\Omega')}$ with C' depending on the volume of Ω' . Thus (3.21) yields

$$\|\operatorname{div} u_t\|_{W_0^{-1,p}(\Omega')} \leq C_0 (\|\nabla v\|_\infty + \|q\|_\infty + \|v\|_\infty)$$

with C_0 depending only on δ , R and Ω' through its (α, β, K) . By (3.19) this yields

$$\|u_t^*\|_{L^p(\Omega')} \leq C_B C_0 (\|\nabla v\|_\infty + \|q\|_\infty + \|v\|_\infty). \quad (3.22)$$

We next estimate $\|u_t^*\|_{W^{1,p}}$. By (3.20) a direct computation shows that

$$\|u_t^*\|_{W^{1,p}(\Omega')} \leq C_0 C_B (\|v\|_\infty + \|v_t\|_\infty) \quad (3.23)$$

since $\operatorname{div} u_t = \operatorname{div} \partial_t(\xi v) = \partial_t(\nabla \xi \cdot v)$ by $\operatorname{div} v = 0$.

We now apply the Gagliardo-Nirenberg inequality (e.g. [25])

$$\|u_t^*\|_\infty \leq c \|u_t^*\|_{L^p(\Omega')}^{1-\sigma} \|u_t^*\|_{W^{1,p}(\Omega')}^\sigma, \quad \sigma = n/p$$

to (3.22) and (3.23) to get

$$\|u_t^*\|_\infty \leq C_1 C_B (\|v\|_\infty + \|v_t\|_\infty)^\sigma (\|\nabla v\|_\infty + \|v\|_\infty + \|q\|_\infty)^{1-\sigma}$$

with C_1 depending only on δ , R and Ω' through its (α, β, K) . We replace u^* by $u^*(\cdot, t) - u^*(\cdot, s)$ and observe that

$$\begin{aligned} \|u_t^*(\cdot, t) - u_t^*(\cdot, s)\|_\infty &\leq C_1 C_B (\|\nabla v(\cdot, t) - \nabla v(\cdot, s)\|_\infty + \|q(\cdot, t) - q(\cdot, s)\|_\infty \\ &\quad + \|v(\cdot, t) - v(\cdot, s)\|_\infty)^{1-\sigma} (2N_T/t \wedge s)^\sigma, \quad t, s > 0, \end{aligned} \quad (3.24)$$

where $t \wedge s = \min(t, s)$. As observed in the end of the proof of Lemma 3.1, we have

$$[\nabla v]_{t, Q_\delta}^{(1/2)} \leq CN_T/\delta.$$

By (3.3) we now conclude that

$$\sup_{x \in \Omega'} [\nabla v]_{t, \Omega' \times (\frac{\delta}{2}, T]}^{(\mu')} + \sup_{x \in \Omega'} [q]_{t, \Omega' \times (\frac{\delta}{2}, T]}^{(\mu')} \leq CN_T/\delta, \quad \mu' = \frac{\mu}{2(1-\sigma)}$$

provided that $\mu' < 1/2$ (i.e. $p > n/(1-\mu)$). Dividing both sides of (3.24) by $|t-s|^{\mu/2}$ and take the supremum for $s, t \geq \delta/2$ to get (3.18) since $u^* = 0$ for $t \leq \delta/2$. \square

4 Uniqueness for the Stokes equations in a half space

The goal of this section is to establish a uniqueness theorem for the Stokes equations in a half space $\mathbf{R}_+^n = \{(x', x_n) | x_n > 0\}$ to characterize the limit of rescaled limits in our blow-up argument. The result presented below is by no means optimal but convenient to apply.

Theorem 4.1 (Uniqueness). *Assume that (v, q) satisfies*

$$v \in C(\bar{\mathbf{R}}_+^n \times (0, T)) \cap C^{2,1}(\mathbf{R}_+^n \times (0, T)), \quad \nabla q \in C(\mathbf{R}_+^n \times (0, T)) \quad (4.1)$$

and

$$\int_0^T \int_{\mathbf{R}_+^n} \{v \cdot (\varphi_t + \Delta \varphi) - \varphi \cdot \nabla q\} \, dx dt = 0 \quad (4.2)$$

for all $\varphi \in C_c^\infty(\mathbf{R}_+^n \times [0, T])$ with (1.2), (1.3) for $\Omega = \mathbf{R}_+^n$. Assume that

$$\sup_{0 < t < T} \|N(v, q)\|_\infty(t) < \infty \quad (4.3)$$

and

$$\sup_{\substack{x \in \mathbf{R}_+^n \\ 0 < t < T}} t^{1/2} x_n |\nabla q(x, t)| < \infty. \quad (4.4)$$

Then $v \equiv 0$ and $\nabla q \equiv 0$.

Remark 4.2. Without decay condition (4.4) for the pressure gradient there is a non-trivial solution. In fact, let $v^i = v^i(x_n, t)$ be the solution of the heat equation

$$\begin{aligned} v_t^i - \partial_{x_n}^2 v^i &= a^i \text{ in } \{x_n > 0\} \times (0, T) \\ v^i &= 0 \text{ on } \{x_n = 0\} \\ v^i &= 0 \text{ on } \{t = 0\} \end{aligned}$$

for $i = 1, \dots, n-1$ with $a^i \in C^1[0, T]$ (independent of x). We set $v = (v^1, \dots, v^{n-1}, 0)$ and $q(x, t) = -\sum_{i=1}^{n-1} a^i(t)x_i$. Then (v, q) solves the Stokes equations (1.1)-(1.4) with $\Omega = \mathbf{R}_+^n$ and $v_0 = 0$. It fulfills (4.3) but it does not satisfies (4.4). This is a nontrivial solution unless $a^i \equiv 0$ for all $i = 1, \dots, n-1$. Note that (4.2) is fulfilled for this (v, q) since (v, q) satisfies (1.1)-(1.4) with $v_0 = 0$. So this example shows that the uniqueness of Theorem 4.1 is no longer true without (4.4).

This result is easily reduced to the uniqueness theorem essentially due to Solonnikov [56]. Although it is stated in a different way [56, Theorem 1.1], his proof based on the duality argument (proving the solvability of the dual problem) yields the following uniqueness result (Lemma 4.3). Note that for a half space the Stokes semigroup is not bounded in L^1 (for each $t > 0$) [12] although the derivative fulfills usual regularizing effect $\|\nabla S(t)v_0\|_{L^1(\mathbf{R}_+^n)} \leq Ct^{-1/2}\|v_0\|_{L^1(\mathbf{R}_+^n)}$ as proved in [32].

Lemma 4.3. *Assume that (v, q) satisfies (4.1)-(4.2) and (1.2)-(1.3) with $\Omega = \mathbf{R}_+^n$. Assume that*

$$\sup_{\delta < t < T} \|N(v, q)\|_{\infty}(t) < \infty \quad (4.5)$$

for any $\delta \in (0, T)$. Assume that $|\nabla q(x, t)| \rightarrow 0$ as $x_n \rightarrow \infty$ for $t \in (0, T)$. If $v(\cdot, t)$ converges $*$ -weakly to 0 in $L^\infty(\mathbf{R}_+^n)$ as $t \downarrow 0$, then $v \equiv 0$, $\nabla q \equiv 0$.

