第 19 回　関数空間セミナー報告集

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（会場：北海道大学）

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Seminar on Function Spaces, 2010

CONTENTS

Representing sequences on parabolic Bergman spaces 1
Y. Hishikawa (Gifu National College of Technology)

Backward shifts on function algebras 5
H. Koshimizu (Shinshu University)

Radon-Nikodym Theorem wtih Generalized Density 9
H. Saito (Tokyo University of Science)

A perturbation theory for defect operators on Hilbert function spaces 14
M. Seto (Shimane University)

Oppenheim's inequality and RKHS 18
A. Yamada (Tokyo Gakugei University)

Nonlinear Analytic Methods for Linear Contractive Mappings in Banach Spaces 23
W. Takahashi (Keio University))

Aluthge interactions of weighted translation semigroups 29
M. R. Lee (Kyungpook National University)

Fixed point theorems in a vector lattice 34
T. Kawasaki

On the Fourier transform on $L^2(\mathbb{R})$ 38
Y. Sawano (Kyoto University)

An argument of a function in $H^{1/2}$ 44
T. Yamamoto (Hokkai-Gakuen University)

Banach spaces of harmonic functions on Riemann surfaces 49
M. Nakai (Emeritus Professor at Nagoya Institute of Technology)

Ranks of invariant subspaces of Hardy space over the bidisk 55
K. H. Izuchi (Korea University)

The Lax conjecture and its related topics 60
H. Nakazato (Hirosaki University)

We consider φ-means 66
Y. Nakasuji (The Open University of Japan)
K. Kumahara (The Open University of Japan)
S.-E. Takahashi (Yamagata University)

Fundamental error estimate inequalities for the Tikhonov regularization using reproducing kernels 73
L. P. Castro (University of Aveiro)
H. Fujisawa (Kyoto University)
S. Saitoh (University of Aveiro)
Y. Sawano (Kyoto University)
A. Yamada (Tokyo Gakugei University)
M. Yamada (Gunma University)

Non-α-normal analytic functions with good integrability 79
R. Aulaskari (University of Eastern Finland)
S. Makhmutov (Sultan Qaboos University)
J. Rättyä (University of Eastern Finland)

On unbounded weighted shifts on directed trees 83
I. B. Jung (Kyungpook National University)

On Hahn Banach theorem in a partially ordered vector space and its applications 89
T. Watanabe (Niigata University)

Wandering subspaces and the Beurling type theorem 95

K. J. Izuchi (Niigata University)
K. H. Izuchi (Korea University)
Y. Izuchi

The spectra of Toeplitz operators on the bidisc 101

T. Nakazi (Hokusei Gakuen University)

On quadratically hyponormal weighted shifts 106

S. H. Park (Kyungpook National University)

Representing sequences on parabolic Bergman spaces

Yôsuke Hishikawa (Gifu National College of Technology)

Let H be the upper half-space of the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}(n \geq 1)$, that is, $H = \{(x,t) \in \mathbb{R}^{n+1}; x \in \mathbb{R}^n, t > 0\}$. For $0 < \alpha \leq 1$, the parabolic operator $L^{(\alpha)}$ is defined by

$$L^{(\alpha)} = \partial_t + (-\Delta_x)^\alpha,$$

where $\partial_t = \partial/\partial t$ and Δ_x is the Laplacian with respect to x. Let $C(H)$ be the set of all real-valued continuous functions on H. A function $u \in C(H)$ is said to be $L^{(\alpha)}$-harmonic if $L^{(\alpha)}u = 0$ in the sense of distributions. For $1 \leq p < \infty$ and $\lambda > -1$, $L^p(\lambda)$ is the set of all Lebesgue measurable functions f on H which satisfy

$$\|f\|_{L^p(\lambda)} := \left(\int_H |f(x,t)|^p t^\lambda dV(x,t)\right)^{\frac{1}{p}} < \infty,$$

where dV is the Lebesgue volume measure on H. The parabolic Bergman space $\boldsymbol{b}^p_\alpha(\lambda)$ is the set of all $L^{(\alpha)}$-harmonic functions u on H which belong to $L^p(\lambda)$. We remark that $\boldsymbol{b}^p_\alpha(\lambda)$ is a Banach space with the norm $\|\cdot\|_{L^p(\lambda)}$, and $\boldsymbol{b}^p_{1/2}(\lambda)$ coincides with the harmonic Bergman space of Ramey and Yi [7]. Also, we note that $\boldsymbol{b}^p_\alpha(\lambda) = \{0\}$ when $\lambda \leq -1$ (see Proposition 4.3 of [3]).

Our aim is the study of representing sequences on parabolic Bergman spaces. In [2], we established the reproducing formula on parabolic Bergman spaces by using fractional derivatives of the fundamental solution of the parabolic operator $L^{(\alpha)}$. The reproducing formula of [2] is given by the integral of parabolic Bergman functions (see Theorem A below). In this talk, we present a discrete version of the reproducing formula, which is given by representing sequences.

We give some notations. For a real number κ, let $\mathcal{D}^\kappa_t = (-\partial_t)^\kappa$ be the fractional differential operator, and $W^{(\alpha)}$ the fundamental solution of the parabolic operator $L^{(\alpha)}$. In [2], the following reproducing formula on $\boldsymbol{b}^p_\alpha(\lambda)$ is given.

THEOREM A (Theorem 4.7 of [2]). *Let* $0 < \alpha \leq 1$, $1 \leq p < \infty$, *and* $\lambda > -1$. *And let* $\kappa > \frac{\lambda+1}{p}$ *be a real number. Then the reproducing formula*

$$u(x,t) = C_\kappa \int_H u(x,t)\mathcal{D}^\kappa_t W^{(\alpha)}(x-y, t+s)t^{\kappa-1}dV(y,s) \tag{1.1}$$

holds for all $u \in \boldsymbol{b}^p_\alpha(\lambda)$ *and* $(x,t) \in H$, *where* $C_\kappa = 2^\kappa/\Gamma(\kappa)$ *and* $\Gamma(\cdot)$ *is the gamma function. Moreover, the reproducing formula* (1.1) *also holds when* $p = 1$ *and* $\kappa = \lambda + 1$.

To state our main results, we give some definitions. Let $0 < \alpha \leq 1$, $1 \leq p < \infty$, $\lambda > -1$, and κ be a real number. Furthermore, let $\mathbb{X} = \{(x_j, t_j)\}$ be a sequence in H. For a sequence of real numbers $\{\eta_j\}$, we define a representing operator $U^\kappa_{p,\mathbb{X}}$ by

$$U^\kappa_{p,\mathbb{X}}(\{\eta_j\})(x,t) = \sum_j \eta_j t_j^{\frac{n}{2\alpha}+\kappa-\left(\frac{n}{2\alpha}+\lambda+1\right)\frac{1}{p}} \mathcal{D}^\kappa_t W^{(\alpha)}(x-x_j, t+t_j), \quad (x,t) \in H. \tag{1.2}$$

By using the representing operator $U^{\kappa}_{p,\mathbb{X}}$, we define a representing sequence on $\boldsymbol{b}^p_\alpha(\lambda)$.

DEFINITION 1. Let $0 < \alpha \le 1$, $1 \le p < \infty$, $\lambda > -1$, and κ be a real number. A sequence $\mathbb{X} = \{(x_j, t_j)\}$ in H is called the $\boldsymbol{b}^p_\alpha(\lambda)$–representing sequence of order κ if $U^{\kappa}_{p,\mathbb{X}} : \ell^p \to \boldsymbol{b}^p_\alpha(\lambda)$ is bounded and onto. Explicitly, a sequence $\mathbb{X}$ is called the $\boldsymbol{b}^p_\alpha(\lambda)$–representing sequence of order κ if the following conditions are satisfied.

(1) For $\{\eta_j\} \in \ell^p$, the function $U^{\kappa}_{p,\mathbb{X}}(\{\eta_j\})$ belongs to $\boldsymbol{b}^p_\alpha(\lambda)$, and there exists a constant $C > 0$ such that $\|U^{\kappa}_{p,\mathbb{X}}(\{\eta_j\})\|_{L^p(\lambda)} \le C\|\{\eta_j\}\|_{\ell^p}$ for all $\{\eta_j\} \in \ell^p$.

(2) For $u \in \boldsymbol{b}^p_\alpha(\lambda)$, there exists $\{\eta_j\} \in \ell^p$ such that $u = U^{\kappa}_{p,\mathbb{X}}(\{\eta_j\})$ on H.

For $0 < \delta < 1$ and $(x, t) \in H$, an α-parabolic cylinder $S^{(\alpha)}_\delta$ is defined by

$$S^{(\alpha)}_\delta(x,t) = \left\{ (y,s) \in H; |y - x| < \left(\frac{2\delta}{1-\delta^2}t\right)^{1/2\alpha}, \frac{1-\delta}{1+\delta}t < s < \frac{1+\delta}{1-\delta}t \right\}.$$

A sequence $\{(x_j, t_j)\}$ in H is said to be δ-separated in the α-parabolic sense if $S^{(\alpha)}_\delta(x_j, t_j) \cap S^{(\alpha)}_\delta(x_i, t_i) = \emptyset$ for $j \ne i$. Furthermore, a sequence $\{(x_j, t_j)\}$ in H is said to be a δ-lattice in the α-parabolic sense if $\{(x_j, t_j)\}$ satisfies the following;

(i) $\cup_j S^{(\alpha)}_\delta(x_j, t_j) = H$.

(ii) For some $0 < \varepsilon < \delta$, $\{(x_j, t_j)\}$ is ε-separated in the α-parabolic sense.

The following theorems are the main results.

THEOREM 1. *Let $0 < \alpha \le 1$, $1 < p < \infty$, $\lambda > -1$, and $\kappa > \frac{\lambda+1}{p}$ be a real number. Furthermore, let $\mathbb{X} = \{(x_j, t_j)\}$ be a sequence in H. Then, $U^{\kappa}_{p,\mathbb{X}}$ satisfies the condition (1) of Definition 1 if and only if for any $0 < \delta < 1$, there exists $K \in \mathbb{N}$ such that $\mathbb{X} = \mathbb{X}_1 \cup \cdots \cup \mathbb{X}_K$ and each sequence $\mathbb{X}_i$ is δ-separated in the α-parabolic sense. The " if " part also holds when $p = 1$.*

THEOREM 2. *Let $0 < \alpha \le 1$, $1 \le p < \infty$, $\lambda > -1$, and $\kappa > \frac{\lambda+1}{p}$ be a real number. Then, there exists $0 < \delta_0 < 1$ such that if a sequence $\mathbb{X}$ in H is a δ-lattice in the α-parabolic sense with $0 < \delta \le \delta_0$, then $\mathbb{X}$ is a $\boldsymbol{b}^p_\alpha(\lambda)$-representing sequence of order κ.*

By Theorem 2 and the open mapping theorem, we obtain the following corollary.

COROLLARY 1. *Let $0 < \alpha \le 1$, $1 \le p < \infty$, $\lambda > -1$, and $\kappa > \frac{\lambda+1}{p}$ be a real number. Then, there exists a sequence $\mathbb{X} = \{(x_j, t_j)\}$ in H such that the following properties hold;*

(1) *For $\{\eta_j\} \in \ell^p$, the function $u = U^{\kappa}_{p,\mathbb{X}}(\{\eta_j\})$ belongs to $\boldsymbol{b}^p_\alpha(\lambda)$, and there exists a constant $C > 0$ such that*

$$\|u\|_{L^p(\lambda)} \le C\|\{\eta_j\}\|_{\ell^p}$$

for all $\{\eta_j\} \in \ell^p$.

(2) *For* $u \in b^p_\alpha(\lambda)$, *there exists* $\{\eta_j\} \in \ell^p$ *such that* $u = U^\kappa_{p,\mathbb{X}}(\{\eta_j\})$ *on* H. *Moreover, there exists a constant* $C > 0$ *independent of* u *such that*

$$\|\{\eta_j\}\|_{\ell^p} \le C\|u\|_{L^p(\lambda)}.$$

We present the definition of $b^p_\alpha(\lambda)$–sampling sequences, which are closely related to $b^p_\alpha(\lambda)$–representing sequences. We give the definition of $b^p_\alpha(\lambda)$–sampling sequences. Let $0 < \alpha \le 1$, $1 \le p < \infty$, $\lambda > -1$, $\nu \in \mathbb{R}$, and $\mathbb{X} = \{(x_j, t_j)\}$ be a sequence in H. We say that $\mathbb{X}$ is a $b^p_\alpha(\lambda)$–sampling sequence of order ν if there exists a constant $C > 0$ such that

$$C^{-1}\|u\|^p_{L^p(\lambda)} \le \sum_j t_j^{\frac{n}{2\alpha}+\lambda+1+\nu p}|\mathcal{D}^\nu_t u(x_j,t_j)|^p \le C\|u\|^p_{L^p(\lambda)}$$

for all $u \in b^p_\alpha(\lambda)$.

THEOREM 3. *Let* $0 < \alpha \le 1$, $1 < p < \infty$, $\lambda > -1$, *and* $\kappa > \frac{\lambda+1}{p}$. *Furthermore, let* $\mathbb{X}$ *be a sequence in* H, *and* q *the exponent conjugate to* p. *Then, the following conditions are equivalent*;

(1) $\mathbb{X}$ *is a* $b^p_\alpha(\lambda)$*–representing sequence of order* κ.

(2) $\mathbb{X}$ *is a* $b^q_\alpha(\lambda)$*–sampling sequence of order* $\kappa - (\lambda + 1)$.

We present some lemmas, which are used in the proof of main theorems. Let $0 < \alpha \le 1$, $1 \le p < \infty$, and $\lambda > -1$. Furthermore, let $\gamma \in \mathbb{N}^n_0$, $\nu \in \mathbb{R}$, and $\mathbb{X} = \{(x_j, t_j)\}$ be a sequence in H. For a function u on H, the operator $T^{\gamma,\nu}_{p,\mathbb{X}}$ is defined by

$$T^{\gamma,\nu}_{p,\mathbb{X}}u = \left\{ t_j^{(\frac{n}{2\alpha}+\lambda+1)\frac{1}{p}+\frac{|\gamma|}{2\alpha}+\nu}\partial^\gamma_x \mathcal{D}^\nu_t u(x_j,t_j) \right\}.$$

The following lemma is necessary and sufficient conditions for the operator $T^{\gamma,\nu}_{p,\mathbb{X}}$ to be a bounded operator from $b^p_\alpha(\lambda)$ to ℓ^p.

LEMMA 1. *Let* $0 < \alpha \le 1$, $1 \le p < \infty$, *and* $\lambda > -1$. *Furthermore, let* $\gamma \in \mathbb{N}^n_0$, $\nu > -\frac{\lambda+1}{p}$, *and* $\mathbb{X} = \{(x_j, t_j)\}$ *be a sequence in* H. *Then, the following statements are equivalent* ;

(1) $T^{\gamma,\nu}_{p,\mathbb{X}} : b^p_\alpha(\lambda) \to \ell^p$ *is bounded.*

(2) *For any* $0 < \varepsilon < 1$, *there exists* $K \in \mathbb{N}$ *such that* $\mathbb{X} = \mathbb{X}_1 \cup \cdots \cup \mathbb{X}_K$ *and each sequence* $\mathbb{X}_i$ *is* ε*-separated in the* α*-parabolic sense.*

The following lemma is used in the proof of Theorem 3.

LEMMA 2. *Let* $0 < \alpha \leq 1$, $1 < p < \infty$, $\lambda > -1$, *and* $\kappa > \frac{\lambda+1}{p}$. *Furthermore, let* $\mathbb{X}$ *be a sequence in* H, *and* q *the exponent conjugate to* p. *If* $U^{\kappa}_{p,\mathbb{X}} : \ell^p \to \boldsymbol{b}^p_\alpha(\lambda)$ *is bounded, then*

$$(U^{\kappa}_{p,\mathbb{X}})^* u = \frac{1}{C_{\lambda+1}} T^{\kappa-(\lambda+1)}_{q,\mathbb{X}} u$$

for all $u \in \boldsymbol{b}^q_\alpha(\lambda)$.

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Backward shifts on function algebras

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Abstract

J. R. Holub [1] introduced the concept of backward shift on Banach spaces. In this note, define a backward quasi-shift as a weak type of backward shift. The main result is the following: An infinite-dimensional function algebra does not admit a backward shift. Also, a function algebra A does not admit a backward quasi-shift, under the assumption that the Choquet boundary of A has at most finitely many isolated points. Finally, we give the examples of the space that does not admit a backward shift, the space that admits a backward quasi-shift but not a backward shift, and the space that admits a backward shift.

This is a joint work with H. Ariizumi and H. Takagi.

1 Introduction

Let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space, and T a bounded linear operator on $\mathcal{H}$. We call T a *backward shift* on $\mathcal{H}$, if there is an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ for $\mathcal{H}$ such that $Te_1 = 0$ and $Te_i = e_{i-1}$ for $i = 2, 3, \ldots$. In [1], J. R. Holub characterized a backward shift on $\mathcal{H}$ without using a basis, and he defined a backward shift on a Banach space, as follows:

Definition. Let $\mathcal{B}$ be an infinite-dimensional Banach space and T a bounded linear operator on $\mathcal{B}$. We write $\ker T$ to denote the kernel $\{f \in \mathcal{B} : Tf = 0\}$. We call T a *backward shift* on $\mathcal{B}$ if T satisfies the following three conditions:

(i) $\ker T$ is one-dimensional.

(ii) $\|Tf\| = \inf\{\|f+g\| : g \in \ker T\} \quad (f \in \mathcal{B})$

(iii) $\bigcup_{n=1}^{\infty} \ker T^n$ is dense in $\mathcal{B}$.

Here we assume that $\mathcal{B}$ is infinite-dimensional, because the finite-dimensional case is easily considered. It is known that every backward shift on an infinite-dimensional space is onto (see [2, Proposition 1.2]).

We also consider a weak type of backward shift. We say that T is a *backward quasi-shift* on $\mathcal{B}$, if T satisfies two conditions (i) and (ii) only, and if T is onto.

Holub asked whether there exists a backward shift on various function spaces ([1]). M. Rajagopalan and K. Sundaresan gave the answer about the Banach space $C(X)$ of all continuous functions on X, equipped with the supremum norm, where X is a compact Hausdorff space.

Theorem A (Rajagopalan and Sundaresan [2, 3]). *If $C(X)$ is infinite-dimensional, then $C(X)$ does not admit a backward shift.*

In this paper, we consider $C(X)$ as the *Banach algebra* of all continuous *complex-valued* functions on X, and extend $C(X)$ to a function algebra. Recall that a *function algebra* A on X is a uniformly closed subalgebra of $C(X)$ which contains the constants and *separates* the points of X, that is, for each pair $x_1, x_2 \in X$ with $x_1 \neq x_2$, there exists $f \in A$ such that $f(x_1) \neq f(x_2)$.

2 Theorems

Here, we state the main results.

Theorem 1 ([4]). *An infinite-dimensional function algebra does not admit a backward shift.*

Clearly, this is a generalization of Theorem A. Here the adjective "infinite-dimensional" is crucially necessary, because a finite-dimensional space always admits a backward shift.

This Theorem was shown by Ariizumi and Takagi in 1998, but have not published. I refine their proof a little, and add to some related results containing the following Theorem in the paper [4].

Theorem 2 ([4]). *Let A be a function algebra, and suppose that the Choquet boundary of A has at most finitely many isolated points. Then A does not admit a backward quasi-shift.*

3 Examples

We examine the existence of backward shift and backward quasi-shift in some concrete spaces. The first example is a function algebra which admits no backward quasi-shift.

Example 1. Let $A(\mathbb{D})$ be the disc algebra, that is, the function algebra of all continuous functions on the closed unit disc which are analytic in the open unit disc. The isometric shifts on $A(\mathbb{D})$ are characterized by T. Takayama and J. Wada [5]. A typical example of it is the multiplication operator S:

$$(Sf)(z) = zf(z) \quad \text{for all } z \text{ and } f \in A(\mathbb{D}).$$

This example suggests to us that the following operator T may be a backward shift on $A(\mathbb{D})$:

$$(Tf)(z) = \begin{cases} \frac{f(z)-f(0)}{z} & \text{if } z \neq 0, \\ f'(0) & \text{if } z = 0, \end{cases} \quad \text{for all } f \in A(\mathbb{D}).$$

It is easy to see that T is onto and satisfies the conditions (i) and (iii) in the definition of backward shift. But T is not a backward shift. Indeed, T does not satisfy (ii), because

$\ker T$ is the subspace of constant functions, and the function $f(z) = z^2 + z$ satisfies that

$$\inf\{\|f+g\| : g \in \ker T\} \leq \left\| f - \frac{1}{2} \right\| = \sqrt{\frac{27}{8}} < 2 = \|Tf\|.$$

Moreover, Theorem 2 implies that $A(\mathbb{D})$ does not admit a backward quasi-shift, because the Choquet boundary of $A(\mathbb{D})$ is the unit circle $\mathbb{T}$ and it has no isolated points.

The next example deals with the L^∞-spaces.

Example 2. Let $L^\infty(\Omega, \mu)$ be the Banach algebra of all equivalence classes of μ-essentially bounded measurable functions on a σ-finite measure space (Ω, μ), with the essential supremum norm. Since $L^\infty(\Omega, \mu)$ is a commutative C^*-algebra with identity, the Gelfand-Naimark theorem implies that $L^\infty(\Omega, \mu)$ is isometrically $*$-isomorphic to $C(X)$, where X is the maximal ideal space of $L^\infty(\Omega, \mu)$. By Theorem A, $L^\infty(\Omega, \mu)$ does not admit a backward shift.

If the measure μ has at most finitely many atoms, then X has at most finitely many isolated points, and so Theorem 2 shows that $L^\infty(\Omega, \mu)$ does not admit a backward quasi-shift.

Next, we consider the space ℓ^∞ as the case that μ has infinitely many atoms. At the same time, we take up some sequence spaces.

Example 3. By ℓ^∞, c and c_0, we denote the Banach space of all sequences that are bounded, converge and converge to zero, respectively. They are equipped with the supremum norm. Suppose that $1 \leq p < \infty$. By ℓ^p, we denote the Banach space of all sequences $x = \{x_n\}$ such that $\sum_{n=1}^\infty |x_n|^p < \infty$, equipped with the norm $\|x\| = (\sum_{n=1}^\infty |x_n|^p)^{\frac{1}{p}}$.

We consider the operator T on those spaces defined by $(x_1, x_2, \ldots) \mapsto (x_2, x_3, \ldots)$. On ℓ^∞ and c, T is seen to be a backward quasi-shift but not a backward shift. Indeed, T does not satisfy (iii).

Moreover, the Gelfand-Naimark theorem showed that ℓ^∞ and c are isometrically $*$-isomorphic to $C(X)$ for some compact Hausdorff space X. By Theorem A, ℓ^∞ and c do not admit a backward shift.