Proof of Theorem 4.1. To apply this uniqueness result it suffices to prove that

$$v(\cdot, t) \rightarrow 0 \quad (*\text{-weakly in } L^\infty) \text{ as } t \downarrow 0.$$

Since (v, q) solves (1.1), multiplying $\varphi \in C_c^\infty(\mathbf{R}_+^n \times [0, T])$ and integration by parts yield

$$\int_\delta^T \int_{\mathbf{R}_+^n} \{v \cdot (\varphi_t + \Delta\varphi) - \varphi \nabla q\} \, dx dt + \int_{\mathbf{R}_+^n} v(x, \delta) \varphi(x, \delta) dx = 0.$$

By (4.2) we easily observe that

$$\int_{\mathbf{R}_+^n} v(x, \delta) \varphi(x, \delta) dx \rightarrow 0$$

as $\delta \rightarrow 0$. In particular, $\int_{\mathbf{R}_+^n} v(x, \delta) \psi dx \rightarrow 0$ for all $\psi \in C_c^\infty(\mathbf{R}_+^n)$. Since v is bounded by (4.3) and $C_c^\infty(\mathbf{R}_+^n)$ is dense in $L^1(\mathbf{R}_+^n)$, this implies $v(\cdot, t) \rightarrow 0$ ($*$ -weakly in L^∞). \square

Remark 4.4. (i) The continuity assumption (in Theorem 4.1 and Lemma 4.3) $v \in C(\bar{\mathbf{R}}_+^n \times (0, T))$ in (4.1) is redundant if one assumes (4.3) or (4.5).

(ii) Without the decay condition on the pressure gradient ∇q as $x_n \rightarrow \infty$, one still concludes that v depends only on x_n and t ; see [56, proof of Theorem 1.1]. Since $\operatorname{div} v = 0$ and v vanishes on the boundary, this implies that the normal component v^n (of v) vanishes identically so that $\partial q / \partial x_n = 0$. Thus v^i ($1 \leq i \leq n - 1$) solves the heat equation with a spatially constant external source term a^i which agrees with the counterexample for uniqueness without decay of ∇q as $x_n \rightarrow \infty$. This observation shows that to establish uniqueness it suffices to assume the decay of $\partial q / \partial x_j$ ($j = 1, \dots, n - 1$) as $x_n \rightarrow \infty$.

We conclude this section by giving a uniqueness result for the heat equation which is very easy to prove.

Lemma 4.5. *Assume that $u \in L_{loc}^1(\mathbf{R}^n \times [0, T])$ satisfies*

$$\int_0^T \int_{\mathbf{R}^n} u(x, t) (\varphi_t(x, t) + \Delta \varphi(x, t)) dx dt = 0 \quad (4.6)$$

for all $\varphi \in C_c^\infty(\mathbf{R}^n \times [0, T])$. Assume that

$$\sup_{t \in (0, T)} \|u\|_\infty(t) < \infty. \quad (4.7)$$

Then $u \equiv 0$.

Proof. We prove this statement by a duality argument. We first observe that (4.6) holds for

$$\psi \in C^\infty(\mathbf{R}^n \times [0, T]) \text{ with } \psi, \nabla \psi, \nabla^2 \psi, \psi_t \in L^1(\mathbf{R}^n \times [0, T]) \quad (4.8)$$

and $\operatorname{spt} \psi \subset \mathbf{R}^n \times [0, T]$. This is easily proved by setting $\varphi = \theta_R \psi$ in (4.6) and by sending $R \rightarrow \infty$, where θ_R is a cut-off function defined in the proof of Theorem 3.4. The procedure is justified by (4.7).

For an arbitrary $f \in C_c^\infty(\mathbf{R}^n \times [0, T])$ we solve

$$\begin{aligned} \psi_t + \Delta \psi &= f \quad \text{in } \mathbf{R}^n \times [0, T), \\ \psi(x, T) &= 0 \quad \text{for } x \in \mathbf{R}^n. \end{aligned}$$

It is not difficult to see that $\psi \in C^\infty(\mathbf{R}^n \times [0, T])$ satisfies (4.8) so we conclude that

$$\int_0^T \int_{\mathbf{R}^n} u f dx dt = 0$$

for all $f \in C_c^\infty(\mathbf{R}^n \times [0, T])$. This implies that $u \equiv 0$. \square

5 Blow-up arguments - a priori L^∞ estimates

In this section we shall prove Theorem 1.2 by a blow-up argument. We then derive Theorem 1.3 which deduces Theorem 1.1 since a bounded domain is admissible (Theorem 2.5).

5.1 A priori estimates under stronger regularity assumption

Proposition 5.1. *The assertion of Theorem 1.2 holds under extra restriction that $v(\cdot, t) \in C^2(\bar{\Omega})$ for $t \in (0, 1)$ and $\|N(v, q)\|_\infty(t)$ is bounded in $(0, 1)$ as a function of t .*

Proof. We argue by contradiction. Suppose that (1.15) were false for any choice of T_0 and C . Then there would exist an \tilde{L}^r -solution (v_m, q_m) of (1.1)-(1.4) with $v_0 = v_{0m} \in C_{c,\sigma}^\infty(\Omega)$ and sequence $\tau_m \downarrow 0$ (as $m \rightarrow \infty$) such that $\|N(v_m, q_m)\|_\infty(\tau_m) > m\|v_{0m}\|_\infty$. There is $t_m \in (0, \tau_m)$ such that

$$\|N(v_m, q_m)\|_\infty(t_m) \geq \frac{1}{2}M_m, \quad M_m = \sup_{0 < t < \tau_m} \|N(v_m, q_m)\|_\infty(t).$$

Note that thanks to our extra assumption M_m is finite. We normalize v_m, q_m by defining $\tilde{v}_m = v_m/M_m, \tilde{q}_m = q_m/M_m$. Then $(\tilde{v}_m, \tilde{q}_m)$ enjoys estimates (1.7)-(1.9). Since $(\tilde{v}_m, \tilde{q}_m)$ is an \tilde{L}^r -solution, we have $\nabla \tilde{q}_m = Q[\Delta \tilde{v}_m]$. Since Ω is admissible, (1.7) implies that there is a dilation and translation invariant constant C_Ω independent of m such that

$$\sup\{t^{1/2}d_\Omega(x)|\nabla \tilde{q}_m(x, t)| \mid x \in \Omega_m, t \in (0, t_m)\} \leq C_\Omega. \quad (5.1)$$

Here we have invoked the assumption $v(\cdot, t) \in C^2(\bar{\Omega})$ to apply the estimate for Q . We rescale $(\tilde{v}_m, \tilde{q}_m)$ around a point $x_m \in \Omega$ satisfying (1.10) to get a blow-up sequence (u_m, p_m) of the form

$$u_m(x, t) = \tilde{v}_m(x_m + t_m^{1/2}x, t_m t), \quad p_m(x, t) = t_m^{1/2}\tilde{q}_m(x_m + t_m^{1/2}x, t_m t).$$