In case of the other spaces c_0 and ℓ^p, the situation is different. It is easily seen that T is a backward shift on c_0 and ℓ^p. Hence c_0 and ℓ^p admit a backward shift.

We obtained an example of the spaces that admit a backward quasi-shift but not a backward shift, and the spaces that admit a backward shift.

In the last example, we ask the question whether there exists a backward shift on a Banach algebra which is not a function algebra.

Example 4. Let W be the Wiener algebra, that is, the Banach algebra of all continuous functions on the unit circle $\mathbb{T}$ whose Fourier series is absolutely convergent, equipped with the norm $\|f\| = \sum_{n=-\infty}^\infty |\widehat{f}(n)|$, where $\widehat{f}(n)$ is the n-th Fourier coefficient of f. Then W becomes a semisimple commutative Banach algebra with identity. It is easily seen that W is linearly isometric to ℓ^1, and ℓ^1 admits a backward shift, as mentioned in Example 3. Hence W admits a backward shift.

Finally, we summarize these observations in the following table.

Is there a backward (quasi-)shift on the space?

Space	Backward shift	Backward quasi-shift
Function algebra	No	Yes or No
Disc algebra	No	No
$C([0,1])$	No	No
$L^\infty[0,1]$	No	No
ℓ^∞	No	Yes
c	No	Yes
c_0	Yes	Yes
$\ell^p \quad (1 \leq p < \infty)$	Yes	Yes
Wiener algebra	Yes	Yes

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Radon-Nikodym Theorem with Generalized Density

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1 Introduction

In this study, we obtain the new formulation of the Radon-Nikodym Theorem without σ-finiteness by Daniell integral. In Daniell integral, Radon-Nikodym density function couldn't be constructed as a function, but "density part" forms special system of measurable functions. We shall call this system folder. Moreover, investigating some non-σ-finite example, we see that Daniell integral differs from Lebesgue integral.

2 Summary of Daniell Integral

In this section, we summarize the Daniell scheme in the following discussion. It should be noted that there are various schemes called Daniell integral, and they are not equivalent each other. For the details, please see [1] and [2].

Definition 2.1 *Let $\mathcal{H}$ be a vector lattice of real valued functions defined on a non-void set Ω, and $\int$ a positive linear functional on $\mathcal{H}$, satisfying: $h_n \searrow 0 \Rightarrow \int h_n \to 0$. We shall say $\mathcal{H}$ is elementary function space, $\int$ is elementary integral. A triplet $(\Omega, \mathcal{H}, \int)$ is called Daniell system.*

A function f is said to belong to $\mathcal{H}^+$ if there exists a sequence of functions $h_n \in \mathcal{H}$ such that $h_n \nearrow f$, where it is permitted that $f(x)$ assumes the value $+\infty$. We define the integral by the formula $\int f = \lim_{n\to\infty} \int h_n$. This integral may be $+\infty$ so we denote by $\mathcal{H}^+_{int}$ the set of $f \in \mathcal{H}^+$ which integral is finite.

Definition 2.2 *Let $Z \subset \Omega$ be called a null set if there exists a $f \in \mathcal{H}^+_{int}$ such that $\infty I(Z) \leqq f$.*

Here, $I(Z)$ is indicator function of the set Z, and $\infty I(Z)$ is regarded as the function taking $+\infty$ at the point of Z.

Definition 2.3 *A function $\varphi : \Omega \to \mathbb{R} \cup \{\pm\infty\}$ which is defined almost everywhere, is said to be measurable function if there exists a sequence of functions $h_n \in \mathcal{H}$ such that $h_n \to \varphi$ (a.e.). Denote by $\mathcal{M}$ the totality of measurable functions.*

An $E \subset \Omega$ is said to be a measurable set if $I(E) \in \mathcal{M}$. We should remark that Ω is necessarily not measurable set (see section 4).

Definition 2.4 *A measurable function φ is said to belong to $\mathcal{L}^+$ if it can be represented as the difference $\varphi = f - g$ (a.e.) between two functions $f \in \mathcal{H}^+$ and $g \in \mathcal{H}^+_{int}$. We define the integral of φ by the formula $\int \varphi = \int f - \int g$. If the above f belongs to $\mathcal{H}^+_{int}$, we call φ is integrable and denote by $\mathcal{L}$ the all of integrable functions.*

This is the summary of Daniell scheme. Moreover, We recall important terms: we say that $\mathcal{H}$ satisfies Stone consition if $h \wedge 1 \in \mathcal{H}$ for any $h \in \mathcal{H}$. Under Stone condition, it is verified that measurable functions $\mathcal{M}$ is closed under multiplication, and a carrier set $\{\varphi \neq 0\}$ is measurable set.

3 Folder and Integration with Folder

In this section, we describe the main concept in this discussion which is called folder. It is inspired by *quasifunction* in Rao[4] and *cross-section* in Zaanen[3]. A cross-section is regarded as the special case of quasifunction. More details can be seen in [4]. In Daniell integral, it is shown that the Radon-Nikodym density forms folder. Please refer to [5] for the elementary properties and these proofs.

Definition 3.1 *We say an $E \subset \Omega$ is elementary measurable set if there exists $\varphi \in \mathcal{H}^+$ such that $E = \{\varphi > 0\}$, and we denote by $\mathcal{E}$ the all of elementary measurable sets.*

The folder $\langle h \rangle$ is defined to be the map: $\mathcal{E} \to \mathcal{M}$; $E \mapsto h_E$, which satisfies that for any $E, F \in \mathcal{E}$, $h_F I(E) = h_{E \cap F}$ (a.e.) holds.

Example 1. A mapping

$$\mathcal{E} \ni E \mapsto I(E)$$

forms folder. We denote this folder by $\langle I \rangle$.

Example 2. Given a measurable function h_0 defined on Ω,

$$\mathcal{E} \ni E \mapsto h_0 I(E)$$

forms folder.

Let φ be a measurable function. We define the product with folder $\varphi\langle h \rangle$ to be a map: $\mathcal{E} \ni E \mapsto \varphi h_E$. Then the folder , in Ex 2, can be expressed by$\langle h \rangle = h_0 \langle I \rangle$. Since it is shown that for any measurable φ there exists $E_0 \in \mathcal{E}$ such that $\{\varphi \neq 0\} \subset E_0$, then we have $\varphi\langle h \rangle = \varphi h_{E_0} \langle I \rangle$.

A folder $\langle h \rangle$ is said to be *locally integrable* or *density* if for any elementary function f, the product $f h_{E_0}$ is integrable, where $E_0 \in \mathcal{E}$ contains $\{f \neq 0\}$. Now we can define the integral:

$$\int f\langle h \rangle := \int f h_{E_0}, \quad (f \in \mathcal{H})$$

for density folder $\langle h \rangle$.

4 Radon-Nikodym Theorem

The following is our main theorem.

Theorem 4.1 (Radon-Nikodym Theorem) *Let $(\Omega, \mathcal{H}, \int)$ be a Daniell system satisfying the Stone condition, and Q be any integral on $\mathcal{H}$ such that $Q \ll \int$. Then there exists a non-negative density folder $\langle h \rangle$, such that for any $f \in \mathcal{H}$,*

$$Q(f) = \int f\langle h \rangle. \tag{1}$$

This $\langle h \rangle$ is determined (a.e.)-uniquely.

Remark 1. $Q \ll \int$ means Q absolutely continuous with respect to $\int$, that is to say for any $\int$-null set is Q-null set.

Remark 2. We say that $\mathcal{H}$ is σ-finite if $1 \in \mathcal{H}^+$. It is equivalent $\Omega \in \mathcal{E}$. It should be remarked that σ-finiteness is not assume in this theorem.

Idea of the proof.

Let $\mathcal{E}_0$ denote the sets of *elementary integrable set* E which satisfy $E = \{\varphi > 1\}$ for some $\varphi \in \mathcal{H}$. Then $E \in \mathcal{E}_0$ is integrable with respect to any integral on $\mathcal{H}$.

Fix $E \in \mathcal{E}_0$, let

$$\int_E f := \int fI(E), \ Q_E(f) := Q(fI(E)), \quad (\forall f \in \mathcal{H})$$

so these integrals play role of finite measure. Now, we can obtain the density function h_E ($a.e.$)-uniquely as usual way and we have

$$Q(fI(E)) = \int fh_E, \quad (\forall f \in \mathcal{H}).$$

These h_E $(E \in \mathcal{E}_0)$ satisfies folder's condition, and it is verifed that above equation holds for $E \in \mathcal{E}$. Then we have $Q(f) = \int f\langle h \rangle$, because we can choose $E \in \mathcal{E}$ such that $\{f \neq 0\} \subset E$ so $Q(fI(E)) = Q(f)$. ∎

More details can be seen in [5].

5 Reseach of Counter-Example

We consider the counter-example of Radon-Nikodym theorem without σ-finiteness. At first, we recall classical example in measure theory. Let Ω be a closed interval $[0,1]$, and denote by $\mathcal{B}$ the Borel sets in $[0,1]$, and μ be a counting-measure on $\mathcal{B}$. Then the only μ-null set is empty set, and every measure on $\mathcal{B}$ is absolutely continuous with respect to μ. Now, Lebesgue measure ν, for instance, is absolutely continuous with respect to μ, but it is well known example that Radon-Nikodym theorem doesn't hold, that is to say, the density function h doesn't exist satisfying $\nu(E) = \int_E h \, d\mu$ for $E \in \mathcal{B}$

Next, we consider this example with Daniell integral. Let $\Omega = [0,1]$, $\mathcal{H}$ the set of real valued functions on $[0,1]$, each of which vanishes outside some finite set, i.e.,

$$f = \sum_{k \in A} a_k I(\{k\}) \quad A : \text{finite set}, \quad a_k \in \mathbb{R}.$$

$\mathcal{H}$ consists of an elementary function space and satisfies the Stone condition. We define the elementary integral as follows:

$$\int f := \sum_{k \in A} a_k.$$

According to this definition, if the set A is finite subset in $[0,1]$ then $I(A) \in \mathcal{H}$, and $\int I(A)$ is nothing else but counting-measure μ, i.e., $\int I(A)$ is the number of elements of A. Furthermore,

- The only null set is empty set with respect to this integral,
- measurable set is countable set,

- measurable set is elementary measurable set,
- $\Omega \notin \mathcal{E}$ (because Ω is uncountable set) so $\mathcal{H}$ is not σ-finite.

Then arbitrary integral Q on $\mathcal{H}$ is absolutely continuous with respect to $\int$. By definition of an integral Q of $f \in \mathcal{H}$, we have

$$Q(f) = \sum_{k \in A} a_k Q(I(k)),$$

and $Q(I(k))$ takes non-negative real value. Here, let $Q(I(k)) = c_k$ $(0 \leqq c_k \in \mathbb{R})$ then,

$$Q(f) = \sum_{k \in A} a_k c_k \int I(k) = \int \sum_{k \in A} a_k I(k) \sum_{m \in \Omega} c_m I(m).$$

Define $h := \sum_{m \in \Omega} c_m I(m)$, then h is a function on Ω and we obtain $Q(f) = \int fh$. But h is not measurable. Let $\langle h \rangle = h\langle I \rangle$ then each $h_E = hI(E)$ is measurable and $Q(f) = \int f\langle h \rangle$ holds. This example tells us there doesn't exist the counting integral and Lebesgue integral couldn't coexist on the same elementary function space.

6 Applications

Theorem 6.1 (Hahn Decomposition) *There exist $\langle I_+ \rangle$ and $\langle I_- \rangle$ satisfying:* $\langle I \rangle = \langle I_+ \rangle + \langle I_- \rangle$ $\quad$ ($|\Phi|$-*a.e.) and,*

$$\Phi_+(f) = |\Phi|(f\langle I_+ \rangle)$$
$$\Phi_-(f) = |\Phi|(f\langle I_- \rangle) \quad (f \in \mathcal{L}^+(|\Phi|)).$$

Here, Φ is signed Daniell integral, $|\Phi|, \Phi_+$, and Φ_- are total, positive, and negative variation, respectively. For the details and definition, please see in [1] and [3].

Remark 3. In general, Ω is necessarily not measurable so it is impossible to decompose into 2 piece of measurable sets as usual. But these folders $\langle I_+ \rangle, \langle I_- \rangle$ play the role of the decomposition.

This application tells us the essential difference between Daniell integral and Lebesgue integral. We would like to consider the following measure Φ:

$$\Phi(\{x\}) = \begin{cases} 1 & (x \geqq 0) \\ -1 & (x < 0) \end{cases} \tag{2}$$

i.e., this measure has a point mass and take $+1$ or -1 with respect to positive or negative point of $\mathbb{R}$, respectively.

At first, we consider this example with Daniell scheme. Let $\Omega = \mathbb{R}$, $\mathcal{H}$ the set of real valued functions on finite subset of Ω. $\Phi : \mathcal{H} \to \mathbb{R}$ is defined by

$$h = \sum_{k \in A} a_k I(\{k\}) \mapsto \Phi(h) := \sum_{k \geq 0} a_k - \sum_{k < 0} a_k$$

Then $(\Omega, \mathcal{H}, \Phi)$ is Daniell system, and it is easily seen

- measurable set is countable set,
- measurable set is elementary measurable set,

- $\forall E \in \mathcal{E}$, $\exists E_1, E_2$ s.t. $I(E) = I(E_1) + I(E_2)$ and

$$\Phi_+(I(E_2)) = \Phi_-(I(E_1)) = 0.$$

It means that E is decomposed into positive part E_1 and negative part E_2. But, this example doesn't work in measure theory.

Let $\mathcal{B}$ be Borel sets in $\mathbb{R}$. For $E \in \mathcal{B}$, we define $E_+ := E \cap [0, \infty)$, $E_- := E \cap (-\infty, 0)$. We would like to define Φ as

$$\Phi(E) := \#(E_+) - \#(E_-),$$

but it is not well-defined, because E_+ and E_- contains uncountable points in general and it occurs $\infty - \infty$. If we demand more proper definition of Φ, then we have to assume much rigid condition.

Moreover, it can be identified the dual space of $\mathcal{L}$ by essentially bounded folder, i.e.,

Theorem 6.2 (Dual space of $\mathcal{L}$.) *let $(\Omega, \mathcal{H}, \int)$ be a Daniell system. For any $T \in \mathcal{L}^*$ (dual space of $\mathcal{L}$), there exists an essentially bounded folder $\langle h \rangle$ such that for any $f \in \mathcal{L}$,*

$$Tf = \int f \langle h \rangle. \tag{3}$$

When this holds, we have $\|T\| = \|\langle h \rangle\|_\infty$ and the mapping $\tau : T \to \langle h \rangle$ is an isometric isomorphism between $\mathcal{L}^*$ and the space of essentially bounded folders, so that they may be identified.

Here $\mathcal{L}$ is a Banach space with respect to the norm defined by $\|f\| = \int |f|$. And we define $\|\langle h \rangle\|_\infty := \sup_{E \in \mathcal{E}} \|h_E\|_\infty$, where $\|\cdot\|_\infty$ is essential supremum as usual. We shall say $\langle h \rangle$ is essentially bounded folder if $\|\langle h \rangle\|_\infty < \infty$. In this theorem, it is not assumed that $\mathcal{H}$ is σ-finite. Besides, if $1 < p, q < \infty$ and $(1/p + 1/q = 1)$, then $(\mathcal{L}^p)^* = \mathcal{L}^q$ holds in the sense of function space. This result was descried in [6].

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A perturbation theory for defect operators on Hilbert function spaces

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Abstract

We deal with families of operators encoding structure of submodules in Hilbert function spaces. We give a partial announcement of results obtained in [5].

1 $H^2(\mathbb{D}^2)$ case

Let $\mathcal{M}$ be a submodule (= an invariant subspace) of the Hardy space $H^2(\mathbb{D}^2)$ over the bidisk, and let $\mathcal{Z}(\mathcal{M})$ denote the zero set of $\mathcal{M}$. Then it is known that the following family of quotient spaces defines a vector bundle over $\mathbb{D}^2 \setminus \mathcal{Z}(\mathcal{M})$ under some appropriate condition:

$$\mathcal{M}/[(z_1-\lambda_1)\mathcal{M}+(z_2-\lambda_2)\mathcal{M}] \ ((\lambda_1,\lambda_2)\in\mathbb{D}^2).$$

In this talk, we discuss problems of a family of self-adjoint operators having the above quotient spaces as their eigenspaces.

Let R_f denote the compression of a Toeplitz operator T_f into $\mathcal{M}$, that is, we set $R_f = P_{\mathcal{M}}T_f|_{\mathcal{M}}$. We define an operator valued function as follows:

$$\Delta_\lambda = I_{\mathcal{M}} - R_{b_{\lambda_1}(z_1)}R^*_{b_{\lambda_1}(z_1)} - R_{b_{\lambda_2}(z_2)}R^*_{b_{\lambda_2}(z_2)} + R_{b_{\lambda_1}(z_1)}R_{b_{\lambda_2}(z_2)}R^*_{b_{\lambda_1}(z_1)}R^*_{b_{\lambda_2}(z_2)},$$

where

$$(b_{\lambda_1}(z_1), b_{\lambda_2}(z_2)) = \left(\frac{z_1-\lambda_1}{1-\overline{\lambda_1}z_1}, \frac{z_2-\lambda_2}{1-\overline{\lambda_2}z_2}\right) \quad (\lambda=(\lambda_1,\lambda_2)\in\mathbb{D}^2).$$

We should mention that Δ_0 has already been studied by Guo-Yang [3] and called the core operator or the defect operator of $\mathcal{M}$. The following theorem is the reason why we have been interested in Δ_λ.

Theorem 1 (Guo-Yang [3] (where $\lambda = 0$))

$$\ker(I_{\mathcal{M}} - \Delta_\lambda) = \mathcal{M}/[(z_1 - \lambda_1)\mathcal{M} + (z_2 - \lambda_2)\mathcal{M}].$$

Definition 1 (Yang [6, 7]) $\mathcal{M}$ is called a Hilbert-Schmidt submodule if Δ_0 is Hilbert-Schmidt.

Theorem 2 (S [5]) Let $\mathcal{M}$ be a submodule of $H^2(\mathbb{D}^2)$.

(i) If Δ_μ is Hilbert-Schmidt for some μ, then Δ_λ is so for any λ.

(ii) If $\mathcal{M}$ is Hilbert-Schmidt then $\|\Delta_\lambda - \Delta_\mu\|_2 \to 0$ $(\lambda \to \mu)$.

Theorem 3 (S [5]) Let $\mathcal{M}$ be a Hilbert-Schmidt submodule. If

$$\dim\ker(I_{\mathcal{M}} - \Delta_\mu) = n > 1$$

for some μ in $\mathbb{D}^2$, then, for any neighborhood U_1 of 1 such that

$$\sigma(\Delta_\mu) \cap \overline{U_1} = \{1\},$$

there exists a neighborhood U_μ of μ such that

$$\sigma(\Delta_\lambda) \cap U_1 = \{1, \sigma_1(\lambda), \ldots, \sigma_{n-1}(\lambda)\}$$

for any λ in U_μ, counting multiplicity.

2 $L_a^2(\mathbb{D})$ case

In this section, we deal with the defect operator of a submodule in Bergman space over $\mathbb{D}$. The Bergman space over $\mathbb{D}$ is defined as follows:

$$L_a^2(\mathbb{D}) = \left\{ f \in \mathrm{Hol}(\mathbb{D}) : \frac{1}{\pi}\int_{\mathbb{D}} |f(z)|^2 \, dxdy < \infty \ (z = x + iy) \right\}.$$

The reproducing kernel is

$$k_\lambda(z) = \frac{1}{(1-\overline{\lambda}z)^2} \quad \text{(the Bergman kernel)},$$

and the operator $S_z : f \mapsto zf$ acting on $L^2_a(\mathbb{D})$ is called the Bergman shift. The definition of submodules in $L^2_a(\mathbb{D})$ is the same as that of $H^2(\mathbb{D}^2)$. We summarize well known facts on submodules of $L^2_a(\mathbb{D})$.

Theorem 4 Let $\mathcal{M}$ be a submodule of $L^2_a(\mathbb{D})$.

(i) $\dim \mathcal{M}/(z-\lambda)\mathcal{M}$ is independent of choice of λ in $\mathbb{D}$ (Richter [4]).

(ii) For every n in $\{1, 2, \ldots, \infty\}$, there exists a submodule $\mathcal{M}$ such that $\dim \mathcal{M}/z\mathcal{M} = n$ (Apostol-Bercovici-Foiaş-Pearcy [1]).

(iii) $\mathcal{M}/z\mathcal{M}$ is a generating set of $\mathcal{M}$ (Aleman-Richter-Sundberg [2]).

The defect operator of a submodule of $L^2_a(\mathbb{D})$ is as follows:

$$\Delta = I_\mathcal{M} - 2R_zR_z^* + R_z^2R_z^{*2},$$

which was introduced by Yang-Zhu [8] (they called this the root operator of $\mathcal{M}$). The definition of Δ_λ is similar to that given in Section 1,

$$\Delta_\lambda = I_\mathcal{M} - 2R_{b_\lambda}R_{b_\lambda}^* + R_{b_\lambda}^2 {R_{b_\lambda}^*}^2,$$

where we set $b_\lambda = (z-\lambda)/(1-\overline{\lambda}z)$. The following theorem was shown in Yang-Zhu [8] in the case where $\lambda = 0$, and their proof can be applied to the general case.

Theorem 5 (Yang-Zhu [8])

$$\ker(I_\mathcal{M} - \Delta_\lambda) = \mathcal{M}/(z-\lambda)\mathcal{M}.$$

The Hilbert-Schmidt class of submodules in $L^2_a(\mathbb{D})$ is defined as same as that given in Section 1.

Theorem 6 (S) Let $\mathcal{M}$ be a Hilbert-Schmidt submodule of $L_a^2(\mathbb{D})$. Then

(i) Δ_λ is Hilbert-Schmidt for any λ in $\mathbb{D}$,

(ii) $\|\Delta_\lambda - \Delta_\mu\|_2 \to 0$ $(\lambda \to \mu)$.

Remark 1 In [8], Yang-Zhu proved that Δ is compact if and only if Δ is in the trace class.

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OPPENHEIM'S INEQUALITY AND RKHS

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ABSTRACT. Applying the norm inequality for RKHSs corresponding to the product of reproducing kernels and using the minimal norm of the Nevanlinna interpolation, we prove Oppenheim's inequality for positive semidefinite matrices, and show equality conditions for it.

1. INTRODUCTION

Oppenheim's inequality [4]: for any positive semidefinite matrices A and B, we have

$$|A \circ B| \geq |A| b_{11} \cdots b_{nn}, \tag{1}$$

where $A = (a_{ij})_{i,j=1}^n$, $B = (b_{ij})_{i,j=1}^n$, and $A \circ B = (a_{ij}b_{ij})$ is the *Hadamard product* of A and B. Our aim is to prove Oppenheim's inequality by using the theory of kernel functions, and to derive equality conditions for Oppenheim's inequality.

A function $k \colon E \times E \to \mathbb{C}$ is called a *positive definite kernel* on E if, for any finite sequence $\{x_i\}_{i=1}^n \subset E$ and for any complex numbers ξ_i $(i = 1, \ldots, n)$, k satisfies the inequality

$$\sum_{i=1}^{n} k(x_i, x_j)\xi_i \overline{\xi}_j \geq 0.$$

One verifies easily that the reproducing kernel of a *reproducing kernel Hilbert space* (RKHS) on E is a positive definite kernel on E. The converse to this fact is important. Indeed, it is well-known that, for each positive definite kernel k on E, there exists a unique RKHS H_k on E whose reproducing kernel is k. By Schur's theorem the sum and the product of two positive definite kernels on E are also positive definite kernels on E. Thus, if H_{k_1} and H_{k_2} are RKHSs on E, then RKHSs $H_{k_1+k_2}$ and $H_{k_1k_2}$ are well-defined. For applications the following norm inequalities are very useful: for every $f \in H_{k_1}$ and $g \in H_{k_2}$,

$$\|f + g\|_{k_1+k_2}^2 \leq \|f\|_{k_1}^2 + \|g\|_{k_2}^2,$$
$$\|fg\|_{k_1k_2} \leq \|f\|_{k_1} \|g\|_{k_2}.$$

Here the norm of H_k is denoted by $\|\cdot\|_k$. For general theory of reproducing kernels, the reader is referred to [1, 5].

2. POSITIVE SEMIDEFINITE MATRIX AND ITS RKHS

Setting $a_{ij} = a(i, j)$, we may regard any positive semidefinite matrix $A = (a_{ij}) \in M_n$ as a positive definite kernel on the set $\{1, \ldots, n\}$, where M_n is the set of $n \times n$ complex matrices. Moreover, we regard a column vector $(x_i)_{i=1}^n$ as a function $x(i) = x_i$ on $\{1, \ldots, n\}$. With this identification it is interesting to know a concrete description of the RKHS H_A. We summarize well-known facts about H_A as follows (cf. [5, pp. 13–14]):

Proposition 1. *Let $A = (a_{ij})$ be a $n \times n$ positive semidefinite matrix. By identifying the matrix A as a positive definite kernel on $\{1, \ldots, n\}$, the RKHS H_A on $\{1, \ldots, n\}$ is given by the vector space* $\operatorname{ran} A$ *equipped with the inner product given by*

$$\langle Ax, Ay \rangle = \sum_{i,j=1}^{n} x_i \overline{y}_j a_{ji},$$

with ${}^t x = (x_1, \ldots, x_n)$ and ${}^t y = (y_1, \ldots, y_n)$. The i-th column vector of A is the reproducing kernel of H_A at i.

The reproducing kernel of H_A at i is denoted by k_i^A. Thus, $A = (k_1^A \ k_2^A \ \ldots \ k_n^A)$.

3. Bergman's formula

First we derive an analog of Bergman's formula for minimal integral [2, p. 26].

Theorem 1. *Let $\{x_j\}_{j=1}^n$ be a linearly independent set of elements of the complex Hilbert space H. Then, for any complex numbers $\{b_j\}_{j=1}^n$, there exists a unique element $f \in H$ which satisfies*

$$\langle f, x_j \rangle = b_j, \quad j = 1, \ldots, n, \tag{2}$$

and minimizes the norm among all $f \in H$ satisfying (2)*. Moreover, if we denote by f_n the extremal element above, then f_n and its norm are given by*

$$f_n = -\frac{1}{G_n} \begin{vmatrix} 0 & x_1 & \ldots & x_n \\ b_1 & \langle x_1, x_1 \rangle & \ldots & \langle x_n, x_1 \rangle \\ \vdots & \vdots & \ddots & \vdots \\ b_n & \langle x_1, x_n \rangle & \ldots & \langle x_n, x_n \rangle \end{vmatrix},$$

and

$$\|f_n\|^2 = -\frac{1}{G_n} \begin{vmatrix} 0 & \overline{b}_1 & \ldots & \overline{b}_n \\ b_1 & \langle x_1, x_1 \rangle & \ldots & \langle x_n, x_1 \rangle \\ \vdots & \vdots & \ddots & \vdots \\ b_n & \langle x_1, x_n \rangle & \ldots & \langle x_n, x_n \rangle \end{vmatrix},$$

where $G_n = \det(\langle x_j, x_i \rangle)_{i,j=1}^n$ is the Gramian of $\{x_j\}_{j=1}^n$.

Remark 1. When H is a RKHS with reproducing kernel k, by setting $x_j = k_{a_j}$, our problem (2) is rewritten as $f(a_j) = b_j$. This is just a Nevanlinna interpolation problem. In this sense we call (2) the interpolation problem.

Specializing the constants $\{b_j\}$ in our interpolation problem (2), we obtain a relation between the minimal norm and the Gramians, which is the main tool for our paper.

Corollary 1. *Let $\{x_j\}_{j=1}^n$ be a set of linearly independent elements in H. Set $b_1 = \cdots = b_{n-1} = 0$, $b_n = 1$ and consider the interpolation problem* (2)*. Then the following hold:*

(i) $\|f_n\|^2 = G_{n-1}/G_n$.
(ii) $f_n = \Phi_n/G_n$.

Here, G_k denotes the Gramian of $\{x_j\}_{j=1}^k$, $(k = 1, \dots, n,\ G_0 = 1)$, and

$$\Phi_n = \begin{vmatrix} \langle x_1, x_1 \rangle & \dots & \langle x_n, x_1 \rangle \\ \vdots & \ddots & \vdots \\ \langle x_1, x_{n-1} \rangle & \dots & \langle x_n, x_{n-1} \rangle \\ x_1 & \dots & x_n \end{vmatrix}.$$

Remark 2. The sequence $\{f_n\}$ obtained above from a linearly independent set $\{x_j\}$ coincides, up to multiplicative constants, with the orthonormal sequence constructed from $\{x_j\}$ by the Gram-Schmidt orthonormalization.

Using the minimal solution to the Nevanlinna interpolation on a RKHS, we shall give a simple proof of the Oppenheim's inequality [4] and show necessary and sufficient conditions for equality (cf. [7]).

To this end we first recall the norm inequality concerning the tensor product RKHS $H_{k^1} \otimes H_{k^2}$ and the RKHS $H_{k^1 k^2}$. Let H_{k^j} $(j = 1, 2)$ be RKHSs on E. Then the tensor product Hilbert space $H_{k^1} \otimes H_{k^2}$ is a RKHS on $E \times E$ whose reproducing kernel at (x, y) in $E \times E$ is $k_x^1 \otimes k_y^2$, where k_x^j $(j = 1, 2)$ denotes the reproducing kernel of H_{k^j} at $x \in E$. Now we have the following inequality (see e.g. [5]): for any $f \in H_{k^1} \otimes H_{k^2}$,

$$\|f \circ \iota\|_{k^1 k^2} \leq \|f\|_{k^1 \otimes k^2}, \tag{3}$$

where the map $\iota \colon E \to E \times E$ is the natural inclusion of E to the diagonal of $E \times E$, that is, $\iota(x) = (x, x)$ for all $x \in E$.

Definition 1. If equality holds in the above inequality (3), the element $f \in H_{k^1} \otimes H_{k^2}$ is called *extremal* ([6]).

Lemma 1. *A function $f \in H_{k^1} \otimes H_{k^2}$ on $E \times E$ is extremal if and only if f belongs to the closed span of the set $\{k_x^1 \otimes k_x^2\}_{x \in E}$.*

We use the following notation. For x, y in a complex vector space, we write $x \sim y$ if there exists a nonzero constant $\alpha \in \mathbb{C}$ with $x = \alpha y$. For positive semidefinite matrix $X \in M_n$, if the set of solutions to the interpolation problem

$$f(1) = \cdots = f(m-1) = 0,\ f(m) = 1 \tag{4}$$

is nonempty for a RKHS H_X, let λ_m^X $(m = 1, \dots, n)$ be the minimal norm of such solutions.

Lemma 2. *Assume that $A \in M_n$ is positive definite and that $B \in M_n$ is positive semidefinite with $b_{mm} > 0$ $(m = 1, \dots, n)$. Then, for $m = 1, \dots, n$, (4) has a solution in $H_{A \circ B}$ and the following inequality holds:*

$$\lambda_m^{A \circ B} \leq \lambda_m^A / \sqrt{b_{mm}}. \tag{5}$$

Equality holds for (5) if and only if the solution $f_m \in H_A$ which satisfies (4) and minimizes the norm is a linear combination of $\{k_i^A \colon k_i^B \sim k_m^B, 1 \leq i \leq m\}$.

For an element σ in the symmetric group S_n of degree n, and for $A = (a_{ij}) \in M_n$, we define the matrix A^σ by $A^\sigma = (a_{\sigma(i)\sigma(j)})$. When σ is the transposition $(i\ j)$ $(1 \leq i, j \leq n)$, the matrix A^σ is obtained from the matrix A by swapping i-th and j-th rows and, simultaneously, i-th and j-th columns. We call such operations of a matrix by *simultaneous exchanges of rows and columns.* In terms of this terminology, A^σ is obtained from the matrix A by a finite number of simultaneous exchanges. One verifies easily the following:

(i) A is positive semidefinite if and only if A^σ is positive semidefinite.
(ii) $|A| = |A^\sigma|$.
(iii) The set of the diagonal entries of A coincides with that of A^σ.

We remark that if
(a) the matrix A is diagonal, or
(b) the matrix B is of rank one,
then equality holds in Oppenheim's inequality (1). Indeed, if (a) holds, this is trivial. So assume that the condition (b) holds. Since B is of rank one and positive semidefinite, B is of the form $B = (w_i \overline{w}_j)$ for some $(w_i) \in \mathbb{C}^n$. Then,

$$|A \circ B| = \det(a_{ij} w_i \overline{w}_j) = |A||w_1 \cdots w_n|^2 = |A| b_{11} \cdots b_{nn},$$

as desired.

Our next Theorem asserts that the condition for equality of Oppenheim's inequality is in general a blend of two conditions (a) and (b) stated in the above remark.

Theorem 2. *If A, $B \in M_n$ are positive semidefinite, then Oppenheim's inequality holds. Equality holds in Oppenheim's inequality if and only if the following conditions hold:*
(i) *$A \circ B$ is singular, or*
(ii) *there exists $\sigma \in S_n$ such that A^σ is block diagonal, i.e.*

$$A^\sigma = \begin{pmatrix} A_{11} & & & \mathbf{0} \\ & A_{22} & & \\ & & \ddots & \\ \mathbf{0} & & & A_{pp} \end{pmatrix},$$

with $A_{ii} \in M_{n_i}$ $(i = 1, \ldots, p)$, $n_1 + \cdots + n_p = n$, and that B^σ satisfies

$$k_1^{B^\sigma} \sim \cdots \sim k_{n_1}^{B^\sigma},\ k_{n_1+1}^{B^\sigma} \sim \cdots \sim k_{n_1+n_2}^{B^\sigma},\ \ldots,\ k_{n_1+\cdots+n_{p-1}+1}^{B^\sigma} \sim \cdots \sim k_n^{B^\sigma}.$$

Theorem 3. *The following are equivalent:*
1) Equality holds in Oppenheim's inequality.
2) The condition (i) or (ii) of Theorem 2 holds.
3) $A \circ B$ is singular, or there exists $B' \in M_n$ such that
(a) B' is positive semidefinite and of rank one,
(b) $A \circ B = A \circ B'$, and
(c) the diagonal entries of B' coincide as a set with that of B.
4) $A \circ B$ is singular, or there exists a diagonal matrix $T = \mathrm{diag}(w_1, \ldots, w_n)$ such that
(a) $A \circ B = TAT^*$, and
(b) $|w_i|^2 = b_{ii}$ $(i = 1, \ldots, n)$.

Remark 3. The equality condition 4) is given in [7, Theorem 1.5]. However, it seems that their argument and results in [7] need slight modifications.

Remark 4. For positive semidefinite matrices $A = (a_{ij})$ and $B = (b_{ij}) \in M_n$, the following inequality holds (Schur's inequality):

$$|A \circ B| + |A||B| \geq |A| b_{11} \cdots b_{nn} + |B| a_{11} \cdots a_{nn}.$$

Oppenheim [4] gave an equality condition for Schur's inequality when both A and B are positive definite. Observe that, when either A or B is singular, Schur's inequality reduces to Oppenheim's inequality. Thus equality condition for Schur's inequality can be reduced to that of Oppenheimer's inequality (cf. [7]).

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Nonlinear Analytic Methods for Linear Contractive Mappings in Banach Spaces

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1 Introduction

Let E be a real Banach space and let C be a nonempty closed convex subset of E. For a mapping $T : C \to C$, we denoted by $F(T)$ the set of fixed points of T. A mapping $T : C \to C$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. In particular, a nonexpansive mapping $T : E \to E$ is called *contractive* if it is linear. That is, a linear contactive mapping $T : E \to E$ is a linear operator satisfying $\|T\| \leq 1$. From [17] we know a weak convergence theorem by Mann's iteration for nonexpansive mappings in a Hilbert space: Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Define a sequence $\{x_n\}$ in C by $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$ such that $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$. Then, $\{x_n\}$ converges weakly to an element z of $F(T)$, where $z = \lim_{n\to\infty} Px_n$ and P is the metric projection of H onto $F(T)$. By Reich [15], such a theorem was extended to a uniformly convex Banach space with a Fréchet differentiable norm. However, we have not known whether the limit point z is characterized under any projections in a Banach space. Recently, using nonlinear analytic methods obtained by [9], [10] and [5], Takahashi and Yao [18] solved such a problem for positively homogeneous nonexpansive mappings in a Banach space. In 1938, Yosida [21] also proved the following mean ergodic theorem for linear bounded operators: Let E be a real Banach space and let T be a linear operator of E into itself such that there exists a constant C with $\|T^n\| \leq C$ for $n \in \mathbb{N}$, and T is weakly completely continuous, i.e., T maps the closed unit ball of E into a weakly compact subset of E. Then, for each $x \in E$, the Cesàro means $S_n x$ converge strongly as $n \to \infty$ to a fixed point of T; see also Kido and Takahashi [12].

In this paper, motivated by these theorems, we study nonlinear analytic methods for linear contractive mappings in Banach spaces. Using these results, we obtain new strong convergence theorems for linear operators in Banach spaces. In the theorems, the limit points are characterized by sunny generalized nonexpansive retractions. Furthermore, we deal with some results which are related to conditional expectations in the probability theory.

2 Preliminaries

Throughout this paper, we assume that a Banach space E with the dual space E^* is real. We also denote by $\langle x, x^* \rangle$ the dual pair of $x \in E$ and $x^* \in E^*$. A Banach space E is said to be strictly convex if $\|x+y\| < 2$ for $x, y \in E$ with $\|x\| \leq 1$, $\|y\| \leq 1$ and $x \neq y$. A Banach space E is said to be smooth provided

$$\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each $x, y \in E$ with $\|x\| = \|y\| = 1$. Let E be a Banach space. With each $x \in E$, we associate the set $J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$. The multi-valued operator $J : E \to E^*$ is called the normalized duality mapping of E. From the Hahn-Banach theorem, $Jx \neq \emptyset$ for each $x \in E$. We know that E is smooth if and only if J is single-valued; see [16]. Let E be a smooth Banach space and let J be the normalized duality mapping of E. We define the function $\phi : E \times E \to \mathbb{R}$ by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$$

for all $x, y \in E$. It is easy to see that $(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2$ for all $x, y \in E$. Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. For an arbitrary point x of E, the set $\{z \in C : \phi(z, x) = \min_{y \in C} \phi(y, x)\}$ is always a singleton. Let us define the mapping Π_C of E onto C by $z = \Pi_C x$ for every $x \in E$, i.e., $\phi(\Pi_C x, x) = \min_{y \in C} \phi(y, x)$ for every $x \in E$. Such Π_C is called the generalized projection of E onto C; see Alber [1]. Let D be a nonempty closed subset of a smooth Banach space E, let T be a mapping from D into itself and let $F(T)$ be the set of fixed points of T. Then, T is said to be generalized nonexpansive if $F(T)$ is nonempty and $\phi(Tx, u) \leq \phi(x, u)$ for all $x \in D$ and $u \in F(T)$. Let C be a nonempty subset of E and let R be a mapping from E onto C. Then R is said to be a retraction, or a projection if $Rx = x$ for all $x \in C$. It is known that if a mapping P of E into E satisfies $P^2 = P$, then P is a projection of E onto $\{Px : x \in E\}$. A mapping $T : E \to E$ with $F(T) \neq \emptyset$ is a retraction if and only if $F(T) = R(T)$, where $R(T)$ is the range of T. The mapping R is also said to be sunny if $R(Rx + t(x - Rx)) = Rx$ whenever $x \in E$ and $t \geq 0$. A nonempty subset C of a smooth Banach space E is said to be a generalized nonexpansive retract (resp. sunny generalized nonexpansive retract) of E if there exists a generalized nonexpansive retraction (resp. sunny generalized nonexpansive retraction) R from E onto C. The following lemmas are in [13].

Lemma 2.1. *Let E be a smooth, strictly convex and reflexive Banach space, let C^* be a nonempty closed convex subset of E^* and let Π_{C^*} be the generalized projection of E^* onto C^*. Then the mapping R defined by $R = J^{-1}\Pi_{C^*}J$ is a sunny generalized nonexpansive retraction of E onto $J^{-1}C^*$.*

Lemma 2.2. *Let E be a smooth, strictly convex and reflexive Banach space and let D be a nonempty closed subset of E. Then, the following are equivalent.*

(1) D is a sunny generalized nonexpansive retract of E;
(2) D is a generalized nonexpansive retract of E;
(3) JD is closed and convex.

Let C be a closed convex subset of a strictly convex and reflexive Banach space E. For an arbitrary point x of E, the set $\{z \in C : \|z - x\| = \min_{y \in C} \|y - x\|\}$ is always a singleton.

Let us define the mapping P_C of E onto C by $z = P_C x$ for every $x \in E$, i.e., $\|P_C x - x\| = \min_{y \in C} \|y - x\|$ for every $x \in E$. Such P_C is called the metric projection of E onto C. Let E be a Banach space and let K be a closed convex cone of E. Then, a mapping $T : K \to K$ is called positively homogeneous if $T(\alpha x) = \alpha T x$ for all $\alpha \geq 0$ and $x \in K$. Let M be a linear closed subspace of E. Then, $T : M \to M$ is called homogeneous if $T(\beta x) = \beta T x$ for all $\beta \in \mathbb{R}$ and $x \in M$. In L^p spaces, $1 \leq p \leq \infty$, we know examples of nonexpansive and positively homogeneous mappings; see, for instance, Wittmann [20]. From Takahashi and Yao [18] we know the following result; see also Honda, Takahashi and Yao [5].

Lemma 2.3. *Let E be a smooth Banach space and let K be a closed convex cone in E. If $T : K \to K$ is a positively homogeneous nonexpansive mapping, then T is generalized nonexpansive. In particular, if $T : E \to E$ is a linear contractive mapping, then T is generalized nonexpansive.*

3 Strong Convergence Theorems

Let Y be a nonempty subset of a Banach space E and let Y^* be a nonempty subset of the dual space E^*. Then, we can define the annihilator $Y^*_\perp$ of Y^* and the annihilator $Y^\perp$ of Y as follows:

$$Y^*_\perp = \{x \in E : f(x) = 0, \ \forall f \in Y^*\} \ \ \textit{and} \ \ Y^\perp = \{f \in E^* : f(x) = 0, \ \forall x \in Y\}.$$

Let $T : E \to E$ be a bounded linear operator. Then, the adjoint mapping $T^* : E^* \to E^*$ is defined as follows: $\langle x, T^* x^* \rangle = \langle Tx, x^* \rangle$ for any $x \in E$ and $x^* \in E^*$. We know that T^* is also a bounded linear operator and $\|T\| = \|T^*\|$. The following results were proved in Takahashi, Yao and Honda [19].

Theorem 3.1. *Let E be a strictly convex, smooth and reflexive Banach space, let T be a linear contractive operator of E into itself, i.e., $T : E \to E$ is a linear operator such that $\|T\| \leq 1$ and let $F(T)$ be the set of fixed points of T. Then $JF(T)$ is a closed linear subspace in E^* and $JF(T) = F(T^*) = \{z - Tz : z \in E\}^\perp$, where $J : E \to E^*$ is the normalized duality mapping and T^* is the adjoint operator of T.*

Theorem 3.2. *Let E be a strictly convex, smooth and reflexive Banach space, let T be a linear contractive operator on E and let $\{S_n : n \in \mathbb{N}\}$ be a sequence of linear contractive operators on E such that $F(T) \subset F(S_n)$ for all $n \in \mathbb{N}$. Suppose $T \circ S_n = S_n \circ T$ for all $n \in \mathbb{N}$. Then, the following are equivalent:*

1. *$\{S_n x\}$ converges to an element of $F(T)$ for each $x \in E$;*
2. *$\{S_n x\}$ converges to 0 for each $x \in (JF(T))_\perp$;*
3. *$\{S_n x - T \circ S_n x\}$ converges to 0 for each $x \in E$.*

Furthermore, if (1) holds, then $\{S_n x\}$ converges to $R_{F(T)} x \in F(T)$, where $R_{F(T)} = J^{-1} \Pi_{JF(T)} J$ and $\Pi_{JF(T)}$ is the generalized projection of E^* onto $JF(T)$.

Using Theorem 3.2, we obtain some strong convergence theorems for linear contractive mappings in a Banach space. In 2003, Bauschk, Deutsch, Hundal and Park [2] showed the following theorem.

Theorem 3.3. *Let T be a linear contractive operator on a Hilbert space H; i.e., $\|T\| \leq 1$, and let M be a linear closed subspace of H. Consider the following statements;*

1. $\lim_{n\to\infty} \|T^n x - P_M x\| = 0$ *for each* $x \in H$*;*
2. $M = F(T)$ *and* $\{T^n x\}$ *converges to* 0 *for each* $x \in M^{\perp}$*;*
3. $M = F(T)$ *and* $T^n x - T^{n+1}x \to 0$ *for each* $x \in E$*.*

Then, all statements are equivalent.

Using Theorem 3.2, we can extend Theorem 3.3 to that of a Banach space.