By the scaling invariance of the Stokes equations (1.1), (1.2) this (u_m, p_m) solves the Stokes equations in a rescaled domain $\Omega_m \times (0, 1]$, where

$$\Omega_m = \{x \in \mathbf{R}^n \mid x = (y - x_m) / t_m^{1/2}, y \in \Omega\}.$$

It follows from (1.7), (5.1) and (1.10) that

$$\sup_{0 < t < 1} \|N(u_m, p_m)\|_{L^\infty(\Omega_m)} \leq 1, \quad (5.2)$$

$$\sup\{t^{1/2}d_{\Omega_m}(x)|\nabla p_m(x, t)| \mid x \in \Omega_m, 0 < t < 1\} \leq C_\Omega, \quad (5.3)$$

$$N(u_m, p_m)(0, 1) \geq 1/4. \quad (5.4)$$

Moreover, for initial data v_{0m} the condition (1.9) implies $\|u_{0m}\|_{L^\infty(\Omega_m)} \rightarrow 0$ (as $m \rightarrow \infty$). The situation is divided into two cases depending on whether or not

$$c_m = d_\Omega(x_m)/t_m^{1/2}$$

tends to infinity as $m \rightarrow \infty$. This c_m is the distance from zero to $\partial\Omega_m$, i.e. $c_m = d_{\Omega_m}(0)$.

Case 1. $\overline{\lim}_{m \rightarrow \infty} c_m = \infty$. We may assume that $\lim_{m \rightarrow \infty} c_m = \infty$ by taking a subsequence. In this case the rescaled domain Ω_m expands to \mathbf{R}^n . Thus for any $\varphi \in C_c^\infty(\mathbf{R}^n \times [0, 1])$ the blow-up sequence (u_m, p_m) satisfies

$$\int_0^1 \int_{\mathbf{R}^n} \{u_m(\varphi_t + \Delta\varphi) - \nabla p_m \cdot \varphi\} \, dxdt = - \int_{\mathbf{R}^n} u_m(x, 0)\varphi(x, 0)dx$$

for sufficiently large $m > 0$. By (5.2) and Proposition 3.2 we have a necessary compactness to conclude that there exists a subsequence of solutions still denoted by (u_m, p_m) such that (u_m, p_m) converges to some (u, p) locally uniformly in $\mathbf{R}^n \times (0, 1]$ together with $\nabla u_m, \nabla^2 u_m, u_{mt}, \nabla p_m$. (Note that the constant C in (3.1) is invariant under dilation and translation of Ω so (3.1) for (u_m, p_m) gives equi-continuity of $\nabla^2 u_m, u_{mt}$ and ∇p_m .) Since for each $R > 0$

$$\inf\{d_{\Omega_m}(x) \mid |x| \leq R\} \rightarrow \infty \quad \text{as } m \rightarrow \infty,$$

the estimate (5.3) implies that $\nabla p = 0$. Thus the limit $u \in C(\mathbf{R}^n \times (0, 1])$ solves

$$\int_0^1 \int_{\mathbf{R}^n} u(\varphi_t + \Delta\varphi) \, dxdt = 0$$

for all $\varphi \in C_c^\infty(\mathbf{R}^n \times [0, 1])$ since $\|u_{0m}\|_{L^\infty(\Omega_m)} \rightarrow 0$ as $m \rightarrow \infty$. Since u is bounded by (5.2), applying the uniqueness of the heat equation (Lemma 4.5) we conclude that $u \equiv 0$. However, (5.4) implies $N(u, p)(0, 1) \geq 1/4$ which is a contradiction so Case 1 does not occur.

Case 2. $\overline{\lim}_{m \rightarrow \infty} c_m < \infty$. By taking a subsequence we may assume that c_m converges to some $c_0 \geq 0$. We may also assume that x_m converges to a boundary point $\hat{x} \in \partial\Omega$. By rotation and translation of coordinates we may assume that $\hat{x} = 0$ and that exterior normal $n_\Omega(\hat{x}) = (0, \dots, 0, -1)$. Since Ω is a uniformly C^3 -domain of type (α, β, K) , the domain Ω is represented locally near \hat{x} of the form

$$\Omega_{loc} = \{(x', x_n) \in \mathbf{R}^n \mid h(x') < x_n < h(x') + \beta, |x'| < \alpha\}$$

with a C^3 -function h such that $\nabla' h(0) = 0, h(0) = 0$, where derivatives up to third order of h is bounded by K . If one rescales with respect to x_m, Ω_{loc} is expanded as

$$\Omega_m \text{ loc} = \{(y', y_n) \in \mathbf{R}^n \mid h(t_m^{1/2}y' + x'_m) < t_m^{1/2}y_n + (x_m)_n < h(t_m^{1/2}y' + x'_m) + \beta, |t_m^{1/2}y'| < \alpha\}.$$

Since $d_\Omega(x_m)/(x_m)_n \rightarrow 1$ as $m \rightarrow \infty$ and $x'_m \rightarrow 0$, this domain $\Omega_m \text{ loc}$ converges to

$$\mathbf{R}_{+,-c_0}^n = \{(x', x_n) \in \mathbf{R}^n \mid x_n > -c_0\}.$$

In fact, if one expresses

$$\Omega_m \text{ loc} = \{(y', y_n) \in \mathbf{R}^n \mid h_m(y') < y_n < \beta_m + h_m(y'), |y'| < \alpha_m\}$$

with $\alpha_m = \alpha/t_m^{1/2}$, $\beta_m = \beta/t_m^{1/2}$, $h_m(y') = h(t_m^{1/2}y' + x'_m)/t_m^{1/2} - (x_m)_n/t_m^{1/2}$, then $h_m \rightarrow -c_0$ locally uniformly up to third derivatives and $\alpha_m, \beta_m \rightarrow \infty$. Note that $|\partial_x^\mu h_m|$ for μ , $1 \leq |\mu| \leq 3$ is uniformly bounded by K .

Thus, (u_m, p_m) solves (1.1)-(1.4) in $\Omega_m \text{ loc} \times (0, 1]$. By (5.2) and Theorem 3.4 we have a necessary compactness to conclude that there exists a subsequence (u_m, p_m) converges to some (u, p) locally uniformly in $\bar{\mathbf{R}}_{+,-c_0}^n \times (0, 1]$ together with $\nabla u_m, \nabla^2 u_m, u_{mt}, \nabla p_m$ as interior case. (Note that Ω_m is still of type (α, β, K) which is uniform in m).

Now we observe that the limit (u, p) solves the Stokes equations (1.1)-(1.4) in a half space with zero initial data in a weak sense. In fact, since (u_m, p_m) fulfills

$$\int_0^1 \int_{\mathbf{R}_{+,-c_0}^n} \{u_m(\varphi_t + \Delta\varphi) - \varphi \cdot \nabla p_m\} dxdt = - \int_{\mathbf{R}_{+,-c_0}^n} u_m(x, 0)\varphi(x, 0)dx$$

for all $\varphi \in C_c^\infty(\mathbf{R}_{+,-c_0}^n \times (0, 1))$. We note that (5.2) and (5.3) are inherited to (u, p) , in particular

$$\sup \left\{ t^{1/2}(x_n + c_0) |\nabla p(x, t)| \mid x' \in \mathbf{R}^{n-1}, x_n > -c_0, t \in (0, 1) \right\} \leq C_\Omega.$$

Since the convergence of u_m is up to boundary, the boundary condition is also preserved. We thus apply the uniqueness to the Stokes equations in a half space (Theorem 4.1) to conclude $u \equiv 0$ and $\nabla p \equiv 0$.

However, (5.4) implies $N(u, p)(0, 0) \geq 1/4$ which is a contradiction so Case 2 does not occur neither.

We have thus proved (1.15). □

5.2 Regularity for \tilde{L}^r -solutions

We shall prove that the extra condition for v in Proposition 5.1 can be removed. For example we have

Proposition 5.2. *Let Ω be a uniformly C^3 -domain in \mathbf{R}^n . Let (v, q) be an \tilde{L}^r -solution of (1.1)-(1.4) for $r > n$. Assume that $v_0 \in D(\tilde{A}_r)$, where \tilde{A}_r is the Stokes operator in $\tilde{L}_\sigma^r(\Omega)$, i.e. $-\tilde{A}_r$ is the generator of the Stokes semigroup in $\tilde{L}_\sigma^r(\Omega)$. Then $v(\cdot, t) \in C^2(\bar{\Omega})$ for all $t > 0$. Moreover, for each $T > 0$ we have*

$$\sup_{0 < t < T} \|N(v, q)\|_\infty(t) < \infty. \quad (5.5)$$

Proof. We shall claim a stronger statement

$$\sup_{0 < t < T} \left\{ \|v\|_{W_{\text{ul}}^{1,r}}(t) + t^{1/2} \|\nabla v\|_{W_{\text{ul}}^{1,r}}(t) + t (\|\nabla^2 v\|_{W_{\text{ul}}^{1,r}}(t) + \|\partial_t v\|_{W_{\text{ul}}^{1,r}}(t) + \|\nabla q\|_{W_{\text{ul}}^{1,r}}(t)) \right\} \leq C \|v_0\|_{D(\tilde{A}_r)} \quad (5.6)$$

with $C = C(T, \Omega, r)$. Here $W_{\text{ul}}^{1,r}$ is a uniformly local $W^{1,r}$ space defined by

$$W_{\text{ul}}^{1,r}(\Omega) = \{f \in L_{\text{ul}}^r(\Omega) \mid \nabla f \in L_{\text{ul}}^r(\Omega)\}, \quad \|f\|_{W_{\text{ul}}^{1,r}} = \|f\|_{L_{\text{ul}}^r} + \|\nabla f\|_{L_{\text{ul}}^r}$$

and

$$L_{\text{ul}}^r(\Omega) = \left\{ f \in L_{\text{loc}}^r(\Omega) \mid \|f\|_{L_{\text{ul}}^r} = \sup_{x \in \Omega} \left(\int_{\Omega_{x,R}} |f(y)|^r dy \right)^{1/r} \right\},$$

where $\Omega_{x,R} = \text{int} B_R(x) \cap \Omega$ and R is a fixed positive number. The norm depends on R but the topology defined by the norm is independent of the choice of R . The norm of the domain $D(\tilde{A}_r)$ is defined by

$$\|u\|_{D(\tilde{A}_r)} = \|u\|_{\tilde{L}^r(\Omega)} + \|\tilde{A}_r u\|_{\tilde{L}^r(\Omega)}, \quad \|u\|_{\tilde{L}^r(\Omega)} = \max(\|u\|_{L^r(\Omega)}, \|u\|_{L^2(\Omega)})$$

when $r \geq 2$. As proved in [14], [16], this norm is equivalent to the norm

$$\|u\|_{\tilde{W}^{2,r}(\Omega)} = \sum_{|l| \leq 2} \|\partial_x^l u\|_{\tilde{L}^r(\Omega)}.$$

Note that once we have proved (5.6), the inequality and $v(\cdot, t) \in C^2(\bar{\Omega})$ follows from the Sobolev embedding. (One can even claim that $\nabla^2 v(\cdot, t)$ is Hölder continuous with exponent $\gamma = 1 - n/r$.)

We shall prove (5.6). We first observe that by analyticity of the semigroup $S(t) = e^{-t\tilde{A}_r}$

$$\sup_{0 < t < T} t \|v_t\|_{D(\tilde{A}_r)}(t) \leq C_1 \|v_0\|_{D(\tilde{A}_r)}$$

since $\tilde{A}_r v_t = \tilde{A}_r e^{-t\tilde{A}_r} \tilde{A}_r v_0$. It is easy to see that

$$\sup_{0 < t < T} \|v\|_{D(\tilde{A}_r)}(t) \leq C_2 \|v_0\|_{D(\tilde{A}_r)} \quad (5.7)$$

with C_j depending only on T, Ω and r . Thus we have proved that

$$\sup_{0 < t < T} (\|v\|_{\tilde{W}^{1,r}(\Omega)}(t) + \|\nabla v\|_{\tilde{W}^{1,r}(\Omega)}(t) + t \|v_t\|_{\tilde{W}^{2,r}(\Omega)}(t)) \leq C_3 \|v_0\|_{D(\tilde{A}_r)} \quad (5.8)$$

since $D(\tilde{A}_r)$ -norm and $\tilde{W}^{2,r}$ -norm is equivalent.

To show (5.6) it remains to prove that

$$\sup_{0 < t < T} t (\|\nabla^2 v\|_{W_{\text{ul}}^{1,r}}(t) + \|\nabla q\|_{W_{\text{ul}}^{1,r}}(t)) \leq C_4 \|v_0\|_{D(\tilde{A}_r)}. \quad (5.9)$$

We take R sufficiently small such that $\Omega_{x,3R} \subset U_{\alpha,\beta,h}(x_0)$ for any $x_0 \in \partial\Omega$. We normalize q by taking

$$\hat{q}(x) = q(x) - \frac{1}{|\Omega''|} \int_{\Omega''} q(x) dx, \quad \Omega'' = \Omega_{x_0,3R}.$$