Theorem 3.4. *Let* E *be a smooth, strictly convex and reflexive Banach space, let* M *be a linear closed subspace of* E *such that there exists a sunny generalized nonexpansive retraction* R *of* E *onto* M *and let* T *be a linear contractive operator on* E*. Then the following are equivalent:*

1. $\{T^n x\}$ *converges to the element* Rx *of* M *for each* $x \in E$*;*
2. $M = F(T)$ *and* $\{T^n x\}$ *converges to* 0 *for each* $x \in (JM)_{\perp}$*;*
3. $M = F(T)$ *and* $T^n x - T^{n+1}x \to 0$ *for each* $x \in E$*.*

Furthermore, if (1) holds, then $R = R_{F(T)} = J^{-1}\Pi_{JF(T)}J$*, where* $\Pi_{JF(T)}$ *is the generalized projection of* E^* *onto* $JF(T)$*.*

Remark 3.5. *If* M *is a linear closed subspace of a Hilbert space* H*, then there exists the metric projection* P *of* H *onto* M*. In a Hilbert space, the metric projection* P *of* H *onto* M *is coincident with the sunny generalized nonexpansive retraction* R_M *of* H *onto* M*.*

Applying Theorem 3.2, we obtain a strong convergence theorem of Mann type [14] for linear contractive mappings in a Banach space.

Theorem 3.6. *Let* E *be a smooth and uniformly convex Banach space and let* T *be a linear contractive operator on* E*. Let* $\{\alpha_n\}$ *be a sequence of real numbers such that* $0 \le \alpha_n \le 1$ *and* $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$*. Then a sequence* $\{x_n\}$ *generated by* $x_1 = x \in E$ *and* $x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n$ *for all* $n = 1, 2, 3, \ldots$ *converges strongly to the element* Rx *of* $F(T)$*, where* $R = R_{F(T)} = J^{-1}\Pi_{JF(T)}J$ *and* $\Pi_{JF(T)}$ *is the generalized projection of* E^* *onto* $JF(T)$*.*

From Theorem 3.2, we can show a mean strong convergence theorem for linear contractive operators in a Banach space; see Yosida [21].

Theorem 3.7. *Let* E *be a smooth, strictly convex and reflexive Banach space and let* T *be a linear contractive operator on* E*. Then, for each* $x \in E$*, the Cesàro means* $S_n x = \frac{1}{n}\sum_{k=1}^{n} T^k x$ *converge strongly to the element* Rx *of* $F(T)$*, where* $R = R_{F(T)} = J^{-1}\Pi_{JF(T)}J$ *and* $\Pi_{JF(T)}$ *is the generalized projection of* E^* *onto* $JF(T)$*.*

Remark 3.8. *In Theorem 3.7, note that the point* $z = \lim_{n\to\infty} S_n x$ *is characterlized by the sunny generalized nonexpansive retraction* $R = R_{F(T)} = J^{-1}\Pi_{JF(T)}J$ *of* E *onto* $F(T)$*. Such a result is still new even if the operator* T *is linear.*

4 Generalized Conditional Expectations

Motivated by Lemmas 2.1 and 2.2, we can define the following nonlinear operator: Let E be a smooth, strictly convex and reflexive Banach space and let J be the normalized duality mapping from E onto E^*. Let Y^* be a linear closed subspace of the dual space E^* of E. Then, the generalized conditional expectation E_{Y^*} with respect to Y^* is defined as follows:

$$E_{Y^*} := J^{-1}\Pi_{Y^*}J,$$

where Π_{Y^*} is the generalized projection from E^* onto Y^*; see [4]. Let E be a normed linear space and let $x, y \in E$. We say that x is orthogonal to y in the sense of Birkhoff-James, denoted by $x \perp y$, if $\|x\| \leq \|x + \lambda y\|$ for all $\lambda \in \mathbf{R}$. We know that for $x, y \in E$, $x \perp y$ if and only if there exists $f \in J(x)$ with $\langle y, f \rangle = 0$; see [16]. In general, $x \perp y$ does not imply $y \perp x$. An operator T of E into itself is called left-orthogonal (resp. right-orthogonal) if for each $x \in E$, $Tx \perp (x - Tx)$ (resp. $(x - Tx) \perp Tx$). The following theorems are in Honda and Takahashi [4].

Theorem 4.1. *Let E be a smooth, strictly convex and reflexive Banach space and let Y^* be a linear closed subspace of the dual space E^*. Then, E_{Y^*} with respect to Y^* is left-orthogonal, i.e., for any $x \in E$,*

$$E_{Y^*}x \perp (x - E_{Y^*}x).$$

Theorem 4.2. *Let E be a smooth, strictly convex and reflexive Banach space and let I be the identity operator of E into itself. Let Y^* be a linear closed subspace of the dual space E^* and let E_{Y^*} be the generalized conditional expectation with respect to Y^*. Then, the mapping $I - E_{Y^*}$ is the metric projection of E onto $Y^*_{\perp}$. Let Y be a linear closed subspace of E and let P_Y be the metric projection of E onto Y. Then, $I - P_Y$ is the generalized conditional expectation $E_{Y^\perp}$ i.e., $I - P_Y = E_{Y^\perp}$.*

Let E be a normed space and let $Y_1, Y_2 \subset E$ be linear closed subspaces. If $Y_1 \cap Y_2 = \{0\}$ and for any $x \in E$ there exists a unique pair $y_1 \in Y_1, y_2 \in Y_2$ such that $x = y_1 + y_2$, and any element of Y_1 is BJ-orthogonal to any element of Y_2, i.e., $y_1 \perp y_2$ for any $y_1 \in Y_1, y_2 \in Y_2$, then we represent the space E as

$$E = Y_1 \oplus Y_2 \text{ and } Y_1 \perp Y_2.$$

The kernel of $T : E \to E$ is denoted by $\ker(T)$, i.e., $\ker(T) = \{x \in E : Tx = 0\}$. Using Theorem 4.2, we have the following theorem [4].

Theorem 4.3. *Let E be a smooth, strictly convex and reflexive Banach space and let Y^* be a linear closed subspace of the dual space E^* of E such that for any $y_1, y_2 \in J^{-1}Y^*$, $y_1 + y_2 \in J^{-1}Y^*$. Then, $J^{-1}Y^*$ is a linear closed subspace of E and the generalized conditional expectation E_{Y^*} is a norm one linear projection from E to $J^{-1}Y^*$. Further, the following hold:*

(i) $E = J^{-1}Y^* \oplus \ker(E_{Y^*})$ and $J^{-1}Y^* \perp \ker(E_{Y^*})$;
(ii) $I - E_{Y^*}$ is the metric projection of E onto $\ker(E_{Y^*})$.

In general, a nonzero linear bounded projection on a Banach space has a norm which is more than or equal to 1. So, a norm one linear projection plays an important role in Functional Analysis. Using Nonlinear Functional Analytic Methods, we derive the following two representation theorems for norm one linear projections; see Honda and Takahashi [3].

Theorem 4.4. *Let E be a smooth, strictly convex and reflexive Banach space and let $P : E \to E$ be a norm one projection with $Y = \{Px : x \in E\}$. Then, JY is a linear closed subspace of E^* and P is the generalized conditional expectation E_{JY} with respect to JY, i.e., $P = J^{-1}\Pi_{JY}J$.*

Theorem 4.5. *Let E be a smooth, strictly convex and reflexive Banach space and let Y^* be a linear closed subspace of E^*. Let P be a projection of E onto $J^{-1}Y^*$ such that $\|Px - m\| \leq \|x - m\|$ for all $x \in E$ and $m \in J^{-1}Y^*$. Then, $J^{-1}Y^*$ is a linear closed subspace of E and P is the generalized conditional expectation E_{Y^*}. Further, P is a norm one linear projection.*

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ALUTHGE ITERATIONS OF WEIGHTED TRANSLATION SEMIGROUPS

Mi Ryeong Lee

ABSTRACT. The problem whether Aluthge iteration of bounded operators on a Hilbert space $\mathcal{H}$ is convergent was introduced in [10]. And the problem whether the hyponormal operators on $\mathcal{H}$ with $\dim \mathcal{H} = \infty$ has a convergent Aluthge iteration under the strong operator toplogy remains an open problem ([11]). In this note we consider symbols with a fractional monotone property which generalizes hyponormality and 2-expansivity on weighted translation semigroups, and prove that if $\{S_t\}$ is a weighted translation semigroup whose symbol has the fractional monotone property, then its Aluthge iteration converges to a quasinormal operator under the strong operator topology.

1. Introduction

This was presented at the international conference: The Seminar on Function Spaces, which was held at Hokkaido University in Japan on December 23-25, 2010. This note is the joint work with C. Burnap, I. B. Jung and J. W. Park, which was published in J. Math. Anal. Appl. Vol. 352 in 2009.

Let $\mathcal{H}$ be a separable infinite dimensional complex Hilbert space and let $B(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. An operator $T \in B(\mathcal{H})$ has a unique polar decomposition $T = U|T|$, where $|T| = (T^*T)^{\frac{1}{2}}$ and U is a partial isometry. For $T = U|T|$ in $B(\mathcal{H})$, the *Aluthge transform* of T is defined by $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ (cf. [**1**],[**10**]). Several operators related to such transforms are well-developed and introduced in detail ([**8**]). For every T in $B(\mathcal{H})$, the sequence of Aluthge iterates of T is defined by $\widetilde{T}^{(0)} = T$ and $\widetilde{T}^{(n+1)} = (\widetilde{T}^{(n)})^{\sim}$ for $n \in \mathbb{N}$. In [**11**] the authors continued to study this sequence $\{\widetilde{T}^{(n)}\}$ of iterates, and discussed the convergence of Aluthge iterations in some special cases. In particular, it was shown in [**4**] that the sequence $\{\widetilde{T}^{(n)}\}_{n=1}^{\infty}$ of iterated Aluthge transforms of T need not converge in the strong operator topology in general. However, it was proved that the sequence $\{\widetilde{T}^{(n)}\}$ (of $n\times n$ complex matrices) converges to a normal operator (cf. [**3**], [**2**]). In this note we discuss Aluthge iteration of a weighted translation semigroup $\{S_t\}$ with symbol ϕ which will be defined below.

Let $\mathbb{R}_+ := (\mathbb{R}_+, \mu)$ be the Lebesgue measure space on the set of nonnegative real numbers and let $L^2 := L^2(\mathbb{R}_+)$ be the Hilbert space of square integrable Lebesgue

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measurable complex valued functions on $\mathbb{R}_+$. Let $B(L^2)$ be the algebra of all bounded linear operators on L^2. A family $\{S_t : t \in \mathbb{R}_+\}$ in $B(L^2)$ is a *semigroup* if $S_0 = I$ and $S_t S_s = S_{t+s}$ for all t and s in $\mathbb{R}_+$. In particular, a *weighted translation semigroup* $\{S_t\}$ on L^2 is defined by

$$(S_t f)(x) = \begin{cases} \frac{\phi(x)}{\phi(x-t)} f(x-t) & \text{if } t \le x, \\ 0 & \text{if } 0 \le x < t, \end{cases}$$

where ϕ is a measurable, almost everywhere non-zero function from $\mathbb{R}_+$ into $\mathbb{C}$ that is called the *symbol* of $\{S_t\}$. A semigroup $\{S_t\}$ is *strongly continuous* if, for each f in L^2, the mapping $t \to S_t f$ is continuous from $\mathbb{R}_+$ into L^2. It follows from [**5**, p.619] that $\{S_t\}$ is strongly continuous on $\mathbb{R}_+$ if and only if there exist $M, \omega > 0$ such that

$$\operatorname{esssup}_{x \in \mathbb{R}_+} \left| \frac{\phi(x+t)}{\phi(x)} \right| \le M e^{\omega t}, \qquad t \in \mathbb{R}_+. \tag{1.1}$$

For brevity we will assume that ϕ is continuous on $\mathbb{R}_+$ throughout this article. Since the weighted translation semigroups with symbols ϕ and $|\phi|$ are unitarily equivalent, we will assume throughout this paper that all symbols of weighted translation semigroups are positive, and also assume that $\{S_t\}$ is a strongly continuous semigroup with symbol ϕ. (See [**9**] for more informations about semigroups.)

2. Fractional monotone properties

Let ϕ be a symbol satisfying (1.1) and let

$$\Phi_t^{(k)}(x) := \begin{cases} \frac{\phi(x+(k+1)t)}{\phi(x+(k-1)t)} & \text{if } t \le x, \\ 0 & \text{if } 0 \le x < t, \end{cases} \tag{2.1}$$

for $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $t \in \mathbb{R}_+$. Then $\{\Phi_t^{(k)}(x)\}_{k=0}^{\infty}$ is a sequence of measurable functions on $\mathbb{R}_+$.

Let ϕ be a symbol satisfying (1.1). The symbol ϕ is called to have the *fractional monotone property* (we write *f.m.p.*) if the sequence $\{\Phi_t^{(k)}(x)\}_{k\ge 0}$ as in (2.1) is monotone pointwise on $[t, \infty)$, for each $t \in \mathbb{R}_+$. And, when $\{\Phi_t^{(k)}(x)\}_{k\ge 0}$ is monotone increasing (decreasing, resp.) pointwise on $[t, \infty)$ and for each $t \in \mathbb{R}_+$, we say that the symbol ϕ has the *fractional monotone increasing* (*decreasing,* resp.) *property* (we write f.m.i.p. (f.m.d.p, resp.)).

Let ϕ be a symbol with the f.m.p. Then, since

$$\frac{\phi(x+(k+1)t)}{\phi(x+(k-1)t)} \le \left\| \frac{\phi(x+2t)}{\phi(x)} \right\|_{\infty} \le M e^{2\omega t}, \quad x \ge t,$$

$\{\Phi_t^{(k)}(x)\}_{k=0}^{\infty}$ is bounded sequence pointwise on $\mathbb{R}_+$ for $t \in \mathbb{R}_+$. Therefore a measurable bounded function $\lim_{k\to\infty} \Phi_t^{(k)}(x)$ exists and we denote it by

$$\Phi_t^{(\infty)}(x) := \lim_{k\to\infty} \Phi_t^{(k)}(x). \tag{2.2}$$

In particular, if ϕ has the f.m.i.p. (or, f.m.d.p.), then $\Phi_t^{(\infty)}(x) = \sup_{k\ge 0} \Phi_t^{(k)}(x)$ (or, $\inf_{k\ge 0} \Phi_t^{(k)}(x)$).

Recall from [**6**, Lemma 3.3] that a weighted translation semigroup $\{S_t\}$ with symbol ϕ is hyponormal if and only if

$$\phi(x-t)\phi(x+t) \geq \phi^2(x), \; x \geq t. \tag{2.3}$$

PROPOSITION 2.1. *Let $\{S_t\}$ be a weighted translation semigroup with symbol ϕ. Then the following assertions are equivalent:*

(i) *$\{S_t\}$ is hyponormal;*
(ii) *ϕ has the f.m.i.p.;*
(iii) *$\log \phi$ is convex.*

Recall from [**7**] that T in $B(\mathcal{H})$ is *k-expansive* if

$$\sum_{0\leq p\leq k} (-1)^p \binom{k}{p} \|T^p h\|^2 \leq 0, \quad h \in \mathcal{H}.$$

A simple computation shows that the k-expansivity of $\{S_t\}$ is equivalent to the inequality

$$\sum_{0\leq p\leq k} (-1)^p \binom{k}{p} \phi^2(x+pt) \leq 0, \quad x \in \mathbb{R}_+, \, t \in \mathbb{R}_+. \tag{2.4}$$

PROPOSITION 2.2. *Let $\{S_t\}$ be a weighted translation semigroup with symbol ϕ. Then $\{S_t\}$ is 2-expansive if and only if $\phi^2(x)$ is concave, and thus 2-expansivity implies that* $\log \phi$ *is concave.*

COROLLARY 2.3. *If* $\log\phi$ *is concave, then ϕ has f.m.d.p. Thus the symbol ϕ of any 2-expansive weighted translation semigroup $\{S_t\}$ has the f.m.d.p.*

REMARK 2.4. There are several classes of operators with weak hyponormality, for example, p-paranormal operators, absolutely p-paranormal operators, $A(p)$-class operators, etc. (These definitions will be defined below.) The symbols of these weighted translation semigroups have f.m.p., too. Recall that T is *p-hyponormal* if $(T^*T)^p \geq (TT^*)^p$; *p-paranormal* if $\||T|^p U|T|^p x\| \geq \||T|^p x\|^2$ for all unit vectors $x \in \mathcal{H}$; *absolute p-paranormal* if $\||T|^p Tx\| \geq \|Tx\|^{p+1}$ for all unit vectors $x \in \mathcal{H}$; and *$A(p)$-class* if $(T^*|T|^{2p}T)^{1/(p+1)} \geq |T|^2$, $(0 < p < \infty)$ (cf., [**8**],[**12**]). It is known that "p-hyponormal $\Rightarrow$ $A(p)$ class $\Rightarrow$ absolute p-paranormal"; "p-hyponormal $\Rightarrow$ p-paranormal". In fact, some direct computations show that if $\{S_t\}$ is a weighted translation semigroup, then $\{S_t\}$ is one of the above weak hyponormal semigroups if and only if $\log\phi$ is convex, which is equivalent to that $\{S_t\}$ is hyponormal.

EXAMPLE 2.5. Let us consider a symbol $\phi(x) = 2-x^2$ for $0 \leq x \leq 1$ and $\phi(x) = 1$ for $x \geq 1$ satisfying (1.1). Since $\log\phi$ is not convex, a translation semigroup $\{S_t\}$ with symbol ϕ is not hyponormal. But, $\|S_t\| = \|\phi(x)/\phi(x-t)\|_\infty = 1$ and $\|S_t^n\| = \|\phi(x)/\phi(x-nt)\|_\infty = 1$. Thus $\{S_t\}$ is normaloid.

3. Convergence of Aluthge iterations

Let $\{U_t\}$ be the isometric semigroup in $B(L^2)$ defined by $(U_t f)(x) = f(x-t)$ for $x \geq t$ and 0 otherwise. Then the polar decomposition of a weighted translation semigroup S_t is represented by $U_t\,|S_t|$. Note that $S_t^* f(x) = \frac{\phi(x+t)}{\phi(x)} f(x+t)$ and $|S_t|\, f(x) = \frac{\phi(x+t)}{\phi(x)} f(x)$.

The following is the main theorem of this note.

THEOREM 3.1. *Let $\{S_t\}$ be a weighted translation semigroup whose symbol ϕ has the f.m.p. Then the sequence $\{\widetilde{S}_t^{(n)}\}_{n\geq 1}$ of Aluthge iteration converges to a quasinormal operator Δ_t in $B(L^2)$ under the strong operator topology (SOT), where*

$$\Delta_t f(x) = \begin{cases} \Phi_t^{(\infty)}(x)^{1/2} U_t f(x) & \text{if } t \leq x, \\ 0 & \text{if } 0 \leq x < t. \end{cases} \tag{3.1}$$

We need several lemmas to prove this theorem.

LEMMA 3.2. *Let $\{S_t\}$ be a weighted translation semigroup with symbol ϕ. Suppose that $n \in \mathbb{N}$. Then*

$$\widetilde{S}_t^{(n)} f(x) = \prod_{k=0}^{n-1} \left(\frac{\phi(x+(k+1)t)}{\phi(x+(k-1)t)} \right)^{\binom{n-1}{k}/2^n} f(x-t), \qquad x \geq t \tag{3.2}$$

and 0 otherwise.

The next Lemma and Proposition use the function Φ introduced in (2.1) and (2.2).

LEMMA 3.3. *Suppose that $\{a_k\}_{k=0}^{\infty}$ is a sequence of real numbers that converges to a. Then*

(i) *it holds that*

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} a_k = a;$$

(ii) *if $\{\Phi_t^{(n)}(x)\}_{n\geq 0}$ is a sequence of positive real numbers that converges to $\Phi_t^{(\infty)}(x)$ pointwise on $\mathbb{R}_+$, then $\prod_{k=0}^{n-1} (\Phi_t^{(k)}(x))^{\binom{n-1}{k}/2^n}$ converges to $(\Phi_t^{(\infty)}(x))^{1/2}$ pointwise on $\mathbb{R}_+$ as $n \to \infty$.*

Proof of Theorem 3.1. Suppose that ϕ has the f.m.p. Let $t \in \mathbb{R}_+$ and let $E_t := \{x \in \mathbb{R}_+ : \left|\Phi_t^{(\infty)}(x)\right| \leq \|\Phi_t\|_\infty\}$. Then obviously the complement of E_t has measure zero. To show the SOT-convergence of $\{\widetilde{S}_t^{(n)}\}_{n\geq 1}$, for $f \in L^2$, we consider $\|\widetilde{S}_t^{(n)} f(x) - \Delta_t f(x)\|_{L^2}$, where Δ_t is as in (3.1). Note that $\Phi_t^{(\infty)}(x) = \Phi_t^{(\infty)}(x-t)$ since $\Phi_t^{(\infty)}(x)$ is periodic w.r.t. $t \in \mathbb{R}_+$. And we have that

$$\begin{aligned} \|\widetilde{S}_t^{(n)} f(x) - \Delta_t f(x)\|_{L^2}^2 &= \int_{[t,\infty)} \left| \prod_{k=0}^{n-1} \left(\Phi_t^{(k)}(x)\right)^{\binom{n-1}{k}/2^n} - \Phi_t^{(\infty)}(x-t)^{1/2} \right|^2 |f(x-t)|^2 \, d\mu \\ &= \int_{\mathbb{R}_+} \left| \prod_{k=0}^{n-1} \left(\Phi_t^{(k+1)}(x)\right)^{\binom{n-1}{k}/2^n} - \Phi_t^{(\infty)}(x)^{1/2} \right|^2 |f(x)|^2 \, d\mu. \end{aligned}$$

Applying Lemma 3.3, and by using the Lebesgue dominated convergence theorem, we see that $\|\widetilde{S}_t^{(n)}f(x) - \Delta_t f(x)\|_{L^2}^2 \to 0$ as $n \to \infty$.

On the other hand, since $\Delta_t^* f(x) = \Phi_t^{(\infty)}(x)^{1/2}\bar{f}(x+t)$, by some computations, we have that

$$(\Delta_t^*\Delta_t)\Delta_t f(x) = \Delta_t(\Delta_t^*\Delta_t)f(x) = \begin{cases} \Phi_t^{(\infty)}(x)^{3/2}\bar{f}(x-t) & \text{if } t \leq x, \\ 0 & \text{if } 0 \leq x < t. \end{cases}$$

Hence Δ_t is quasinormal. Hence the proof is complete. □

EXAMPLE 3.4. We can construct recursively the symbol ϕ of a weighted translation semigroup $\{S_t\}$ whose Aluthge iteration $\{\widetilde{S}_t^{(n)}\}_{n\geq 1}$ converges to a quasinormal operator in $B(L^2)$ under SOT. This means that examples applied by Theorem 3.1 are abound. For example, let $f_0(x) = x+1$ and let $f_n(x) = \frac{1}{n+1}(x-n) + f_{n-1}(n)$, $n \leq x \leq n+1$, for $n \in \mathbb{N}$. Define $\phi(x) = e^{f_n(x)}$ for $n \leq x \leq n+1$ and $n \in \mathbb{N}_0$. Then obviously $\phi(x)$ satisfies (1.1) and $\log \phi(x)$ is concave. Thus, by Theorem 3.1, Aluthge iteration converges to a quasinormal operator Δ_t in $B(L^2)$ under SOT.