It follows from the Poincaré inequality [13, 5.8.1] that

$$\|\hat{q}\|_{L^r(\Omega'')} \leq c \|\nabla q\|_{L^r(\Omega'')} \quad (5.10)$$

with c independent of x_0 . Since Ω is C^3 and (v, q) solves

$$-\Delta v + \nabla q = -v_t, \quad \operatorname{div} v = 0 \quad \text{in } \Omega''$$

with

$$v = 0 \quad \text{on } \partial\Omega'' \cap \partial\Omega,$$

the local higher regularity theory for elliptic systems (see [20, V]) shows that

$$\|\nabla^3 v\|_{L^r(\Omega')} + \|\nabla^2 q\|_{L^r(\Omega')} \leq C(\|v_t\|_{W^{1,r}(\Omega'')} + \|v\|_{W^{1,r}(\Omega'')} + \|\hat{q}\|_{L^r(\Omega'')})$$

with $\Omega' = \Omega_{x_0,2R}$. Here the dependence with respect to t is suppressed. The last term is estimated by (5.10) so we observe that

$$\|\nabla^3 v\|_{L^r(\Omega')} + \|\nabla^2 q\|_{L^r(\Omega')} \leq C(\|v_t\|_{W^{1,r}(\Omega)} + \|v\|_{W^{1,r}(\Omega)} + \|\nabla q\|_{L^r(\Omega)}) \quad (5.11)$$

with C depending only on Ω , R and r but independent of $x_0 \in \partial\Omega$. If $x_0 \in \Omega$ is taken so that $B_{2R}(x_0) \subset \Omega$, then interior higher regularity theory yields (5.11) with $\Omega' = B_R(x_0)$ (by taking $\Omega'' = B_{2R}(x_0)$). Since Ω is covered by $\Omega_{x_0,2R}$, $x_0 \in \partial\Omega$ and $B_R(x_0)$ with $x_0 \in \Omega$ such that $B_{2R}(x_0) \subset \Omega$, the estimate (5.11) implies that

$$\|\nabla^3 v\|_{L^r_{\text{int}}(\Omega)} + \|\nabla^2 q\|_{L^r_{\text{int}}(\Omega)} \leq C(\|v_t\|_{W^{1,r}(\Omega)} + \|v\|_{W^{1,r}(\Omega)} + \|\nabla q\|_{L^r(\Omega)}). \quad (5.12)$$

Since $\nabla q = Q[\Delta v]$ implies

$$\|\nabla q\|_{\tilde{L}^r(\Omega)} \leq C' \|\Delta v\|_{\tilde{L}^r(\Omega)},$$

with $C' = C'(\Omega, r)$, the estimate (5.12) together with (5.8) now yields (5.9). \square

Proof of Theorem 1.2. Combining Propositions 5.1 and 5.2 yields Theorem 1.2 since $C_{c,\sigma}^\infty(\Omega)$ is included in $D(\tilde{A}_r)$. \square

5.3 Analyticity of the Stokes semigroup in $C_{0,\sigma}$

We shall prove Theorem 1.3. To show C_0 -property of the semigroup we prepare

Proposition 5.3. *Let Ω be a uniformly C^2 -domain in \mathbf{R}^n . Let (v, q) be an \tilde{L}^r -solution of (1.1)-(1.4) with $r > n$ and $v_0 \in D(\tilde{A}_r)$. Then*

$$\lim_{t \downarrow 0} \|v(\cdot, t) - v_0\|_\infty = 0. \quad (5.13)$$

In other words

$$\lim_{t \downarrow 0} \|e^{-t\tilde{A}_r} v_0 - v_0\|_\infty = 0.$$

Proof. By the Gagliardo-Nirenberg inequality we have

$$\|v(t) - v_0\|_{L^\infty(\Omega)} \leq C \|v(t) - v_0\|_{L^r(\Omega)}^{1-\theta} \|v(t) - v_0\|_{W^{1,r}(\Omega)}^\theta \quad (5.14)$$

with $\theta = 1 - n/r$, where $v(t) = v(\cdot, t)$. Since

$$\|f\|_{W^{1,r}(\Omega)} \leq \|f\|_{W^{2,r}(\Omega)} \leq \|f\|_{\tilde{W}^{2,r}(\Omega)} \leq C' \|f\|_{D(\tilde{A}_r)},$$

we have by (5.7) that

$$\|v(t) - v_0\|_{W^{1,r}(\Omega)} \leq C' \left(\|v(t)\|_{D(\tilde{A}_r)} + \|v_0\|_{D(\tilde{A}_r)} \right) \leq C'' \|v_0\|_{D(\tilde{A}_r)} \quad (5.15)$$

Since $e^{-t\tilde{A}_r}$ is strongly continuous in \tilde{L}^r , (5.14) with (5.15) yields (5.13). \square

Proof of Theorem 1.3. By a priori estimate (1.15) the operator $S(t)$ is uniquely extended to a bounded operator $\tilde{S}(t)$ in $C_{0,\sigma}$ at least for a small t , say $t \in [0, T_0)$. Since $S(t)$ is a semigroup in \tilde{L}^r , we have

$$\tilde{S}(t_1)\tilde{S}(t_2) = \tilde{S}(t_1 + t_2) \quad \text{as far as } t_1 + t_2 < T_0. \quad (5.16)$$

We extend $\tilde{S}(t)$ to $t \geq T_0$ by $\tilde{S}(t) = \tilde{S}(t_1) \cdots \tilde{S}(t_m)$ so that $t_i \in (0, T_0)$ and $t_1 + \cdots + t_m = t$. This is well-defined in the sense that $\tilde{S}(t)$ is independent of the division of t by the semigroup property (5.16). Thus we are able to define the Stokes semigroup $\tilde{S}(t)$ for all $t \geq 0$ which we simply write by $S(t)$ (since it agrees with $S(t)$ on $C_{0,\sigma} \cap \tilde{L}^r$.) Our estimate (1.15) is inherited to $S(t)$. Moreover, by the semigroup property, the estimate (1.15) yields $\|S(t)v_0\|_\infty \leq C_T \|v_0\|_\infty$ with C_T independent of $v_0 \in C_{0,\sigma}(\Omega)$ and $t \in (0, T)$ for arbitrary $T > 0$. Since $dS(t)/dt = S(t-s)dS(s)/ds$ for $s \in (0, t)$, the estimate (1.15) together with an L^∞ bound for $S(t)$ yields

$$\sup_{0 < t < T} t \left\| \frac{d}{dt} S(t)v_0 \right\|_\infty \leq C'_T \|v_0\|_\infty$$

with a constant C'_T independent of $v_0 \in C_{0,\sigma}(\Omega)$. This implies that $S(t)$ is an analytic semigroup in $C_{0,\sigma}(\Omega)$.

It remains to prove that $S(t)$ is a C_0 -semigroup in $C_{0,\sigma}(\Omega)$. Since $C_{c,\sigma}^\infty(\Omega)$ is dense in $C_{0,\sigma}(\Omega)$, for each $v_0 \in C_{0,\sigma}(\Omega)$ there is $v_{0m} \in C_{c,\sigma}^\infty(\Omega)$ such that $v_{0m} \rightarrow v_0$ in $L^\infty(\Omega)$. Since $\|S(t)v_0\|_\infty \leq C_T\|v_0\|_\infty$ for $0 < t < T$ we have

$$\begin{aligned} \|S(t)v_0 - v_0\|_\infty &\leq \|S(t)v_0 - S(t)v_{0m}\|_\infty + \|S(t)v_{0m} - v_{0m}\|_\infty + \|v_{0m} - v_0\|_\infty \\ &\leq (C_T + 1)\|v_{0m} - v_0\|_\infty + \|S(t)v_{0m} - v_{0m}\|_\infty. \end{aligned}$$

By Proposition 5.3 sending $t \downarrow 0$ yields

$$\overline{\lim}_{t \downarrow 0} \|S(t)v_0 - v_0\|_\infty \leq (C_T + 1)\|v_{0m} - v_0\|_\infty.$$

Letting m to infinity we conclude that $S(t)$ is a C_0 -semigroup in $C_{0,\sigma}(\Omega)$. \square

Since a bounded domain is admissible, Theorem 1.3 yields Theorem 1.1.

Remark 5.4. (i) In general, we do not know whether or not $S(t)$ is a bounded analytic semigroup in the sense that

$$\left\| \frac{d}{dt} S(t)v_0 \right\|_\infty \leq \frac{C}{t} \|v_0\|_\infty \quad (5.17)$$

for some C independent of $t > 0$. When Ω is bounded, one can claim such boundedness. In fact, multiplying v with (1.1) and integrating by parts we obtain an energy equality

$$\frac{1}{2} \frac{d}{dt} \|v\|_{L^2}^2(t) + \|\nabla v\|_{L^2}^2(t) = 0.$$

Since Ω is bounded, the Poincaré inequality implies

$$\|\nabla v\|_{L^2}^2 \geq \nu \|v\|_{L^2}^2$$

with some $\nu > 0$. Thus

$$\|S(t)v_0\|_{L^2}^2 \leq e^{-2\nu t} \|v_0\|_{L^2}^2.$$

If Ω is sufficiently smooth, by the Sobolev inequality and the property of the Stokes semigroup in L^2 (see [51, III.2.1]) we have

$$\|S(t)v_0\|_{L^\infty} \leq C_1 \|S(t)v_0\|_{W^{2k,2}} \leq C_2 \|A_2^k S(t)v_0\|_{L^2}$$

for an integer $k > n/4$ with C_j ($j = 1, 2, \dots$) independent of t and $v_0 \in L_\sigma^2(\Omega)$. Since $S(t)$ is analytic semigroup in L_σ^2 , this yields

$$\|S(t)v_0\|_{L^\infty} \leq C_3 \|S(t-1)v_0\|_{L^2} \quad \text{for } t \geq 1.$$

We have thus proved that

$$\|S(t)v_0\|_{L^\infty} \leq C_4 e^{-\nu t} \|v_0\|_{L^2} \leq C_5 e^{-\nu t} \|v_0\|_{L^\infty}, \quad t \geq 1. \quad (5.18)$$

Similarly,

$$\begin{aligned} \left\| \frac{d}{dt} S(t)v_0 \right\|_{L^\infty} &\leq C_1 \left\| \frac{d}{dt} S(t)v_0 \right\|_{W^{2k,2}} \leq C_2 \|A_2^{k+1} S(t)v_0\|_{L^2} \\ &\leq C_6 e^{-\nu t} \|v_0\|_{L^\infty} \quad \text{for } t \geq 1. \end{aligned}$$

Since

$$\left\| \frac{d}{dt} S(t)v_0 \right\|_{\infty} \leq \frac{C_7}{t} \|v_0\|_{\infty} \quad \text{for } t \leq 1,$$

this yields (5.17). Thus $S(t)$ is a bounded analytic semigroup in $C_{0,\sigma}(\Omega)$ and $L^\infty_\sigma(\Omega)$ (see in next the section) when Ω is a smoothly bounded domain. If one uses L^r -theory ($r > n$) instead of L^2 -theory, the result is still valid for a bounded domain with C^3 boundary.

(ii) Since we have (5.18) for $t \geq T_0 > 0$, our a priori estimate (1.15) in particular implies that

$$\|S(t)v_0\|_{\infty} \leq C \|v_0\|_{\infty} \quad \text{for all } t > 0, v_0 \in C_{0,\sigma}(\Omega)$$

with C depending only on Ω when Ω is bounded. This type of results is often called a maximum modulus result which is available in the literature.

The maximum modulus theorem is first stated in [63] when Ω is a bounded, convex domain with smooth boundary for $v_0 \in C_{e,\sigma}^\infty(\Omega)$. Later a full proof is given in [54]. It is extended by [55] for a general bounded domain with C^2 boundary. It is extended by [41] for $v_0 \in C_{0,\sigma}(\Omega)$ but $\partial\Omega$ is assumed to be $C^{2+\gamma}$ with $\gamma \in (0, 1)$.

By our extension to L^∞_σ space in the next section we conclude that

$$\|S(t)v_0\|_{\infty} \leq C \|v_0\|_{\infty}, \quad v_0 \in L^\infty_\sigma(\Omega)$$

for all $t > 0$ with C depending only on Ω when Ω is bounded and of C^3 boundary.

(iii) We are curious that whether our semigroup $S(t)$ is $\pi/2$ -type analytic semigroup (i.e. it is extendable as a holomorphic semigroup in $\text{Re } t > 0$). Our results say that $S(t)$ is an ε -type analytic semigroup for some $\varepsilon > 0$. If we are able to prove (1.15) for $\text{Re } t \in (0, T_0)$ with $|\arg t| < \alpha$ for $\alpha \in (0, \pi/2)$ where analyticity is valid, then we conclude that $S(t)$ is $\pi/2$ -analytic semigroup. This idea would work provided that the Schauder type estimate for complex t with $|\arg t| < \varepsilon$ would be available. It is of course likely but there seems to be no explicit reference.

(iv) A closer examination of the proof of Proposition 5.1 shows that it suffices to apply an estimate

$$\sup_{x \in \Omega} d_\Omega(x) |Q[\nabla \cdot f](x)| \leq C \|f\|_{L^\infty(\Omega)}$$

which is weaker than (1.13) in the sense that the norm in the right hand side is over Ω not only over $\partial\Omega$.

6 Results for L_σ^∞

In this section we shall prove that the Stokes semigroup is a (non C_0 -)analytic semigroup in $L_\sigma^\infty(\Omega)$ when Ω is bounded as stated in Theorem 1.4.

6.1 Approximation

We begin with an approximation result when Ω is star-shaped (with respect to some point $a \in \mathbf{R}^n$, i.e. $\lambda(\Omega - a) \subset \Omega - a$ for all $\lambda \in (0, 1)$).

Lemma 6.1 (Approximation). *Let Ω be a bounded, star-shaped domain in \mathbf{R}^n . There exists a constant $C = C_\Omega$ such that for any $v \in L_\sigma^\infty(\Omega)$ there exists a sequence $\{v_m\}_{m=1}^\infty \subset C_{c,\sigma}^\infty(\Omega)$ such that*

$$\|v_m\|_\infty \leq C \|v\|_\infty \quad (6.1)$$

and

$$v_m \rightarrow v \quad \text{a.e. in } \Omega \quad (6.2)$$

as $m \rightarrow \infty$. If in addition $v \in C(\bar{\Omega})$, the convergence is locally uniform in Ω . If in addition $v = 0$ on $\partial\Omega$, the convergence is uniform in $\bar{\Omega}$.