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FIXED POINT THEOREMS IN A VECTOR LATTICE

TOSHIHARU KAWASAKI

ABSTRACT. In this talk we introduce a topology in a vector lattice and we show Takahashi's, Fan-Browder's, Shauder-Tychonoff's and Kirk's fixed point theorems in a vector lattice.

1. TOPOLOGY IN A VECTOR LATTICE

First we introduce a topology in a vector lattice introduced by [4]; see also [6, 7].

Let X be a vector lattice. $e \in X$ is said to be an unit if $e \wedge x > 0$ for any $x \in X$ with $x > 0$. Let $\mathcal{K}_X$ be the class of units of X. In the case where X is the set of real numbers $\mathbf{R}$, $\mathcal{K}_{\mathbf{R}}$ is the set of positive real numbers. Let X be a vector lattice with unit and let Y be a subset of X. Y is said to be open if for any $x \in Y$ and for any $e \in \mathcal{K}_X$ there exists $\varepsilon \in \mathcal{K}_{\mathbf{R}}$ such that $[x - \varepsilon e, x + \varepsilon e] \subset Y$. Let $\mathcal{O}_X$ be the class of open subsets of X.

Let X be a vector lattice with unit and Y a vector lattice. Let $\mathcal{U}_Y^s(\mathcal{K}_X, \geq)$ be the class of $\{v_e \mid e \in \mathcal{K}_X\}$ satisfying the following conditions:

(U1) $v_e \in Y$ with $v_e > 0$;
(U2)^d $v_{e_1} \geq v_{e_2}$ if $e_1 \geq e_2$;
(U3)^s For any $e \in \mathcal{K}_X$ there exists $\theta(e) \in \mathcal{K}_{\mathbf{R}}$ such that $v_{\theta(e)e} \leq \frac{1}{2} v_e$.

Let $x_0 \in Z \subset X$ and $f : Z \longrightarrow Y$. f is said to be continuous at x_0 if there exists $\{v_e\} \in \mathcal{U}_Y^s(\mathcal{K}_X, \geq)$ such that for any $e \in \mathcal{K}_X$ there exists $\delta \in \mathcal{K}_{\mathbf{R}}$ such that for any $x \in Z$ if $|x - x_0| \leq \delta e$, then $|f(x) - f(x_0)| \leq v_e$.

Let X and Y be vector lattices with unit, $Z \subset X$ and $f : Z \longrightarrow Y$. Suppose that there exists $P \subset Y$ satisfying the following conditions:

(P1) P is open and convex;
(P2) If $x \in P$ and $x \leq y$, then $y \in P$;
(P3) $0 \notin P$;
(P4) $\{x \mid x > 0\} \subset P$.

Let $\mathcal{P}_Y$ be the class of the above P's. f is said to be upper semi-continuous with respect to $P \in \mathcal{P}_Y$ if $\{x \mid y - f(x) \in P\} \in \mathcal{O}_X \cap Z$ for any $y \in Y$. f is said to be lower semi-continuous with respect to $P \in \mathcal{P}_Y$ if $\{x \mid f(x) - y \in P\} \in \mathcal{O}_X \cap Z$ for any $y \in Y$. f is said to be semi-continuous with respect to $P \in \mathcal{P}_Y$ if it is upper and lower semi-continuous with respect to $P \in \mathcal{P}_Y$.

A vector lattice is said to be Archimedean if it holds that $x = 0$ whenever there exists $y \in X$ with $y \geq 0$ such that $0 \leq rx \leq y$ for any $r \in \mathcal{K}_{\mathbf{R}}$.

Let X be an Archimedean vector lattice. Then there exists a positive homomorphism f from X into $\mathbf{R}$, that is, f satisfies the following conditions:

(H1) $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ for any $x, y \in X$ and for any $\alpha, \beta \in \mathbf{R}$;

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(H2) $f(x) \geq 0$ for any $x \in X$ with $x \geq 0$;

see [7, Example 3.1]. Suppose that there exists a homomorphism f from X into $\mathbf{R}$ satisfying the following condition instead of (H2):

$(\text{H2})^s$ $f(x) > 0$ for any $x \in X$ with $x > 0$.

For a vector lattice endowed with the topology above we can show the following lemmas; see [6, 7].

Lemma 1.1. *Let X be an Archimedean vector lattice with unit and $\{x_1, \ldots, x_n\}$ a subset of X. Then $co\{x_1, \ldots, x_n\}$ is homeomorphic to a compact and convex subset of $\mathbf{R}^n$.*

Lemma 1.2. *Let X be an Archimedean vector lattice with unit, Y a vector lattice with unit, $Z \subset X$ and f a mapping from Z into Y. Suppose that there exists a homomorphism from X into $\mathbf{R}$ satisfying condition $(\text{H2})^s$ and that $\mathcal{P}_Y \neq \emptyset$.*

Then f is semi-continuous with respect to any $P \in \mathcal{P}_Y$ if it is continuous at any $x \in Z$.

Lemma 1.3. *Let X be an Archimedean vector lattice with unit, Y a vector lattice with unit, $x_0 \in Z \subset X$ and f a mapping from Z into Y. Suppose that there exists a homomorphism from X into $\mathbf{R}$ satisfying condition $(\text{H2})^s$.*

Then f is continuous at x_0 in the sense of topology if it is continuous at x_0.

2. Takahashi's and Fan-Browder's fixed point theorems

By using the lemmas above, we can show the following theorem, which is a vector lattice version of Takahashi's fixed point theorem.

Theorem 2.1. *Let X be a Hausdorff Archimedean vector lattice with unit, Y a compact subset of X and Z a convex subset of Y. Suppose that a mapping f from Z into 2^Y satisfies*

(0) *$f^{-1}(y)$ is convex for any $y \in Y$,*

and there exists a mapping g from Z into 2^Y satisfying the following conditions:

(1) *$g(z)$ is a subset of $f(z)$ for any $z \in Z$;*
(2) *$g^{-1}(y)$ is non-empty for any $y \in Y$;*
(3) *$g(z)$ is an open subset of X for any $z \in Z$.*

Then there exists $z_0 \in Z$ such that $z_0 \in f(z_0)$.

In the above theorem, putting $Z = Y$ and $g = f$, the following theorem is obtained. It is Fan-Browder's fixed point theorem in a vector lattice.

Theorem 2.2. *Let X be a Hausdorff Archimedean vector lattice with unit and Y a compact convex subset of X. Suppose that a mapping f from Y into 2^Y satisfies the following conditions:*

(1) *$f^{-1}(y)$ is non-empty and convex for any $y \in Y$;*
(2) *$f(y)$ is an open subset of X for any $y \in Y$.*

Then there exists $y_0 \in Y$ such that $y_0 \in f(y_0)$.

3. Schauder-Tychonoff's fixed point theorem

By using the lemmas above and Theorem 2.2, we obtain the following.

Theorem 3.1. *Let X be a Hausdorff Archimedean vector lattice with unit and Y a compact convex subset of X. Suppose that $f : Y \longrightarrow X$ is continuous. Then it holds that* (1) *or* (2).

(1) *There exists $y_0 \in Y$ such that $f(y_0) = y_0$.*
(2) *There exists $x_0 \in Y$ such that $f(x_0) \neq x_0$ and $|x_0 - f(x_0)| - |y - f(x_0)| \notin P$ for any $P \in \mathcal{P}_X$ and for any $y \in Y$.*

By using the theorem above we can show the following Schauder-Tychonoff's fixed point theorem in a vector lattice.

Theorem 3.2. *Let X be a Hausdorff Archimedean vector lattice with unit and Y a compact convex subset of X. Suppose that $f : Y \longrightarrow Y$ is continuous. Then there exists $y_0 \in Y$ such that $f(y_0) = y_0$.*

4. Fixed point theorem for a nonexpansive mapping

Let X be a vector lattice and Y a subset of X. A mapping f from Y into Y is said to be nonexpansive if $|f(x) - f(y)| \leq |x - y|$ for any $x, y \in Y$. In this section we consider a fixed point theorem for a nonexpansive mapping.

Let X be a Hausdorff Archimedean vector lattice with unit and Y a subset of X. We say that Y has the normal structure if for any compact convex subset K, which contains two points at least, of Y there exists $x \in K$ such that

$$\bigvee_{y \in K} |x - y| < \bigvee_{x,y \in K} |x - y|.$$

Theorem 4.1. *Let X be a Hausdorff Archimedean vector lattice with unit and K a non-empty compact convex subset of X. Suppose that K has the normal structure. Then every nonexpansive mapping from K into K has a fixed point.*

5. Fixed point theorem for the commutative family of nonexpansive mappings

For any nonexpansive mapping f from K into K let $F_K(f)$ be the set of fixed points of f.

Theorem 5.1. *Let X be a Hausdorff Archimedean vector lattice with unit, K a compact convex subset of X and $\{f_i \mid i = 1, \ldots, n\}$ the finite commutative family of nonexpansive mappings from K into K. Suppose that there exists a homomorphism from X into $\mathbf{R}$ satisfying condition* (H2)^s *and K has the normal structure. Then $\bigcap_{i=1}^{n} F_K(f_i)$ is non-empty.*

Theorem 5.2. *Let X be a Hausdorff Archimedean vector lattice with unit, K a compact convex subset of X and $\{f_i \mid i \in I\}$ the commutative family of nonexpansive mappings from K into K. Suppose that there exists a homomorphism from X into $\mathbf{R}$ satisfying condition* (H2)^s *and K has the normal structure. Then $\bigcap_{i \in I} F_K(f_i)$ is non-empty.*

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ON THE FOURIER TRANSFORM ON $L^2(\mathbb{R})$

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ABSTRACT. We shall indicate how to use the new decomposition formula of the phase space to prove a deep, fundamental and well-known theorem for Fourier analysis.

1. 序

フーリエ解析の講義ではめったに証明が扱われることがないが，次の定理は非常に深淵である．

定理 1.1. $f \in L^2(\mathbb{R})$ とする．このとき，ほとんどすべての $\xi \in \mathbb{R}$ に対して，

(1)
$$\mathcal{F}f(\xi) = \lim_{R\to\infty} \int_{-R}^{R} f(x) \exp(-2\pi i x \cdot \xi)\, dx$$

は収束している．

この定理はカールソンの定理「$L^2(\mathbb{T})$-関数のフーリエ級数は概収束する」という定理と深い関係がある．

この定理は調和解析のいくつかの道具をもとにして証明される．

(1) 極大作用素の理論
(2) 関数空間論とウエーブレット理論 (もしくは，関数空間論とアトム分解理論)
(3) タイルの理論

(1) に関して少し詳しく見ていく．

(a) $f \in L^2(\mathbb{R})$ に対して，**プランシュレルの定理を用いて** $\mathcal{F}f \in L^2(\mathbb{R})$ を定義する．
(b) 定理を書き換える (ねじる)．

定理 1.2. $f \in L^2(\mathbb{R})$ とすると，ほとんどすべての $\xi \in \mathbb{R}$ に対して

(2)
$$f(x) = \lim_{R\to\infty} \int_{-R}^{R} \mathcal{F}f(\xi) \exp(2\pi i x \cdot \xi)\, d\xi$$

が成り立つ．

(c) この形に書き換えたうえで，極大作用素の理論に置き換える．

(3)
$$\left|\left\{x \in \mathbb{R} : \sup \left|\int_{-R}^{R} \mathcal{F}f(\xi) \exp(2\pi i x \cdot \xi)\, d\xi\right| > \lambda\right\}\right| \lesssim \frac{1}{\lambda^2} \|f\|_2{}^2$$

(d) $\int_{-\infty}^{R} + \int_{R}^{\infty} - \int_{-\infty}^{\infty} = \int_{-R}^{R}$ であるから，対称性によって，

(4)
$$\left|\left\{x \in \mathbb{R} : \sup \left|\int_{-\infty}^{R} \mathcal{F}f(\xi) \exp(2\pi i x \cdot \xi)\, d\xi\right| > \lambda\right\}\right| \lesssim \frac{1}{\lambda^2} \|f\|_2{}^2$$

を示せばよい．ここで，

(5) $$\int_{-\infty}^{R} \mathcal{F}f(\xi)\exp(2\pi i x\cdot\xi)\,d\xi = \mathcal{F}^{-1}[\chi_{-\infty}^{R}\mathcal{F}f](x)$$

とみなす．

(e) $M_R f(x) = \exp(2\pi i R x) f(x)$ とおく．フーリエ変換と関数の平行移動の関係に留意して，

(6) $$\mathcal{F}^{-1}[\chi_{(-\infty,R]}\mathcal{F}f](x) = c\,M_{-R}HM_R f(x)$$

と書き換える．ここで，H はヒルベルト変換である．

以上のような流れによって，(1) の議論が進む．

本稿では，このうち (2) と (3)(の一部) を取り上げよう．

2. 記号

定義 2.1.

(1) 立方体とは暗黙のうちに座標軸に平行な辺からなる閉立方体を指す．ただし 2 進立方体に限っては，閉立方体ではない．

(2) $r>0,\ x\in\mathbb{R}^n$ とする．立方体に関して次の記号を用いる．

$$Q(r) := \{y\in\mathbb{R}^n\ :\ \max(\,|y_1|,|y_2|,\cdots,|y_n|\,)\le r\}$$
$$Q(x,r) := \{y\in\mathbb{R}^n\ :\ \max(\,|x_1-y_1|,|x_2-y_2|,\cdots,|x_n-y_n|\,)\le r\}$$

さらに，$k>0$ のとき，$k\,Q(x,r) := Q(x,k\,r)$ と定める．

(3) $\nu\in\mathbb{Z}$ と $m=(m_1,m_2,\cdots,m_n)\in\mathbb{Z}^n$ に対して，$Q_{\nu m} := \prod_{j=1}^{n}\left[\frac{m_j}{2^\nu},\frac{m_j+1}{2^\nu}\right)$ と定める．このような立方体を 2 進立方体といい，$\mathcal{D}$ でその全体を表す．

(4) 通常の立方体もしくは 2 進立方体 Q に対して，$c(Q)$ で中心を，$\ell(Q)$ で $|Q|^{\frac{1}{n}}$ を表すことにする．

(5) タイルとは $Q_{\nu m}\times Q_{-\nu m'}$ をしている $\mathbb{R}^{2n}$ の部分集合である．ここで，$\nu\in\mathbb{Z},\ m,m'\in\mathbb{Z}^n$ である．

(6) $\vec{1} := (1,1,\cdots,1)$ と表す．

(7) $\mathbb{R}^n$ には辞書式順序 $\ll$ を入れる．つまり，

(7) $$x=(x_1,x_2,\cdots,x_n)\ll y=(y_1,y_2,\cdots,y_n),\ x\ne y$$

とは，ある $j=1,2,\cdots,n$ に対して，$x_1=y_1, x_2=y_2,\cdots,x_{j-1}=y_{j-1}$, $x_j<y_j$ となっていることである．

(8) 2 進立方体 Q に対して，これを 2^n 等分して，得られた立方体が

(8) $$c(Q_{(1)})\ll c(Q_{(2)})\ll\cdots\ll c(Q_{(2^n)})$$

を満たすように，$Q_{(1)},Q_{(2)},\cdots,Q_{(2^n)}$ と名前を付ける．

定義 2.2 ([4, 5, 6, 8, 9, 10])**.**

(1) $\nu\in\mathbb{Z},\ m,m'\in\mathbb{Z}^n$ とする．タイル $s=Q_{\nu m}\times Q_{-\nu m'}$ に対して，$I_s := Q_{\nu m}$ と $\omega_s := Q_{-\nu m'}$ とおく．

(2) 2 つのタイル $u,v\in\mathbb{D}$ が順序 $u\le v$ を満たしているとは，$I_u\subset I_v$, $\omega_u\supset\omega_v$ となっていることをいう．

(3) 木とは組 $(\mathbb{T},t)$ であって，$\mathbb{T}$ は $\mathbb{D}$ の有限部分集合であり，$t\in\mathbb{D}$ はタイルで，すべての $s\in\mathbb{T}$ に対して，$t\ge s$ となるものである．さらに，このとき，$\omega_{\mathbb{T}} := \omega_t$，$I_{\mathbb{T}} := I_t$ と書くことにする．

(4) $1 \le i \le 2^n$ とする．木 $(\mathbb{T}, t)$ が i-木であるとは，すべての $s \in \mathbb{T}$ に対して $\omega_{t(i)} \subset \omega_{s(i)}$ を満たすことである．

この場合 t は $\mathbb{T}$ の頂点と呼ばれる．$\mathbb{T}$ の頂点は一意的ではないことに注意したい．頂点を特定する場合は，$(\mathbb{T}, t)$ を木という．

今度は，関数 $\Phi \in \mathcal{S}$ を

(9)
$$\chi_{Q(9/100)} \le \Phi \le \chi_{Q(1/10)}$$

となるようにとる．

以下の定義を与える．

定義 2.3.

(1) $a, \xi \in \mathbb{R}^n$，$\lambda > 0$ とする．

$$\begin{cases} T_a f(x) := f(x-a) \\ M_\xi f(x) := \exp(2\pi i \xi \cdot x) f(x) \\ D_\lambda f(x) := \lambda^{-n/2} f(\lambda^{-1} x) \end{cases}$$

と定める．

(2) $\varphi := \mathcal{F}^{-1}\Phi$ と定める．

(3) $\Psi := \Phi - \Phi(2\cdot)$ とおく．

(4) 立方体 Q に対して，$\Phi_Q(\xi) := \Phi\left(\dfrac{\xi - c(Q)}{\ell(Q)}\right)$ とおく．

(5) [10] タイル $s \in \mathbb{D}$ に対して，$\varphi_s(x) := M_{c(\omega_{s(1)})} T_{c(I_s)} D_{\ell(I_s)} \varphi(x)$ とおく．

3. 関数空間論とウエーブレット理論

基本的な計算によって，以下のことが示される．

補題 3.1.

(1) 立方体 Q に対して，

(10)
$$\chi_{\frac{27}{25}Q} \le \Phi_{6Q} \le \chi_{\frac{6}{5}Q}$$

が成り立つ．

(2) タイル s に対して，

(11)
$$\mathcal{F}\varphi_s = T_{c(\omega_{s(1)})} M_{-c(I_s)} D_{\ell(\omega_s)} \Phi.$$

が成り立つ．特に，$\mathrm{supp}(\mathcal{F}\varphi_s) \subset \dfrac{1}{5}\omega_{s(1)}$ である．

(10) から Φ_{6Q} は χ_Q とほぼ同じであるとわかる．一方で，(11) から，φ_s のフーリエ変換の台は $c(\omega_{s(1)})$ に集中しているとわかる．

次の補題はプランシュレルの定理から証明される．

補題 3.2. $\xi \in \mathbb{R}^n$ とする．このとき，

(12)
$$\left(\sum_{s \in \mathbb{D} : \omega_{s(2^n)} \ni \xi} |\langle f, \varphi_s \rangle_{L^2}|^2 \right)^{\frac{1}{2}} \lesssim \|f\|_2 \quad (f \in L^2)$$

である．

モデル作用素という次の作用素を考える．

定義 3.3. モデル2進極大作用素とは

$$A_{\xi,\mathbb{P}}f(x) := \sum_{s\in\mathbb{P}\,:\,,\omega_{s(2^n)}\ni\xi} \langle f,\varphi_s\rangle_{L^2}\varphi_s,\ \mathbb{P}\subset\mathbb{D},\ \xi\in\mathbb{R}^n \tag{13}$$

で与えられる作用素である．

次の補題は，関数空間論の定理を集約して得られる命題であると言えよう．$B(L^2)$ で L^2-有界である作用素のなすバナッハ空間を表すとする．

補題 3.4 ([10]). 作用素 $A_{\xi,\mathbb{P}}$ は $\mathbb{P}\subset\mathbb{D}$ *and* $\xi\in\mathbb{R}^n$ に関して L^2-有界である．つまり，

$$\|A_{\xi,\mathbb{P}}\|_{B(L^2)} \lesssim 1. \tag{14}$$

証明. [12] の分子分解を $\dot{F}^0_{22}\simeq L^2$ として用いる．φ_s の台の条件は $\mathcal{F}\varphi_s$ の台が0を含んでいないことより問題がない．微分不等式などは φ_s の定義から明らかである．係数条件は補題 3.2 そのものである． □

「almost-orthogonality」と補題 3.2 を用いた証明もある．

4. タイルの理論

定義 4.1. $l\in\mathbb{Z}$ とする．l-世代の (モデル) 作用素 $A_{\eta,l}$ は

$$A_{\eta,l}f(x) := \sum_{s\in\mathbb{D}\,:\,\omega_{s(2^n)}\ni\eta,\ |I_s|=2^{ln}} \langle f,\varphi_s\rangle_{L^2}\varphi_s \tag{15}$$

で与えられる．

補題 4.2. 適当な $m\in C^\infty_c(\mathbb{R}^n\setminus\{0\})$ が存在して

$$\lim_{N\to\infty}\int_{Q_N} M_{-\eta}A_{\eta,l}M_\eta f\,\frac{d\eta}{|Q_N|} = \mathcal{F}^{-1}[m(2^l\cdot)\cdot\mathcal{F}f],\ f\in L^2 \tag{16}$$

が，$2Q_N\subset Q_{N+1}$ $(N\in\mathbb{N})$ となる $\{Q_N\}_{N\in\mathbb{N}}$ に対して，成り立つ．

証明. $B(L^2)$ における一様有界族 $\left\{\int_{Q_N} M_{-\eta}A_{\eta,l}M_\eta\,\frac{d\eta}{|Q_N|}\right\}_{N\in\mathbb{N}}$ を扱うので，

$$f\in\mathcal{S}_0 = \bigcap_{\alpha\in{\mathbb{N}_0}^n}\left\{f\in\mathcal{S}\,:\,\int_{\mathbb{R}^n} x^\alpha f(x)\,dx = 0\right\} \tag{17}$$

としてよい．

$$\mathcal{F}\left[\int_{Q_N} M_{-\eta}A_{\eta,l}M_\eta\,\frac{d\eta}{|Q_N|}\right]\mathcal{F}^{-1}f = \int_{Q_N}\mathcal{F}M_{-\eta}A_{\eta,l}M_\eta\mathcal{F}^{-1}f\,\frac{d\eta}{|Q_N|} \tag{18}$$

を考えよう．$Q_l=Q_l(\eta)$ で，$\ell(Q_l)=2^{-l}$ かつ $\eta\in Q_{l(2^n)}$ となる2進立方体を表すとする．Q_l の存在はわからないが，あったとしてもそれは一意的である．