Proof. Since Ω is star-shaped, we may assume that

$$\lambda\bar{\Omega} \subset \Omega \quad \text{for all } \lambda \in [0, 1)$$

by a translation. We extend that $v \in L_\sigma^\infty(\Omega)$ by zero outside Ω and observe that the extension (still denoted by v) is in $L_\sigma^\infty(\mathbf{R}^n)$ with $\text{spt } v \subset \bar{\Omega}$. We set $v_\lambda(x) = v(x/\lambda)$ and observe that $\text{spt } v_\lambda \subset \lambda\bar{\Omega} \subset \Omega$. Since $v_\lambda \rightarrow v$ a.e. as $\lambda \uparrow 1$, it is easy to find the desired sequence by mollifying v_λ i.e. $v_\lambda * \eta_\varepsilon$. Here C in (6.1) can be taken 1. \square

To establish the above approximation result for a general bounded domain we need a localization lemma.

Lemma 6.2 (Localization). *Let Ω be a bounded domain with Lipschitz boundary in \mathbf{R}^n . Let $\{G_k\}_{k=1}^N$ be an open covering of $\bar{\Omega}$ in \mathbf{R}^n and $\Omega_k = G_k \cap \Omega$. Then there exists a family of bounded linear operators $\{\pi_k\}_{k=1}^N$ from $L_\sigma^\infty(\Omega)$ into itself satisfying $u = \sum_{k=1}^N \pi_k u$ and for each $k = 1, \dots, N$*

- (i) $\pi_k u|_{\Omega_k} \in L_\sigma^\infty(\Omega_k)$, $\pi_k u|_{\Omega \setminus \Omega_k} = 0$ for $u \in L_\sigma^\infty(\Omega)$,
- (ii) $\pi_k u \in C(\bar{\Omega}_k)$ and $\pi_k u|_{\partial\Omega_k \setminus \partial\Omega} = 0$ for $u \in C(\bar{\Omega}) \cap L_\sigma^\infty(\Omega)$,
- (iii) $\pi_k u|_{\partial\Omega_k} = 0$ if $u|_{\partial\Omega_k} = 0$ for $u \in C(\bar{\Omega}) \cap L_\sigma^\infty(\Omega)$.

Proof. We shall prove by induction with respect to N . If $N = 1$, the result is trivial by taking π_1 as the identity.

Assume that the result is valid for N . We shall prove the assertion when the number of cover is $N + 1$. We set

$$D = \bigcup_{k=2}^{N+1} \Omega_k, \quad U = \bigcup_{k=2}^{N+1} G_k$$

and observe that $\Omega = \Omega_1 \cup D$ and $\{G_1, U\}$ is a covering of $\bar{\Omega}$.

Let $\{\xi_1, \xi_2\}$ be a partition of unity of Ω associated with $\{G, U\}$, i.e. $\xi_j \in C_c^\infty(\mathbf{R}^n)$ with $0 \leq \xi_j \leq 1$, $\text{spt } \xi_1 \subset G_1$, $\text{spt } \xi_2 \subset U$, $\xi_1 + \xi_2 = 1$ in $\bar{\Omega}$. For $E = \Omega_1 \cap D$ let B_E denotes the Bogovskii operator. We set

$$\pi_1 u = \begin{cases} u \xi_1 - B_E(u \cdot \nabla \xi_1) & \text{in } E, \\ u \xi_1 & \text{in } \Omega_1 \setminus D, \\ 0 & \text{in } \Omega \setminus \Omega_1. \end{cases}$$

Since $u \in L_\sigma^\infty(\Omega)$ and $\xi_1 = 0$ in $\Omega \setminus \Omega_1$, $\nabla \xi_1 = 0$ in $\Omega_1 \setminus D$, we see

$$\int_E u \cdot \nabla \xi_1 dx = \int_\Omega u \cdot \nabla \xi_1 dx = 0. \quad (6.3)$$

By the Sobolev inequality and (3.13) we observe that

$$\begin{aligned} \|B_E(u \cdot \nabla \xi_1)\|_{L^\infty(E)} &\leq C \|B_E(u \cdot \nabla \xi_1)\|_{W^{1,p}(E)} \quad (p > n) \\ &\leq CC_B \|u \cdot \nabla \xi_1\|_{L^p(E)} \leq CC_B \|\nabla \xi_1\|_{L^p(E)} \|u\|_{L^\infty(\Omega)} \end{aligned}$$

with a constant C independent of u and ξ_1 . We thus observe that

$$\|\pi_1 u\|_{L^\infty(\Omega_1)} \leq C_1 \|u\|_{L^\infty(\Omega)} \text{ for all } u \in L_\sigma^\infty(\Omega)$$

with C_1 independent of u .

By (6.3) we see $\text{div} B_E(u \cdot \nabla \xi_1) = u \cdot \nabla \xi_1$ in E . Moreover, $B_E(u \cdot \nabla \xi_1) = 0$ on $\partial(\Omega_1 \cap D)$. Thus for each $\varphi \in L_{loc}^1(\bar{\Omega}_1)$ with $\nabla \varphi \in L^1(\Omega_1)$ we have

$$\begin{aligned} \int_{\Omega_1} \pi_1 u \cdot \nabla \varphi dx &= \int_{\Omega_1} u \xi_1 \cdot \nabla \varphi dx - \int_E B_E(u \cdot \nabla \xi_1) \cdot \nabla \varphi dx \\ &= \int_{\Omega_1} u \xi_1 \cdot \nabla \varphi dx + \int_E (u \cdot \nabla \xi_1) \varphi dx \\ &= \int_\Omega u \cdot \nabla (\xi_1 \varphi) dx = 0. \end{aligned}$$

By the Poincaré inequality if $\varphi \in \hat{W}^{1,1}(\Omega_1)$ then $\varphi \in L_{loc}^1(\bar{\Omega}_1)$ (not only $\varphi \in L_{loc}^1(\Omega_1)$). Thus the above identity implies that $\pi_1 u|_{\Omega_1} \in L_\sigma^\infty(\Omega_1)$. By definition $\pi_1 u = 0$ in $\Omega \setminus \Omega_1$. If $u \in C(\bar{\Omega})$, it is easy to see that the term $B_E(u \cdot \nabla \xi_1)$ is always Hölder continuous by the Sobolev embeddings.