とりあえず，このような Q_l があったとしよう．フーリエ級数展開と (11) を用いて，

$$
\begin{aligned}
&\mathcal{F}M_{-\eta}A_{\eta,l}M_\eta\mathcal{F}^{-1}f\\
&=\sum_{s\in\mathbb{D}\,:\,\omega_{s(2^n)}\ni\eta,\,|I_s|=2^{ln}}\langle M_\eta\mathcal{F}^{-1}f,\varphi_s\rangle_{L^2}\mathcal{F}M_{-\eta}\varphi_s\\
&=\sum_{s\in\mathbb{D}\,:\,\omega_{s(2^n)}\ni\eta,\,|I_s|=2^{ln}}\langle f,\mathcal{F}M_{-\eta}\varphi_s\rangle_{L^2}\mathcal{F}M_{-\eta}\varphi_s\\
&=\sum_{s\in\mathbb{D}\,:\,\omega_{s(2^n)}\ni\eta,\,|I_s|=2^{ln}}\langle f,T_{c(\omega_{s(1)})-\eta}M_{c(I_s)}D_{\ell(\omega_s)}\Phi\rangle_{L^2}T_{c(\omega_{s(1)})-\eta}M_{c(I_s)}D_{\ell(\omega_s)}\Phi\\
&=f\cdot\left|\Phi\left(\frac{\cdot+\eta-c(Q_{l(1)})}{\ell(Q_l)}\right)\right|^2
\end{aligned}
$$

を得る．

(19) $$m_l:=2^{ln}\int_{Q_{l+1,\vec{1}}}\left|\Phi\left(2^l(\cdot+\eta-2^{-2}\vec{1})\right)\right|^2d\eta=\int_{\frac{\vec{1}}{2}+Q(\frac{1}{4})}\left|\Phi\left(2^l\cdot+\zeta\right)\right|^2d\zeta$$

とおく．この不等式を代入して，

(20) $$\mathcal{F}\left(\int_{Q_N}M_{-\eta}A_{\eta,l}M_\eta\frac{d\eta}{|Q_N|}\right)\mathcal{F}^{-1}f=m_l\cdot f$$

となる．よって，$m:=\displaystyle\int_{\frac{\vec{1}}{2}+Q(\frac{1}{4})}|\Phi(\cdot+\zeta)|^2\,d\zeta$ とおけばよいことがわかった． □

系 4.3. 補題 4.2 と同じ記号のもとで，

(21) $$M(\xi):=\sum_{l=-\infty}^{\infty}m(2^l\xi)\quad \xi\in\mathbb{R}^n$$

と表せば，

(22) $$\lim_{L\to\infty}\sum_{l=-L}^{L}\left(\lim_{N\to\infty}\int_{Q_N}M_{-\eta}A_{\eta,l}M_\eta f\frac{d\eta}{|Q_N|}\right)=\mathcal{F}^{-1}(M\cdot\mathcal{F}f),$$

がすべての $f\in L^2$ に対して，L^2-収束の意味合いで成り立つ．

$SO(n)$ は直交行列で行列式が 1 であるものの全体である．$SO(n)$ はコンパクトであるから，ハール測度 μ をとれる．正規化して，μ は確率測度であるとしよう．ユニタリー作用素 $\rho:SO(n)\to U(L^2)$ を

(23) $$\rho(A)f:=f(A^{-1}\cdot),\ f\in L^2$$

で定める．

系 4.4. 補題 4.2 と同じ記号のもとで，

(24) $$\alpha:=\int_{SO(n)}\int_0^1 M(2^\kappa A\,(1,0,0,\cdots,0))\,d\kappa\,d\mu(A)$$

と書くことにする．すると，

$$\alpha\,id_{L^2}=\int_{SO(n)}\int_0^1\left(\sum_{l=-\infty}^{\infty}\lim_{N\to\infty}\int_{Q_N}\rho(A^{-1})D_{2^{-\kappa}}M_{-\eta}A_{\eta,l}M_\eta D_{2^\kappa}\rho(A)\frac{d\eta}{|Q_N|}\right)d\kappa\,d\mu(A),$$

がすべての $f\in L^2$ に対して，L^2-収束の意味合いで成り立つ．

注意 4.5. [10] において Pramanik と Terwilleger は

$$\rho(A^{-1})D_{2^{-\kappa}}T_{-y}M_{-\eta}A_{\eta,l}M_{\eta}T_yD_{2^\kappa}\rho(A) \tag{25}$$

を考えた．しかし，系 4.4 でみたように，$\mathbb{R}^n_y$ に関する平均を考えなくて済む．このことは，次節の応用において反映される．

5. 応用

$\mathbb{R}^n$ で考える．$\mathbb{N}_0 = \{0, 1, 2, \cdots\}$ として，

$$c_{\alpha,\beta}(a) := \sup_{x\in\mathbb{R}^n,\,\xi\in\mathbb{R}^n} |\xi|^{|\alpha|-|\beta|}|\partial_\xi^\alpha\partial_x^\beta a(x,\xi)| < \infty,\ \alpha,\beta\in\mathbb{N}_0^n \tag{26}$$

を満たしている擬微分作用素のシンボル a を考える．

$f \in \mathcal{S}_0$ に対して，

$$a(X,D)f(x) = \int_{\mathbb{R}^n} a(x,\xi)\mathcal{F}f(\xi)\exp(2\pi i x\cdot\xi)\,d\xi \tag{27}$$

と定める．$a(X,D)f \in \mathcal{S}$ が証明される．対応 $f \in \mathcal{S}_0 \mapsto a(X,D)f \in \mathcal{S}$ が L^2-有界になるような仮定をおく．

例 5.1. $a \in S^0$ なら，確かにこの仮定は満たされる．

このとき，次の定理が成り立つ．

定理 5.2 ([11, Theorem 1]). $1 < p < \infty$ とするとき，

$$\left\|\sup_{\xi\in\mathbb{R}^n}|M_{-\xi}a(x,D)M_\xi f|\right\|_p = \left\|\sup_{\xi\in\mathbb{R}^n}|a(x,D-\xi)f|\right\|_p \lesssim_p \|f\|_p \tag{28}$$

が成り立つ．

この定理は [2, 3] の結果を含んでいる．

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An Argument of a Function in $H^{1/2}$

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This is a joint work with Takahiko Nakazi.

Abstract. Let $H^{1/2}$ be the Hardy space on the open unit disc. Let h be a 1/2-strongly outer function in $H^{1/2}$. Let φ be a unimodular function on the unit circle $\mathbf{T}$ such that $\varphi = \bar{z}^n|h|/h$, $n \geq 0$. We study a nonzero function f in the Hardy space $H^{1/2}$ such that φf is nonnegative a.e. on $\mathbf{T}$. Then we generalize a theorem of Neuwirth-Newman-Helson-Sarason with a simple proof.

1 Hardy Space

Definition. If $0 < p < \infty$ we denote by H^p the class of analytic functions $f(z)$ on the open unit disc $\mathbf{D} = \{|z| < 1\}$ for which the integrals

$$M_p(r, f) = \left\{\int_0^{2\pi} |f(re^{i\theta})|^p d\theta/2\pi\right\}^{1/p}$$

are bounded as $r \to 1$. H^∞ is the class of all bounded analytic functions $f(z)$ on D.
($f \in H^\infty \Leftrightarrow M_\infty(r, f) = \max_{0\leq\theta<2\pi} |f(re^{i\theta})|$ are bounded as $r \to 1$.)

The following theorem is well known. We can consider H^p functions on the unit circle $\mathbf{T} = \{|z| = 1\}$. H^p on $\mathbf{T}$ is exactly the class of L^p functions on $\mathbf{T}$ whose Fourier coefficients vanish for all $n < 0$.

Theorem. For each $f(z)$ in H^p, the nontangential limit $f(e^{i\theta})$ exists almost everywhere, $f(e^{i\theta}) \in L^p(\mathbf{T})$ and $\log|f(e^{i\theta})|$ is integrable unless $f(z) = 0$.

Definition. If $Q \in H^\infty$ satisfies $|Q(e^{i\theta})| = 1$ a.e. θ, then Q is said to be an inner function. If $g \in H^p$ satisfies

$$g(z) = \exp\left\{\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log|g(e^{it})|dt/2\pi\right\},$$

then g is said to be an outer function. The following theorem is well known.

Inner-outer factorization theorem.
Every nonzero function $f(z)$ in H^p has a unique factorization of the form $f(z) = Q(z)g(z)$, where $Q(z)$ is an inner function, and $g(z)$ is an outer function for H^p.

Hence, every nonzero function $f(z)$ in $H^{1/2}$ has a unique factorization of the form $f(z) = Q(z)g(z)$, where $Q(z)$ is an inner function, and $g(z)$ is an outer function for $H^{1/2}$. By the definition of the outer function, every nonzero function $f(z)$ in $H^{1/2}$ has a unique factorization of the form $f(z) = Q(z)k(z)^2$, where $Q(z)$ is an inner function, and $k(z)$ is an outer function for H^1.

Definition. Let $0 < p \leq \infty$. If a nonzero function h in H^p satisfies the following condition (*), then h is said to be a p-strongly outer function.

(*) If f is a nonzero function in H^p such that $f/h \geq$ a.e. on $\mathbf{T}$, then $f = \gamma h$ for some positive constant γ.

A 1-strongly outer function is also called as a rigid function and if it has a unit norm then it is an exposed point of the unit ball of H^1.

If an h in H^1 satisfies h^{-1} is in H^1 or $\mathrm{Re}h \geq 0$, then h is a 1-strongly outer function.

Theorem.([3]) If $0 < p < 1/2$, then there is no p-strongly outer function.

Remark. h is a p-strongly outer function $\Rightarrow$ h is an outer function in H^p.
Proof. Suppose $h \in H^p$ is not an outer function. Then there exists a nonconstant inner function Q and an outer function $g \in H^p$ such that $h = Qg$. If $f = (1+Q)^2 g$, then $f/h = |1+Q|^2 \geq 0$, because $|Q| = 1$ a.e. on $\mathbf{T}$. This is a contradiction. □

Neuwirth-Newman-Helson-Sarason (1967)
If f is in $H^{1/2}$ and $f \geq 0$ a.e. on $\mathbf{T}$, then f is a constant.

This theorem implies that a constant 1 is a 1/2-strongly outer function, and

If $0 < p < 1/2$, then a constant 1 is not a p-strongly outer function, because
$f(z) = z/(z+1)^2 \Rightarrow f \in \cap_{p<1/2} H^p$, $f \geq 0$ a.e. on $\mathbf{T}$.

In the following theorem, if $n = 0$ then we consider that $\prod_{j=1}^{0}(z - a_j)(1 - \overline{a_j}z) = 1$.

Helson-Sarason (1967)
Let f be a nonzero function in $H^{1/2}$ and $n \geq 0$. If $\bar{z}^n f \geq 0$ a.e. on $\mathbf{T}$, then there are complex numbers a_j such that $|a_j| \leq 1$ $(1 \leq j \leq n)$ and $f = \gamma \prod_{j=1}^{n}(z - a_j)(1 - \overline{a_j}z)$ where γ is a positive constant.

Simple proof of Helson-Sarason's theorem.
Suppose $f \neq 0$. By the inner outer factorization theorem, there is an inner function Q and an outer function $k \in H^1$ such that $f = Qk^2$. Since $|Q| = 1$ a.e. on $\mathbf{T}$,
$\bar{z}^n f \geq 0 \Rightarrow \bar{z}^n Q k^2 = |k|^2 \Rightarrow \bar{z}^n Q k = \bar{k} \Rightarrow z^n \overline{Qk} = k$.
Since $z^n \overline{H^1} \cap H^1 = \mathrm{span}\{1, z, ..., z^n\}$, k is a polynomial such that $\deg k = n_0 \leq n$. Hence there are complex numbers a_j $(1 \leq j \leq n_0)$ and c such that

$$k = c \prod_{j=1}^{n_0}(z - a_j).$$

Therefore

$$f = Qk^2 = z^n |k|^2 = |c|^2 z^{n-n_0} \prod_{j=1}^{n_0}(z - a_j)(1 - \overline{a_j}z).$$

Let $a_j = 0$ $(n_0 + 1 \le j \le n)$. Then

$$f = |c|^2 \prod_{j=1}^{n} (z - a_j)(1 - \overline{a_j} z).$$

If $|a_j| > 1$ then

$$(z - a_j)(1 - \overline{a_j} z) = |a_j|^2 (z - \frac{1}{\overline{a_j}})(1 - \frac{1}{a_j} z).$$

Hence we can take $|a_j| \le 1$ for all j. $\square$

2 Problem and Main theorem

Problem. Suppose $\varphi = \bar{z}^n |h|/h, n \ge 0$ and h is a 1/2-strongly outer function. Find f in $H^{1/2}$ such that $\varphi f \ge 0$ a.e. on $\mathbf{T}$.

Remark. If

$$f = \gamma h \prod_{j=1}^{n} (z - a_j)(1 - \overline{a_j} z)$$

then

$$\varphi f = \gamma |h| \prod_{j=1}^{n} |z - a_j|^2 \ge 0.$$

By the following theorem, we show that the converse is also true.

Theorem 1. ([5]) Suppose $\varphi = \bar{z}^n |h|/h$, $n \ge 0$, and h is a 1/2-strongly outer function. Let f be a nonzero function in $H^{1/2}$. If $\varphi f \ge 0$ a.e. on $\mathbf{T}$, then there are complex numbers a_j such that $|a_j| \le 1$ $(1 \le j \le n)$ and $f = \gamma h \prod_{j=1}^{n} (z - a_j)(1 - \overline{a_j} z)$ where γ is a positive constant.

3 Proof of Theorem 1

We use Lemma 1 to prove Theorem 1.

Lemma 1. ([5]) Let $h = h_0^2$ be a 1/2-strongly outer function and $n \ge 0$.
Suppose $\bar{z}^n \overline{h_0}/h_0 = \overline{Qk}/k$ where Q is an inner function and k is an outer function in H^1.
$\Rightarrow$ Q is a finite Blaschke product with $\deg Q \le n$.

Proof of Lemma 1. Suppose $Q = q_1 \cdots q_{n+1}$ and q_j is a nonconstant inner function for $1 \le j \le n+1$. Then there is an inner function q_{j1} such that

$$\overline{q_j} = \frac{\overline{1 - \overline{q_j(0)} q_j}}{q_j - q_j(0)} \cdot \frac{1 - q_j(0)\overline{q_j}}{1 - \overline{q_j(0)} q_j} = \bar{z} \overline{q_{j1}} \frac{1 - q_j(0)\overline{q_j}}{1 - \overline{q_j(0)} q_j}.$$

Hence

$$\bar{z}^n \frac{\bar{h}_0}{h_0} = \bar{Q} \frac{\bar{k}}{k} = \bar{z}^{n+1} \prod_{j=1}^{n+1} \overline{q_{j1}} \cdot \frac{\prod_{j=1}^{n+1} (1 - q_j(0)\overline{q_j})}{\prod_{j=1}^{n+1} (1 - \overline{q_j(0)} q_j)} \cdot \frac{\bar{k}}{k}$$

Let $h_1 = k\prod_{j=1}^{n+1}(1-\overline{q_j(0)}q_j)$. Then h_1 is an outer function, and there is an inner function Q_1 such that

$$\frac{\overline{h_0}}{h_0} = \bar{z}Q_1\frac{\overline{h_1}}{h_1} = \frac{\overline{(1+z)(1+Q_1)h_1}}{(1+z)(1+Q_1)h_1}.$$

Since h_0^2 is 1/2-strongly outer, $h_0^2 = \gamma(1+z)^2(1+Q_1)^2h_1^2$, for some $\gamma > 0$.
Hence $z(1+Q_1)^2h_1^2/h_0^2 \geq 0$. Since h_0^2 is 1/2-strongly outer, $h_0^2 = \gamma_1 z(1+Q_1)^2h_1^2$, for some $\gamma_1 > 0$. This contradicts that h_0 is outer. Therefore Q is a finite Blaschke product with $\deg Q \leq n$. □

Proof of Theorem 1. Since h is a 1/2-strongly outer function, $h = h_0^2$ for an outer function h_0 in H^1. Since $\varphi = \bar{z}^n|h|/h = \bar{z}^n\overline{h_0}/h_0$,

$$z^n h_0 = \overline{\varphi h_0} \Rightarrow z^n h_0 \in H^1 \cap \overline{\varphi H^1} \Rightarrow \{z^j h_0\}_{j=0}^n \subset H^1 \cap \overline{\varphi H^1}.$$

Since f is a nonzero function in $H^{1/2}$, there is an inner function q and an outer function k in H^1 such that $f = qk^2$. Since $\varphi f \geq 0$ a.e. on $\mathbf{T}$,

$$\varphi = \frac{|f|}{f} = \bar{q}\frac{\bar{k}}{k}.$$

Since $qk = \overline{\varphi k}$, $qk \in H^1 \cap \overline{\varphi H^1}$. Hence $H^1 \cap \overline{\varphi H^1}$ contains $\{z^j h_0\}_{j=0}^n$ and qk. Since $h_0(0) \neq 0$, there exists an analytic polynomial p_n with $\deg p_n = n_0 \leq n$ and $s \in H^1$ such that $qk - p_n h_0 = z^{n+1}s$. Suppose $s \neq 0$. If g is the outer part of s then $0 \neq z^{n+1}g \in H^1 \cap \overline{\varphi H^1}$. Therefore there exists a function $\psi \in H^1$ such that $z^{n+1} = \overline{\varphi\psi}$. Since $|\varphi| = 1$, $\psi = Qg$ for some inner function Q. Thus $\bar{z}^n\overline{h_0}/h_0 = \varphi = \bar{z}^{n+1}\overline{Qg}/g$. Let $Q_1 = z^{n+1}Q$. Then $\bar{z}^n\overline{h_0}/h_0 = \overline{Q_1 g}/g$. By Lemma 1, Q_1 is a finite Blaschke product with $\deg Q_1 \leq n$. This contradiction implies that $s = 0$, and so $qk = p_n h_0$. Hence there are complex numbers α_j $(1 \leq j \leq n_0)$ such that $qk = ch_0\prod_{j=1}^{n_0}(z-\alpha_j)$ where c is a complex constant, $|\alpha_j| < 1$ $(1 \leq j \leq n_1)$ and $|\alpha_j| \geq 1$ $(n_1+1 \leq j \leq n_0)$ and so $q = \prod_{j=1}^{n_1}(z-\alpha_j)/(1-\overline{\alpha_j}z)$. Hence $k = ch_0\prod_{j=1}^{n_1}(1-\overline{\alpha_j}z)\prod_{j=n_1+1}^{n_0}(z-\alpha_j)$. Therefore

$$f = qk^2 = c^2h_0^2\prod_{j=1}^{n_1}(z-\alpha_j)(1-\overline{\alpha_j}z)\prod_{j=n_1+1}^{n_0}(z-\alpha_j)^2.$$

Since $\varphi f \geq 0$ and $\bar{z}^{n_1}\prod_{j=1}^{n_1}(z-\alpha_j)(1-\overline{\alpha_j}z) \geq 0$, we have

$$c^2\bar{z}^{n-n_1}\prod_{j=n_1+1}^{n_0}(z-\alpha_j)^2 \geq 0.$$

Hence

$$c^2\bar{z}^{n-n_1}\prod_{j=n_1+1}^{n_0}(z-\alpha_j)^2 = |c|^2\prod_{j=n_1+1}^{n_0}|z-\alpha_j|^2,$$

and so

$$c\prod_{j=n_1+1}^{n_0}(z-\alpha_j) = \bar{c}z^{n-n_0}\prod_{j=n_1+1}^{n_0}(1-\overline{\alpha_j}z).$$

Therefore $n = n_0$ and $|\alpha_j| = 1$ $(n_1+1 \leq j \leq n_0)$. □

4 Corollary.

Lemma 2. If h is in $H^{1/2}$ and h^{-1} is in H^∞ then h is a 1/2-strongly outer function.

Corollary 1. ([5]) Suppose F is a nonnegative function such that qF belongs to $H^{1/2}$ for some inner function q. If $q = 1$ then F is a nonnegative constant. If $q = \prod_{j=1}^n (z - b_j)(1 - \overline{b_j} z)$ and $|b_j| < 1$ $(1 \leq j \leq n)$ then there are complex numbers a_j such that $|a_j| \leq 1$ $(1 \leq j \leq n)$ and

$$F = \gamma \prod_{j=1}^n \frac{(z - a_j)(1 - \overline{a_j} z)}{(z - b_j)(1 - \overline{b_j} z)},$$

where γ is some positive constant.

Proof of Corollary 1. If $q = 1$ then F is a nonnegative constant because 1 is a 1/2-strongly outer function. If $f = qF$, then f belongs to $H^{1/2}$. Since $q = z^n \prod_{j=1}^n |1 - \overline{b_j} z|^2 / (1 - \overline{b_j} z)^2$ and $f/q \geq 0$, we have $f/(z^n \prod_{j=1}^n (1 - \overline{b_j} z)^{-2}) \geq 0$. By Lemma 2, $\prod_{j=1}^n (1 - \overline{b_j} z)^{-2}$ is a 1/2-strongly outer function. By Theorem 1, there are complex numbers a_j such that $|a_j| \leq 1$ $(1 \leq j \leq n)$ and

$$f = \gamma \prod_{j=1}^n (z - a_j)(1 - \overline{a_j} z)(1 - \overline{b_j} z)^{-2},$$

where γ is some positive constant, and so

$$F = \gamma \prod_{j=1}^n \frac{(z - a_j)(1 - \overline{a_j} z)}{(z - b_j)(1 - \overline{b_j} z)}. \quad \square$$

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BANACH SPACES OF HARMONIC FUNCTIONS ON RIEMANN SURFACES [1]

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There are two categories of Banach spaces of harmonic functions on Riemann surfaces: one consists of *Hardy spaces* of exponents between 1 and ∞ and their subspaces; the other consists of the *Dirichlet space* and its subspaces. We will solve the problem whether these spaces are *reflexive* or not and also the same one for *separability*. We will give applications of the result obtained to the inverse inclusion problem in the classification theory of Riemann surfaces, which was one of the motivation of our present study beyond our mere curiosity.

1. Inclusion relations of Banach spaces. We denote by $H(R)$ the linear space of real valued harmonic functions on a Riemann surface R. For each $1 \leqq p < \infty$ we let $HM_p(R)$ be the set of $u \in H(R)$ for which the subharmonic function $|u|^p$ admits the least harmonic majorant $\hat{u} \in H(R)^+$ on R. It forms a Banach space equipped with the norm $\|u\|_p := \hat{u}(a)^{1/p}$ with $a \in R$ an arbitrarily chosen but then fixed reference point in R. For $p = \infty$ we set $HM_\infty(R)$ for the linear space of bounded harmonic functions u on R. It forms a Banach space equipped with the supremum norm $\|u\|_\infty$. The Banach spaces $HM_p(R)$ are referred to as the *Hardy space* of exponent $p \in [1, \infty]$. Observe that

$$HM_p(R) \supset HM_q(R) \quad (1 \leqq p \leqq q \leqq \infty). \tag{1.1}$$

The space $HM_1(R)$ is also denoted by $HP(R)$. It forms a complete vector lattice and because of the identity $HP(R) = H(R)^+ \ominus H(R)^+$ each member of $HP(R)$ is said to be *essentially positive* on R. Similarly $HM_\infty(R)$ is denoted by $HB(R)$, which is together with $HP(R)$ the traditional notation in the classification theory of Riemann surfaces (cf. e.g. [1], [2], [9]). Since $HB(R) \subset HP(R)$ and $HP(R)$ is a complete vector lattice, we can consider the monotone envelope $HB'(R)$ of $HB(R)$ so that $u \in HP(R)$ belongs to $HB'(R)$ if and only if $u = \lim_{\mathbb{N}\ni n\to\infty}(u \wedge n) \vee (-n)$ locally uniformly on R, where $\vee$ and $\wedge$ are lattice operations in $HP(R)$. Functions

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in $HB'(R)$ are referred to as being *quasibounded.* It is viewed to be a Banach space as a Banach subspace of $HP(R)$.

As a Banach space in another category we denote by $HD(R)$ the linear space of Dirichle finite harmonic functions u on R. Here the Dirichlet finiteness of u means that the Dirichlet integral $D(u) := \int_R du \wedge *du$ of u taken over R is finite. If both of u and v are in $HD(R)$, then we can define their mutual Dirichlet integral $D(u,v) := \int_R du \wedge *dv$ as a finite number. Then $HD(R)$ is not only a Banach space but also a Hilbert space under the inner product $(u,v)_{HD} := u(a)v(a) + D(u,v)$. The Hilbert space $HD(R)$ is said to be the *Dirichlet space.* We also consider the linear space $HBD(R) := HB(R) \cap HD(R)$. It forms a Banach space under the norm $\|u\|_{HBD} := \|u\|_\infty + \sqrt{D(u)}$. Then we have the following table of inclusion relations.

$$\begin{array}{ccccc} & \subset & HB(R) \subset HM_p(R) & \subset & \\ HBD(R) & & & & HB'(R) \subset HP(R) \\ & \subset & HD(R) & \subset & \end{array}$$

Here the exponent p in the above table is restricted to $1 < p < \infty$. Every space $HX(R)$ in the above table is a subspace of $HP(R)$ as vector lattices. Let $HX(R)$ and $HY(R)$ be any tow spaces in the above table. The inclusion $HX(R) \subset HY(R)$ merely means the set theoretical inclusion but automatically $HX(R)$ is a subspace of $HY(R)$ as vector lattices. The Banach space structure of $HX(R)$ has nothing to do with that of $HY(R)$ in general. But if $HX(R)$ happens to be identical with some different $HY(R)$ as sets, then the Banach open mapping principle assures that Banach spaces $HX(R)$ and $HY(R)$ are homeomorphically isomorphic.

At this point we add a word on the base Riemann surface R for the six Banach spaces $HX(R)$ $(X = P,\ B',\ M_p\ (1 < p < \infty),\ B,\ D,\ BD)$. It is a traditional use of terminology to call R *open* or *closed* accordind as R is noncompact or compact, respectively. The total space of harmonic functions $H(R) = \mathbb{R}$ if and only if R is closed; "if part" is nothing but the rephrasing of the maximum principle but "only if part" is not at all trivial. Anyway it is quite reasonable to assume that R is open in our present study. Open Riemann surfaces are classified into two categories: *hyperbolic* and *parabolic.* An open Riemann surface R is said to be hyperbolic (or noparabolic) if R carries the Green function with arbitrarily given pole ζ in R, which is the minimal positive solution of the Poisson equation $-\Delta u = Dirac_\zeta$ on R, where $Dirac_\zeta$ is the Dirac measure supported by the poin ζ. Usually the class of parabolic (i.e. nonhyperbolic) Riemann surfaces is designated by the notation $\mathcal{O}_G$. There are many characterizations of the class $\mathcal{O}_G$, among which we state the following. We say that the *minimum principle* is valid for R if for any subregion $S \subset R$ and any $u \in H(S)^+ \cap C(\overline{S})$ we have $\inf_S u = \inf_{\partial S} u$. Then $R \in \mathcal{O}_G$ if and only if the minimum principle is valid for R. Using this characterization we can off hand see that if $R \in \mathcal{O}_G$, then $HP(R) = \mathbb{R}$ and a fortiori the six $HX(R) = \mathbb{R}$. For this reason, avoiding the trivial situation, R should be restricted to hyperbolic Riemann surfaces when we are concerned with the 6 Banach spaces $HX(R)$. Of course $HX(R) = \mathbb{R}$ can happen even for hyperbolic $R \in \mathcal{O}_G$. The hyperbolicity of R assures that the universal covering surface of R is the unit disc $\mathbb{D}$, i.e. there is a locally homeomorphic analytic mapping π (the projection) of $\mathbb{D}$ onto R with the associated Fuchsian group G of Möbius transformations of $\mathbb{D}$ such that

$$(1.2) \qquad H(R) \circ \pi = H(\mathbb{D})/G := \{u \in H(\mathbb{D}) : u \circ g = u \text{ for any } g \in G\}.$$

2. Reflexivity and separability. Our concern in this study is about the interplay between the Banach space structures of 6 spaces $HX(R)$ in the table in §1 and the conformal structures of base Riemann surfaces R. Especially we observe the reflexivity and the separability of $HX(R)$ and we ask how these properties of $HX(R)$ reflect on and are reflected by the conformal structures of R. Some of these properties are free from the choice of R but some of them heavily depend upon R. To get a certain view of the situation we first make an experimental consideration by taking R as a special Riemann surface the unit disc $\mathbb{D}$. The result is indicated in the following table, in which we let $1 < p < \infty$:

Space	$HP(\mathbb{D})$	$HB'(\mathbb{D})$	$HM_p(\mathbb{D})$	$HB(\mathbb{D})$	$HD(\mathbb{D})$	$HBD(\mathbb{D})$
Reflexivity	no	no	yes	no	yes	no
Separability	no	yes	yes	no	yes	no

We denote by $M(\partial\mathbb{D})$ the Banach space of Radon measures on the circle $\partial\mathbb{D}$ equipped with the norm $\|\mu\|$ given by the total mass $|\mu|(\partial\mathbb{D})$ of the total variation measure $|\mu|$ of each $\mu \in M(\partial\mathbb{D})$. Then we have the isometrically isomorphic representation of $HP(R) = HM_1(R)$ as $M(\partial\mathbb{D})$. The space $HB'(\delta)$ is represented as $L^1(\partial\mathbb{D})$ as identical Banach spaces, where the measure associated to the Lebesgue spaces on $\partial\mathbb{D}$ is $d\omega := d\theta/2\pi$ for the point $e^{i\theta}$. Similarly $HM_p(\mathbb{D}) = L^p(\partial\mathbb{D})$ $(1 < p \leqq \infty)$. Using these isometrically isomorphic representations of relevant spaces we can deduce the required properties in the above table. The Hilbert space $HD(R)$ is the easiest to handle. Fourier expansion method is also very helpful. The space $HBD(\mathbb{D})$ seems to be the hardest to treat. Let $\Lambda(\partial\mathbb{D})$ be the set of every Borel function u on $\partial\mathbb{D}$ such that

$$(2.1)\quad \|u\|_\Lambda = \text{ess.sup}_{\zeta\in\partial\mathbb{D}}|u(\zeta)| + \left(2\pi\int_{\partial\mathbb{D}}\left(\int_{\partial\mathbb{D}}\frac{|u(\zeta)-u(\xi)|^2}{|\zeta-\xi|^2}d\omega(\zeta)\right)d\omega(\xi)\right)^{1/2}$$

is finite. Then $HBD(\mathbb{D})$ is represented by the Banach space $(\Lambda(\partial\mathbb{D}), \|\cdot\|_\Lambda)$. Thus the problem can be transformed into the real analytic problem to prove that $\Lambda(\partial\mathbb{D})$ is neither reflexive nor separable. However we have not yet been successful in this direct procedure but fortunately successful in entirely different indirect method (cf. §3 below).

3. Conclusions. We now give our result for the case of general Riemann surfaces R also in the following table form. As in the above case of special $\mathbb{D}$ the exponents p in the Hardy spaces $HM_p(R)$ are restricted to $1 < p < \infty$ since $HM_1(R) = HP(R)$ and $HM_\infty = HB(R)$ are independently treated.

Space	$HP(R)$	$HB'(R)$	$HM_p(R)$	$HB(R)$	$HD(R)$	$HBD(R)$
Reflexivity	no*	no*	yes	no*	yes	no***
Separability	no**	yes	yes	no*	yes	no***

In the above table no* means no if and only if the space is of infinite dimension; no** means no if and only if the harmonic dimension of R, i.e. the cardinal number of the set of extremal rays of the positive cone $H(R)^+$, is strictly greater than the cardinal number $\aleph_0$ of countably infinite sets; no*** means no if and only if the Royden harmonic boundary of R contains at least one point of capacity zero (cf. below). We repeat once more the above explanations on the meanings of no*, no**, and no*** in the statements of four theorem forms.

THEOREM 3.1. *The Banach space $HX(R)$ is reflexive if and only if the space $HX(R)$ is of finite linear dimension for $X = P$, B', and B (cf [7] for $X = B$).*

THEOREM 3.2. *The following four conditions are equivalent by pairs: firstly, the Banach space $HBD(R)$ is reflexive; secondly, the Banach space $HBD(R)$ is separable; thirdly, $HBD(R) = HD(R)$; fourthly and lastly, every point of the Royden harmonic boundary $\delta_\mathcal{R}$ of R is of positive capacity.*

THEOREM 3.3. *The Banach space $HB(R)$ is separable if and only if the space $HB(R)$ is of finite linear dimension (cf. [8]).*

THEOREM 3.4. *The Banach space $HP(R)$ is separable if and only if the harmonic dimension of the base Riemann surface R is at most countably infinite.*

Generalizing the disc case, we observe that $HP(R) = M(\delta_1)$, where δ_1 is the Martin minimal boundary of R;

$$(3.5) \qquad HB'(R) = L^1(\delta_{\mathcal{W}}) = L^1(\partial\mathbb{D})/G \text{ (automorphic representation),}$$

where $\delta_{\mathcal{W}}$ is the Wiener harmonic boundary of R (cf. e.g. [2], [9]), which is a Stonean space, and the Lebesgue spaces over $\delta_{\mathcal{W}}$ are considered with respect to the harmonic measure $d\omega$ on $\delta_{\mathcal{W}}$, and $R = \mathbb{D}/G$ is the universal covering representation of R with the associated Fuchsian group G;

$$(3.6) \qquad HM_p(R) = L^p(\delta_{\mathcal{W}}) = L^p(\partial\mathbb{D})/G \text{ (automorphic representation)}$$

for each exponent $1 < p < +\infty$; $HB(R) = C(\delta_{\mathcal{W}})$. In particular, to derive representation identities (3.5) and (3.6), the following simple relation, the so called *Fuchsian invariance of Poisson forms*, plays a decisive role. Let $P(z,\zeta)$ be the Poisson kernel so that $P(z,\zeta) := \Re((\zeta+z)/(\zeta-z)) = (1-|z|^2)/|\zeta-z|^2$ and $h \in G$ so that $h(z) := \lambda(z-c)/(1-\bar{c}z)$ with $|\lambda| = 1$ and $|c| < 1$ is a Möbius transfomation in G. Then we have

$$(3.7). \qquad P(h(z),h(\zeta))d\arg h(\zeta) = P(z,\zeta)d\arg\zeta.$$

The identity assures that, for harmonic functions

$$u(z) = \frac{1}{2\pi}\int_{\partial\mathbb{D}} P(z,\zeta)f(\zeta)d\arg\zeta$$

on $\mathbb{D}$ with $f \in L^1(\partial\mathbb{D})$, $u \in HB'(\mathbb{D})/G$ if and only if $f \in L^1(\partial\mathbb{D})/G$. Using these isometrically isomorphic representations, the proof of the above table can be completed except for two spaces $HD(R)$ and $HBD(R)$. For these two spaces no convenient easily manageable representations cannot be expected. Because of $HD(R)$ being a Hilbert space and the existence of triangulation of R we can somehow handle $HD(R)$. The Royden harmonic boundary $\delta_{\mathcal{R}}$ of R (cf. e.g. [2], [9]) and the theory of capacities on it (cf. [6]) play the decisive role in the proof concerning $HBD(R)$. Anyway the proof of the conclusion for $HBD(R)$ is not so straightforward as those for other spaces.

4. Applications. The problem of clarifying the conformal structure of R when two classes $HX(R)$ and $HY(R)$ in the table of §1 coincide with each other is said to be an *inverse inclusion problem* in the classification theory of Riemann surfaces. H. Masaoka and S. Segawa [4] obtained the following result: the coincidense relation $HB(R) = HP(R)$ is equivalent to $\dim HB(R) = \dim HP(R) < \infty$. Suppose $HB(R) = HP(R)$, then the table in §1 shows that $HB(R) = HB'(R)$. We give an easy proof for the essential part of the Masaoka-Segawa theorem. From the table in §3, it follows that $HB(R)$ is separable along with the separability of $HB'(R)$ so that $HB(R)$ is of finite dimension. Very simple, isn't it.

Recently, again H. Masaoka [3] proved that (cf. also [5]) the coincidense relation $HB(R) = HD(R)$ is equivalent to $\dim HB(R) = \dim HD(R) < +\infty$. Once more we look at the table in §3 to give a short proof to the essential part of the Masaoka theorem. Since $HD(R)$ is separable (reflexive, resp) so is $HB(R)$, which occurs only when $\dim HB(R) < \infty$. Again extremely simple proof is provided (cf. [7], [8]).

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RANKS OF INVARIANT SUBSPACES OF THE HARDY SPACE OVER THE BIDISK

KOU HEI IZUCHI

Let $T : H \to H$ be a bounded linear operator on a Hilbert space H. For a subset E of H, we denote by $[E]_H$ the smallest invariant subspace of H for T containing E. We denote by $rank\, H$ the smallest number of elements in E satisfying $[E]_H = H$, that is, $rank\, H$ is the smallest number of generators of H as an invariant subspace for T. An element $x \in H$ is called cyclic if $[x]_H = H$. Sometimes the space $H \ominus TH$ is called a wandering subspace of H. It is easy to see that $rank\, H \geq dim\,(H \ominus TH)$. If $[H \ominus TH]_H = H$, then we have $rank\, H = dim\,(H \ominus TH)$. In the case that $H = H^2(\mathbb{D})$, the Hardy space over the open unit disk $\mathbb{D}$ and $T = T_z$, the unilateral shift, by the Beurling theorem [3] we have $rank\, M = 1$ for every invariant subspace M of $H^2(\mathbb{D})$ for T_z. In the case that $H = \mathcal{D}$, the Dirichlet space over $\mathbb{D}$ and $T = T_z$ the multiplication operator by the coordinate function z, by the Richter theorem [12] we have $rank\, M = dim\,(M \ominus T_z M) = 1$ for every invariant subspace M of $\mathcal{D}$. In the case that $H = L^2_a(\mathbb{D})$, the Bergman space over $\mathbb{D}$ and $T = \mathcal{B}$, the Bergman shift, by the Aleman-Richter-Sundberg theorem [1] we have $rank\, M = dim\,(M \ominus \mathcal{B}M)$ for every invariant subspace M of $L^2_a(\mathbb{D})$, see also [16], and it is known that $dim\,(M \ominus \mathcal{B}M)$ ranges from 1 to ∞, see [2, 7, 8, 9]. For a given H and T, one of the basic problems is to determine the rank of M for every invariant subspace M of H for T. In many cases, this problem is quite difficult.

Let $H^2 = H^2(\mathbb{D}^2)$ be the Hardy space over the bidisk $\mathbb{D}^2$. We identify a function in H^2 with its boundary function on the distinguished boundary Γ^2 of $\mathbb{D}^2$, so we think of H^2 as a closed subspace of the Lebesgue space $L^2(\Gamma^2)$. We denote by $\|f\|$ the norm of $f \in H^2$, and by $\langle \cdot, \cdot \rangle$ the inner product in H^2. Let z, w be variables in $\mathbb{D}^2$. We denote by $H^2(z)$ the z-variable Hardy space, and we think of $H^2(z)$ as a closed subspace of H^2. A function $\varphi(z)$ in $H^2(z)$ is called inner if $|\varphi(z)| = 1$ a.e. on Γ. An example of inner functions is a Blaschke product;

$$b(z) = \prod_{n=1}^{\infty} \frac{-\overline{\alpha}_n}{|\alpha_n|} \frac{z - \alpha_n}{1 - \overline{\alpha}_n z},$$

where $\{\alpha_n\}_{n\geq 1}$ is a sequence in $\mathbb{D}$ satisfying $\sum_{n=1}^{\infty}(1-|\alpha_n|)<\infty$ and we consider $-\overline{\alpha}_n/|\alpha_n|=1$ if $\alpha_n=0$. Another example is a singular inner function;

$$S_\mu(z)=\exp\Big(-\int_{\partial\mathbb{D}}\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\mu(e^{i\theta})\Big),$$

where μ is a positive bounded singular measure on $\partial\mathbb{D}$, see [6, 10] for the study of $H^2(z)$.

Let T_z, T_w be multiplication operators on H^2 by z and w. A closed subspace M of H^2 is called invariant if $T_zM\subset M$ and $T_wM\subset M$. The structure of invariant subspaces of H^2 over the bidisk is extremely complicated. For a closed subspace E of H^2, we denote by P_E the orthogonal projection from H^2 onto E. See books [5, 13] for the study of the Hardy space H^2 over $\mathbb{D}^2$.

Let M be an invariant subspace of H^2. By the Wold decomposition theorem, we have

$$M=\sum_{n=0}^{\infty}\oplus(M\ominus wM)w^n.$$

So many properties of invariant subspaces M are considered to be encoded in ones of $M\ominus wM$. So to study the structure of invariant subspaces of H^2, $M\ominus wM$ is one of the most important spaces. To study $M\ominus wM$, Yang [17] defined the *fringe operator* $\mathcal{F}_z$ on $M\ominus wM$ by

$$\mathcal{F}_zf=P_{M\ominus wM}(T_zf),\quad f\in M\ominus wM,$$

and studied the properties of $\mathcal{F}_z$, see [17, 18, 19]. As mentioned in the first paragraph, *rank* $(M\ominus wM)$ is the smallest number of generators of $M\ominus wM$ as an invariant subspace for $\mathcal{F}_z$. Similarly we can define *rank* M by the smallest number of generators of M as an invariant subspace of H^2. It is not difficult to see that *rank* $(M\ominus wM)\leq$ *rank* M, so if *rank* $(M\ominus wM)=\infty$, then *rank* $M=\infty$. It seems fairly difficult to determine *rank* $(M\ominus wM)$ and *rank* M generally.

Let $\{\varphi_n(z)\}_{n\geq 0}$ be a sequence of inner functions such that $\varphi_n(z)/\varphi_{n+1}(z)$ is a nonconstant inner function for every $n\geq 0$. It is not an essential condition that $\varphi_n(z)/\varphi_{n+1}(z)$ is nonconstant for *every* $n\geq 0$. Moreover we assume that $\{\varphi_n(z)\}_{n\geq 0}$ does not have nonconstant common inner factors. This assumption is also not essential in this paper. But for the sake of simplicity we assume these conditions. We set

$$\zeta_n(z)=\frac{\varphi_n(z)}{\varphi_{n+1}(z)},\quad n\geq 0.$$

Then $\{\zeta_n(z)\}_{n\geq 0}$ is a sequence of nonconstant inner functions and

$$\varphi_n(z) = \prod_{j=n}^{\infty} \zeta_j(z), \quad n \geq 0.$$

We define

$$(\#) \qquad \mathcal{M} = \sum_{n=0}^{\infty} \oplus w^n \varphi_n(z) H^2(z) = \sum_{n=0}^{\infty} w^n \varphi_n(z) H^2.$$

Then $\mathcal{M}$ is an invariant subspace of H^2. This type of invariant subspaces of H^2 was studied in [13, 14, 15]. In [13], Rudin essentially showed an example of $\mathcal{M}$ satisfying $rank\,\mathcal{M} = \infty$. A motivation of this study comes from Rudin's work. The form of $\mathcal{M}$ is so simple, nevertheless the number of $rank\,\mathcal{M}$ is unclear until now. At the first glance, it seems $rank\,\mathcal{M} = \infty$. But it is not correct. We consider $rank\,\mathcal{M}$ when $\varphi_0(z)$ is a Blaschke product.

We have

$$\begin{aligned} \mathcal{M} \ominus w\mathcal{M} &= \sum_{n=0}^{\infty} \oplus w^n \big(\varphi_n(z) H^2(z) \ominus \varphi_{n-1}(z) H^2(z)\big) \\ &= \sum_{n=0}^{\infty} \oplus w^n \varphi_n(z) \big(H^2(z) \ominus \zeta_{n-1}(z) H^2(z)\big). \end{aligned}$$

Assume that $\varphi_0(z)$ is a Blaschke product. Then $\zeta_n(z)$ is also a Blaschke product for every $n \geq 0$. For each $\alpha \in \mathbb{D}$, let

$$\mathbb{N}_\alpha = \{n \in \mathbb{N} : \zeta_{n-1}(\alpha) = 0\},$$

where $\mathbb{N}$ is the set of positive integers. We denote by $\#\mathbb{N}_\alpha$ the number of elements in $\mathbb{N}_\alpha$. In [14], Seto proved the following;

Lemma 1. *For* $\alpha \in \mathbb{D}$

$$dim\,\big((\mathcal{M} \ominus w\mathcal{M}) \ominus (\mathcal{F}_z - \alpha I)(\mathcal{M} \ominus w\mathcal{M})\big) = \#\mathbb{N}_\alpha + 1.$$

Lemma 2. *Let T be a bounded linear operator on a Hilbert space H and $\alpha \in \mathbb{C}$. Then the rank of H for T is larger than or equal to $dim\,(H \ominus (T - \alpha I)H)$.*

By these two lemmas, we have the following;

Lemma 3.

$$\sup_{\alpha \in \mathbb{D}} \#\mathbb{N}_\alpha + 1 \leq rank\,(\mathcal{M} \ominus w\mathcal{M}) \leq rank\,\mathcal{M}.$$

Hence if $\sup_{\alpha\in\mathbb{D}} \#\mathbb{N}_\alpha = \infty$, then $rank\,(\mathcal{M}\ominus w\mathcal{M}) = rank\,\mathcal{M} = \infty$. So we assume that

$$m_0 := \max_{\alpha\in\mathbb{D}} \#\mathbb{N}_\alpha < \infty.$$

The following is the main theorem in this paper.