For $u \in L_\sigma^\infty(\Omega)$ we set

$$\pi_D u = \begin{cases} u \xi_2 - B_E(u \cdot \nabla \xi_2) & \text{in } E, \\ u \xi_2 & \text{in } D \setminus \Omega_1, \\ 0 & \text{in } \Omega \setminus D. \end{cases}$$

By definition

$$u = \pi_1 u + \pi_D u$$

and as for π_1 this π_D satisfies all properties of π_k in (i), (ii), (iii) with Ω_k replaced by D . Since \bar{D} is covered by $\{G_k\}_{k=2}^{N+1}$, by our induction assumption there is a bounded linear operator $\{\hat{\pi}_k\}_{k=2}^{N+2}$ in $L_\sigma^\infty(D)$ satisfying $v = \sum_{k=2}^{N+1} \hat{\pi}_k v$ and (i), (ii), (iii) with u replaced by v and with π_k replaced by $\hat{\pi}_k$ for $k = 2, \dots, N+1$. If we set

$$\pi_1 = \pi_1, \quad \pi_k = \hat{\pi}_k \cdot \pi_D \quad (k = 2, \dots, N+1),$$

then it is rather clear that this π_k satisfies all desired properties. \square

Lemma 6.3 (Approximation). *The assertion of Lemma 6.1 is still valid when Ω is a bounded domain with Lipschitz boundary in \mathbf{R}^n .*

Proof. If Ω is a bounded domain with Lipschitz boundary, then it is known that there is an open covering $\{G_k\}_{k=1}^N$ of $\bar{\Omega}$ such that $\Omega_k = G_k \cap \Omega$ is bounded, star-shaped with respect to an open ball $B_k(\bar{B}_k \subset \Omega)$ (i.e. star-shaped with respect to any point of B_k) and G_k has a Lipschitz boundary; see [20, III.3, Lemma 4.3]. In the sequel we only need the property that G_k is bounded and star-shaped with respect to a point.

We apply Lemma 6.2 and set $u_k = \pi_k u$ to observe that $u_k|_{\Omega_k} \in L_\sigma^\infty(\Omega_k)$ and $u_k|_{\Omega \setminus \Omega_k} = 0$. Since Ω_k is star-shaped, by Lemma 6.1 there is $\{u_{k,j}\}_{j=1}^\infty \subset C_{c,\sigma}^\infty(\Omega_k)$ such that

$$\|u_{k,j}\|_{L^\infty(\Omega_k)} \leq \|u_k\|_{L^\infty(\Omega_k)}, \quad u_{k,j} \rightarrow u_k \text{ a.e. in } \Omega.$$

(The constant C in (6.1) can be taken 1.) We still denote the zero extension of $u_{k,j}$ on $\Omega \setminus \Omega_k$ by $u_{k,j}$.

If we set $u_m = \sum_{k=1}^N u_{k,m}$, by construction $u_j \in C_{c,\sigma}^\infty(\Omega)$ and

$$u_m \rightarrow \sum_{k=1}^N u_k = u \quad \text{a.e. in } \Omega \text{ and}$$

$$\|u_m\|_{L^\infty(\Omega)} \leq \sum_{k=1}^N \|u_{k,m}\|_{L^\infty(\Omega)} \leq \sum_{k=1}^N \|u_k\|_{L^\infty(\Omega)} \leq \left(\sum_{k=1}^N \|\pi_k\| \right) \|u\|_{L^\infty(\Omega)},$$

where $\|\pi_k\|$ denotes the operator norm of π_k in $L_\sigma^\infty(\Omega)$. We thus conclude that there is a desired approximate sequence $\{u_m\}_{m=1}^\infty$ for $u \in L_\sigma^\infty(\Omega)$.

If $u \in C(\bar{\Omega})$ ($\cap L_\sigma^\infty(\Omega)$), then $u_k \in C(\bar{\Omega}_k)$ and $u_k|_{\partial\Omega_k \setminus \partial\Omega} = 0$. Thus for any compact set $K_k \subset \Omega_k$ such that $d_\Omega(K_k) = \inf_{x \in K_k} d_\Omega(x) > 0$ we see that $u_{k,m}$ converges to u_k uniformly in K_k by Lemma 6.1 as $m \rightarrow \infty$. Let K be a compact set in Ω . Then $d(K_k) \geq d(K) > 0$ for $K_k = \bar{\Omega}_k \cap K$. Thus

$$\begin{aligned} \|u - u_m\|_{L^\infty(K)} &\leq \sum_{k=1}^N \|u_k - u_{k,m}\|_{L^\infty(K)} \\ &= \sum_{k=1}^N \|u_k - u_{k,m}\|_{L^\infty(K_k)} \rightarrow 0 \quad (\text{as } m \rightarrow \infty). \end{aligned}$$

Thus we have proved that u_m converges to u locally uniformly in Ω . If $u|_{\partial\Omega} = 0$ so that $u_k|_{\partial\Omega_k} = 0$, then $u_{k,m}$ converges to u_k uniformly in $\bar{\Omega}_k$ by Lemma 6.1. Arguing in the same way by replacing K by $\bar{\Omega}$, we conclude that u_m converges to u uniformly in $\bar{\Omega}$. \square

Remark 6.4. This lemma in particular implies that

$$C_{0,\sigma}(\Omega) = \{v \in C(\bar{\Omega}) \cap L^\infty(\bar{\Omega}) \mid \operatorname{div} v = 0 \text{ in } \Omega, v = 0 \text{ on } \partial\Omega\}$$

when Ω is bounded. This give an alternate and direct proof of a result of [41], where the maximum modulus result for the stationary problem is invoked.

Proof of Theorem 1.4. Since Ω is bounded so that $L_\sigma^\infty \subset L_\sigma^r$ for any $r > 1$, $S(t)$ is well-defined from L_σ^∞ to L_σ^r . It suffices to transfer the estimate for $v = S(t)v_0$ in (1.15) to the case $v_0 \in L_\sigma^\infty(\Omega)$. By Lemma 6.1 there is a sequence $v_{0m} \in C_{c,\sigma}^\infty(\Omega)$ approximating v_0 . Our estimate (1.15) implies that

$$\sup_{0 < t < T_0} \left\{ \|v_m\|_\infty(t) + t(\|v_{mt}\|_\infty + \|\nabla^2 v_m\|_\infty)(t) \right\} \leq C \|v_{0m}\|_\infty$$

is valid for such v_{0m} by Theorem 1.2. Here T_0 and C is independent of m . Since $v_{0m} \rightarrow v_0$ in L^r by (6.2) and the Lebesgue dominated convergence theorem, we see that $v_m \rightarrow v$ in L^r uniformly in $t \in [0, T]$; note that $S(t)$ is a semigroup in L_σ^r . Thus we obtain

$$\sup_{0 < t < T_0} \left\{ \|v\|_\infty(t) + t(\|v_t\|_\infty + \|\nabla^2 v\|_\infty)(t) \right\} \leq C \overline{\lim}_{m \rightarrow \infty} \|v_{0m}\|_\infty.$$

By (6.2) one is able to replace the right hand side by a constant multiple of $\|v_0\|_\infty$, so we obtain the desired estimate for claiming the analyticity of $S(t)$ in $L_\sigma^\infty(\Omega)$.

This semigroup $S(t)$ is a non C_0 -semigroup. Indeed, suppose the contrary to get

$$S(t)v_0 \rightarrow v_0 \quad \text{in } L^\infty \quad (\text{as } t \downarrow 0)$$

for all $v_0 \in L^\infty_\sigma(\Omega)$. Our estimate for $\nabla^2 v$ implies that $S(t)v_0$ ($t > 0$) is at least continuous in $\bar{\Omega}$. However, if $S(t)v_0$ converges uniformly, then v_0 must be continuous which is a contradiction. \square

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