Theorem 1. *If $\varphi_0(z)$ is a Blaschke product, then*

$$rank\,\mathcal{M} = \sup_{\alpha\in\mathbb{D}} \#\mathbb{N}_\alpha + 1.$$

To prove this, we find functions $G_1, G_2, \cdots, G_{m_0}$ in $\mathcal{M}$ satisfying

$$[G_1, G_2, \cdots, G_{m_0}, \varphi_0(z)]_{H^2} = \mathcal{M}.$$

The key point is how to prove

$$w\varphi_1(z)H^2 \subset [G_1, G_2, \cdots, G_{m_0}, \varphi_0(z)]_{H^2}.$$

The ideas of the proof are in defining functions $G_1, G_2, \cdots, G_{m_0}$ in $\mathcal{M}$ for which $G_1, G_2, \cdots, G_{m_0}, \varphi_0(z)$ make many one variable outer functions and $G_1, G_2, \cdots, G_{m_0}$ fill the gap of spaces.

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The Lax conjecture and its related topics

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Abstract The c-numerical range of an $n \times n$ matrix T ($c \in \mathbf{R}^n$) can be realized as the classical numerical range of some $(n!) \times (n!)$ matrix S.

1. Hyperbolic differential operators and the related conjectures

We consider a partial differential operatror D in the Euclidean space with complex constant coefficients

$$D = \sum_{i_1+i_2+\ldots+i_m \leq n} a_{i_1,i_2,\ldots,i_m} \frac{\partial^{i_1+i_2+\cdots+i_m}}{\partial x_1^{i_1} \partial x_2^{i_2} \cdot \partial x_m^{i_m}}.$$

We consider its principal part and the corresponding characteristic polynomial

$$P(\xi_1, \xi_2, \ldots, \xi_m) = \sum_{i_1+i_2+\ldots+i_m=n} a_{i_1,i_2,\ldots,i_m} \xi_1^{i_1} \xi_2^{i_2} \cdots \xi_m^{i_m}.$$

The operator D is said to be *hyperbolic* with respect to $\xi = (1, 0, \ldots, 0)$ if $P(1, 0, \ldots, 0) \neq 0$ and the factorization

$$P(t, \xi_2, \ldots, \xi_m) = c_0(t - \lambda_1(\xi_2, , \xi_m))(t - \lambda_2(\xi_2, \ldots, \xi_m)) \cdots (t - \lambda_n(\xi_2, \ldots, \xi_m))$$

holds for some real numbers $\lambda_1, \ldots, \lambda_n$ for any real point $(\xi_2, \ldots, \xi_m) \in \mathbf{R}^{m-1}$, whete c_0 is a constant.

A typical example of such a polynomial is given by

$$P(\xi_1, \xi_2, \ldots, \xi_m) = -\xi_1^2 + \xi_2^2 + \xi_3^2 + \cdots + \xi_m^2.$$

Suppose that $H_2, H_3, \ldots, H_m$ are $(m-1)$-ple of $n \times n$ Hermitian matrices. Then the form,

$$P(t, \xi_2, \ldots, \xi_m) = \det(tI_n + \xi_2 H_2 + \cdots + \xi_m H_m)$$

is hyperbolic with respect to $(1, 0, \ldots, 0)$. In 1981, Czech mathematician Fiedler conjectured that the converse is true for $m = 3$. Let $P(t, x, y)$ be a real ternary form hyperbolic with respect to $(1,0,0)$ and satisfies $F(1,0,0) = 1$. He conjectured there exist a pair of Hermitian matrices H_2, H_3 satisfying $F(t,x,y) = \det(tI_n + xH_2 + yH_3)$. His conjectured is motivated by the study of numerical ranges of matrices. Before Fiedler's one, in 1958, P. D. Lax conjectured a stronger property:

A real ternary form $P(t,x,y)$ hyperbolic with rspect to $(1,0,0)$ satisfying $P(1,0,0) = 1$ is writen as

$$P(t,x,y) = \det(tI_n + xS_1 + yS_2)$$

by some real symmetric matrices S_1, S_2.

For $m = 4$, we can consider analogous problem of these conjectures. An analogous conjecture of Fiedler's one for $m = 4$ is open now. However it is well-known that the analogy of the Lax conjecture for $m = 4$ is false. There is a triple system of 3×3 Hermitian matrices for which any real 3×3 symmetric matrices S_1, S_2, S_3 do not satisfy

$$\det(tI_3 + xH_1 + yH_2 + zH_3) = \det(tI_3 + xS_1 + yS_2 + zS_3).$$

In [4] a concrete example is given as the following.

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, H_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}, H_3 = \begin{pmatrix} 0 & 1+i & 1-i \\ 1-i & 0 & 1-i \\ 1+i & 1+i & 0 \end{pmatrix}.$$

The absence of the symmetric matrices S_1, S_2, S_3 satisfying the condition can be proved by a combinatorial method.

2. New results on the numerical ranges

In 2005, the Lax conjecture was proved to be true by Lewis, Parrilo and Ramana [9]. Their proof based on a deep result of an Israelite mathematician Vinnikov [10] (also [7]). Vinnikov used transcendel theta functions and Riemann surface theory to treat related theory of real plane curves. In case the curve $F(1,x,y) = 0$ is a rational curve, Henrion [6] provided an alternating concrete method to consctrut real symmnetric matrices S_1, S_2.

The solution of the Lax conjecture is applicable to solve some problems in the theory of generalized numerical ranges of matrices. Let T be an $n \times n$ complex matrix. The numerical range $W(T)$ of T was introduced by Toeplitz as the set

$$W(T) = \{\xi^* T\xi : \xi \in \mathbf{C}^n = 1\}.$$

In 1919, Hausdorff proved that the range $W(T)$ was convex. In 1951, a German mathematician Kippenhahn proved that the numerical range $W(T)$ is the convex hull of the real affine part of the dual curve of the algebraic curve

$$F(t,x,y) = \det(tI_n + x/2(T+T^*) - iy/2(T-T^*)) = 0.$$

As an immediate result of this theorem and the solution of the Lax conjecture, we have the following.

Theorem Suppose that A is an $n \times n$ complex matrix. Then there exists an $n \times n$ symmetric complex matrix B satisfying $W(A) = W(B)$.

As a generalization of $W(T)$, a c-numerical range of T was introduced as

$$W_c(T) = \{\sum_{j=1}^{n} c_j \xi_j^* T \xi_j : \{\xi_1, \xi_2, \ldots, \xi_n\} \text{ is an orthonormal basis of } \mathbf{C}^n\},$$

for a real vector $c = (c_1, c_2, \ldots, c_n) \in \mathbf{R}^n$. In 1975, Westwick proved that the range $W_c(T)$ is also convex. In this year, Chien and I provided a progress on Westwick's result.

Theorem[CN-2011] Suppose that T is an $n \times n$ complex matrix and $c = (c_1, c_2, \ldots, c_n)$ is an arbitrary vector in $\mathbf{R}^n$. Then there exists a complex symmetric $(n!) \times (n!)$ matrix S satisfying

$$W_c(T) = W(S).$$

In 1983, Au-Yeung and Tsing proved that the range

$$W_c(H_1, H_2, H_3) = \{(\sum_{j=1}^{n} c_j \xi_j^* H_1 \xi_j, \sum_{j=1}^{n} c_j \xi_j^* H_2 \xi_j, \sum_{j=1}^{n} c_j \xi_j^* H_3 \xi_j) :$$

$$\{\xi_1, \xi_2, \ldots, \xi_n\} \text{ is an orthonormal basis of } \mathbf{C}^n\}$$

is convex for every triple of $n \times n$ Hermtian matrices H_1, H_2, H_3 for $n \geq 3$. If the generalized Fiedler's conjecture for $m = 4$ is affirmatively solved, it would implies that the existence of $(n!) \times (n!)$ Hermitian matrices K_1, K_2, K_3 satisfying $W_c(H_1, H_2, H_3) = W(K_1, K_2, K_3)$. The validity of this relation is still open.

To prove the convexity of the range $W_c(T)$, Westwick used Morse theory. In [?] an analogous result for a Krein space operator was provided by using Morse theory. We consider a complex vector space $\mathbf{C}^n$ with an indefinite inner product

$$[(\xi_1, \xi_2, \ldots, \xi_n)^T, (\eta_1, \eta_2, \ldots, \eta_n)^T] = (\xi_1\overline{\eta_1} + \cdots + \xi_p\overline{\eta_p}) - (\xi_{p+1}\overline{\eta_{p+1}} + \cdots + \xi_n\overline{\eta_n}).$$

A vector space $\mathbf{C}^n$ with such an indefinite inner product is called a *Krein space.* The Krein space numerical ranges of an operator T on $(\mathbf{C}^n, [\cdot, \cdot])$ is defined as the following:

$$W_+^J(T) = \{[Tx, x]/[x, x] : x \in \mathbf{C}^n, [x, x] > 0\},$$
$$W_-^J(T) = \{[Tx, x]/[x, x] : x \in \mathbf{C}^n, [x, x] < 0\}.$$

The study of an infinite dimensional vector space with an indefinite inner product was actively performed by a Russian mathematician Selim G. Krein. However we restrict our attension to the finite dimensional space. Some fundamental properties of the Krein space numerical ranges were found by Rodman et al in 1980's or 1990's. The convexity of the ranges $W_+^J(T)$, $W_-^J(T)$ was proved. These ranges are usually unbounded. Sometimes their union

$$W^J(T) = W_+^J(T) \cup W_-^J(T)$$

is discussed. This range is usually disconnected and non-convex. A linear operator T on a Krein space satisfying $[Tx, y] = [x, Ty]$ for all $x, y \in \mathbf{C}^n$ is said to be J-Hermitian. Unfortunately J-Hermtian operators are not necessarily semi-simple. Those may have imaginary eigenvalues. A linear operator U on $\mathbf{C}^n$ is J-unitary if $[Ux, Uy] = [x, y]$ for all $x, y \in \mathbf{C}^n$. The eigenvalues of a J-Hermitian operator T are real if T is diagonalizable by a J-unitary operator U: $UTU^{-1} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. In this case, there exists a basis $\{x_1, \ldots, x_p, x_{p+1}, \ldots, x_n\}$ of $\mathbf{C}^n$ satisfying

$$[x_1, x_1] = \cdots = [x_p, x_p] = 1, \quad [x_{p+1}, x_{p+1}] = \cdots = [x_n, x_n] = -1,$$
$$[x_j, x_k] = 0 \quad \text{for}\, j \neq k$$

and

$$Tx_j = \lambda_{\sigma(j)} x_j$$

for some permutation σ. For a J-unitarily diagonalizable J-Hermitian operator T, we set

$$\begin{aligned}\sigma_+(T) &= \{\lambda \in \mathbf{R} : Tx = \lambda x, \text{for some}\, x \in \mathbf{C}^n, [x, x] > 0\} \\ &= \{\lambda_1(T) \geq \lambda_2(T) \geq \cdots \geq \lambda_p(T)\}, \\ \sigma_-(T) &= \{\lambda \in \mathbf{R} : Tx = \lambda x, \text{for some}\, x \in \mathbf{C}^n, [x, x] < 0\}, \\ &= \{\lambda_{p+1}(T) \geq \lambda_{p+2}(T) \geq \cdots \geq \lambda_n(T)\}.\end{aligned}$$

We call that Hermitian operators T, S on a finite-dimensional Hilbert space $\mathbf{C}^n$ with eigenvalues

$$\lambda_1(T) \geq \lambda_2(T) \geq \cdots \geq \lambda_n(T), \lambda_1(S) \geq \lambda_2(S) \geq \cdots \geq \lambda_n(S)$$

satisfies the following trace inequality

$$\sum_{j=1}^{n} \lambda_j(T)\lambda_{n+1-j}(S) \leq \mathrm{tr}(TS) \leq \sum_{j=1}^{n} \lambda_j(T)\lambda_j(S).$$

This inequality was proved by Richter in 1958. Analogous inequlity holds for Krein space operators under some conditions.

Theorem [Bebiano, N. Providência, Lemos, Soares; 2005] Let the spectra of J-unitarily diagonalizable non-scalar J-Hermitian operators T, S on $(\mathbf{C}^n, [\cdot,\cdot])$ generate the intervals

$$[\lambda_p(T), \lambda_1(T)] = \mathrm{Conv}(\sigma_+^J(T)), [\lambda_n(T), \lambda_{p+1}(T)] = \mathrm{Conv}(\sigma_-^J(T)$$

$$[\lambda_p(S), \lambda_1(S)] = \mathrm{Conv}(\sigma_+^J(S)), [\lambda_n(S), \lambda_{p+1}(S)] = \mathrm{Conv}(\sigma_-^J(S))$$

(I) Suppose that the spectra satisfy (i) $\lambda_{p+1}(T) \leq \lambda_p(T), \lambda_{p+1}(S) \leq \lambda_p(S)$ or (ii) $\lambda_{p+1}(T) \leq \lambda_p(T), \lambda_1(S) \leq \lambda_n(S)$.

In case (i), the equation

$$W_S^J(T) = \{\mathrm{tr}(SUTU^{-1}) : U \text{is } J-\text{unitary}\}$$

$$= [\sum_{j=1} \lambda_j(T)\lambda_{p+1-j}(S) + \sum_{j=p+1}^{n} \lambda_j(T)\lambda_{n+p+1-j}(S), \quad +\infty),$$

holds. In case (ii), the equation

$$W_S^J(T) = \{\mathrm{tr}(SUTU^{-1}) : U \text{is } J-\text{unitary}\} = (-\infty, \quad \sum_{j=1}^{n} \lambda_j(T)\lambda_j(S)]$$

holds. (II) Suppose that at least one of J-unitarily diagonalizable non-scalar J-Hermitian operators T, S, for instance, T satisfies

$$[\lambda_p(T), \lambda_1(T)] \cap [\lambda_n(T), \lambda_{p+1}] \supset [a, b]$$

for some $a < b$. Then $W_S^J(T) = (-\infty, \infty)$.

Example. In the case (I): If $\lambda_1(S) = 1$, $\lambda_2(S) = \cdots = \lambda_n(S) = 0$, then

$$W_S^J(T) = W_+^J(T) = [\lambda_p(T), +\infty).$$

If $\lambda_{p+1}(S) = 1, \lambda_{p+2}(S) = \cdots = \lambda_n(S) = \lambda_1(S) = \cdots = \lambda_p(S) = 0$, then

$$W_S^J(T) = W_-^J(T) = (-\infty, \lambda_{p+1}(T)].$$

Theorem [Bebiano, N, Providência ; [1], [2]] Suppose that S is a J-unitarily diagonalizable J-Hermitian operator on $(\mathbf{C}^n, [\cdot,\cdot])$ satisfying

$$[\lambda_p(S), \lambda_1(S)] \cap [\lambda_n(S), \lambda_{p+1}(S)] = \emptyset,$$

and T is an operator expressed as the sum $T_1 + iT_2$ by J-Hermitian operators T_1, T_2 with some angle θ for which $T_0 = \cos\theta T_1 + \sin\theta T_2$ satisfies

$$[\lambda_p(T_0), \lambda_1(T_0)] \cap [\lambda_n(T_0), \lambda_{p+1}(T_0)] = \emptyset.$$

Then the range

$$W_S^J(T) = \{\mathrm{tr}(SUTU^{-1}) : U \text{is } J-\text{unitary}\}$$

is a closed convex set.

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WE CONSIDER φ–MEANS (φ–平均を考える)

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Twenty-two years ago, I discussed the following question with my respected colleague Osamu Hatori:

Which come first, the equality or the inequality?

Though I forget the details of our discussion at that time, I remember that my answer was "equality" and that Hatori's opinion was difficult for me. He may have said that "inequality" was primary or he may not.

Later I encountered such a chicken-and-egg question in the plenary talk by Prof. Mikio Sato in the meeting of Mathematical Society of Japan. He talked about the question:

Which come first, the function or the equation?

He said that he could not give an exact answer to it, as well as to the question "Which come first, the real number or the complex number?"

Recently I met Professor Hatori at the conference held in Shinshu University, and the equality-and-inequality question came up among us again. This time, I was aware of the nonsense of the discussion about such a question. Thus, it took twenty-two years for me to conclude this discussion, as if Junji Kinoshita said that four hundred years passed till people know Shakespeare's true motive of "The merchant of Venice".

The other day, when I talked about inequalities in the seminar at Toho University, I heard an attendance saying

"There are many, many equalities, but few inequalities that are essential."

I cannot verify the truth of this statement, but I feel so. For example, we take up Jensen's inequality as one of the essential inequalities:

Jensen's inequality: Let (Ω, μ) be a probability space, and let I be an interval in the real line. If f is a real-valued function on Ω with range in I and if δ is a convex function on I, then

$$\delta\left(\int_\Omega f\, d\mu\right) \le \int_\Omega (\delta \circ f)\, d\mu. \tag{1}$$

As is well known, Jensen's inequality yields Hölder's one. Recently, it is pointed out that a lot of Hua-type inequalities are the incarnation of Jensen's inequality.

In case that μ is discrete, the inequality (1) tells us nothing but the convexity of δ. Now, let us split this convexity into two parts; a numerator and a denominator. Indeed, we write $\delta = \frac{\psi}{\varphi}$. Then Jensen's inequality will assert that

$$\nabla_\varphi \le \nabla_\psi \quad \Longrightarrow \quad M_\varphi(f) \le M_\psi(f),$$

where ∇_φ is the quasi-arithmetic mean induced by φ and $M_\varphi(f)$ denotes the φ–mean of f. This says that Jensen's inequality shows the order-preserving of some of mean functions. This result also explains the geometric aspect of the refinement of a φ–mean.

We have studied this subject as follows: When we were considering the refinement of a φ–mean, Mr. Nakasuji introduced the condition (M). We carried over the condition (M) to the field of means and found the above order-preserving. I think that our idea is based on "the fundamental principle of differential and integral calculus so-called Bisekibun no Genri". Moreover, our idea reminds me the Furuta inequality. Prof. Takayuki Furuta split the power exponent of positive operator into two fractional powers to discover his famous inequality. It was epoch-making and developed the fruitful theory of operator means. Here we must say that our result is not worthy of Huruta's inequality.

We wrote about the above topic in the paper [6]. This talk is the survey of it.

上記和訳：

２２年前畏友羽鳥理さんと「等式が先か、不等式が先か」と言うことを議論した事があります。その時、僕は等式派だったのですが、無論理由などは全く記憶にありません。彼は不等式派のようなそうでないようなその時は彼の主張が良く分かりませんでした。

丁度、佐藤幹夫先生があるとき学会の総合講演で「関数が先か、方程式が先か」を論じられ、「実数が先か、複素数が先か、虚実曰く云々......」と結局はよくわからなかったように。

最近信州のある研究集会で羽鳥さんに会ったとき、「等式が先か、不等式が先か」と言う議論は無意味であるという事に気付かされました。あたかも「ベニスの悪徳商人の正体が４００年の時を経て知らされた」と木下順二が言ったように。

先日東邦大でのセミナーの折、ある人に等式は沢山あるが、不等式は本質的に少ないという意味の事を言われ、真意の程は別として共感を覚えました。その本質的な不等式の一つに Jensen の不等式があると思います。これは確率空間 (Ω, μ) 及び

区間 I が与えられたとき、関数 $f:\Omega\to I$ 及び I 上の凸関数 δ に対して、

$$\delta\left(\int f d\mu\right)\leq\int\delta\circ f d\mu$$

が成り立つ事を主張するものです。Hölder の不等式は言うに及ばず、最近では Hua の不等式までが Jensen の不等式の変身である事が分かっています。

さてこれは μ を離散に取り、f を動かせば δ の凸性を直接表しています。しかしこの凸性を分解すると面白い事が見えて来ます。つまり $\delta=\frac{\psi}{\varphi}$ と分解するのです。このとき、Jensen の不等式は

$$\nabla_{\varphi}\leq\nabla_{\psi}\Rightarrow M_{\varphi}(f)\leq M_{\psi}(f)$$

を主張しています。ここで ∇_{φ} は φ が導く擬相加平均で、$M_{\varphi}(f)$ は f の φ–mean と呼ばれる擬相加平均を表しています。従って Jensen の不等式 はある種の平均関数はその順序を保存すると述べている事になります。この結果から φ–mean の細分に関する幾何学的性質が明らかになって行きます。実は時系列的に言えば、最初 φ–means の細分を考えるとき、中筋さんが条件 (M) を考えられたのですが、これを平均の世界に焼き直すと順序保存が見えて来たという訳です。いずれにしてもこれは所謂「微積分の原理」 の一つの実現と考えられます。

ここで思い出すのは、Furuta 不等式の創始者古田孝之先生が正作用素の指数を分数に分解する事によって、作用素不等式の分野に大きな発展をもたらした事です。最も我々の作業はその万分の一にも達しませんが。

以下は講演者達による論文 [6] を纏めたものであります。

和訳終

We are interested in means of real-valued measurable functions induced by strictly monotone functions. These means are somewhat different from continuously differentiable means, i. e. , C^1–means introducing by J. I. Fujii, etc. [1], but they include many known numerical means. Here we first give a new interpretation of Jensen's inequality by such a mean and we next consider some geometric properties of such means, as an application of it.

Here we denote by (Ω,μ), I and f a probability space, an interval of $\mathbb{R}$ and a real-valued measurable function on Ω with $f(\omega)\in I$ for almost all $\omega\in\Omega$, respectively. Let $C(I)$ be the real linear space of all continuous real-valued functions defined on I. Let $C^+_{sm}(I)$ (resp. $C^-_{sm}(I)$) be the set of all $\varphi\in C(I)$ which is strictly monotone increasing (resp. decreasing) on I. Then $C^+_{sm}(I)$ (resp. $C^-_{sm}(I)$) is a positive (resp. negative) cone of $C(I)$. Put $C_{sm}(I)=C^+_{sm}(I)\cup C^-_{sm}(I)$. Then $C_{sm}(I)$ denotes the set of all strictly monotone continuous functions on I

Let $C_{sm,f}(I)$ be the set of all $\varphi\in C_{sm}(I)$ with $\varphi\circ f\in L^1(\Omega,\mu)$. Let φ be an arbitrary function of $C_{sm,f}(I)$. Since $\varphi(I)$ is an interval of $\mathbb{R}$ and μ is a probability measure on Ω, it follows that

$$\int (\varphi \circ f) d\mu \in \varphi(I).$$

Then there exists a unique real number$M_\varphi(f) \in I$ such that $\int (\varphi \circ f) d\mu = \varphi(M_\varphi(f))$. Since φ is one-to-one, it follows that

$$M_\varphi(f) = \varphi^{-1}\left(\int (\varphi \circ f) d\mu\right).$$

We call $M_\varphi(f)$ a φ–quasi-arithmetic mean of f with respect to μ (or simply, φ–mean of f). A φ–mean of f has the following invariant property :

$$M_\varphi(f) = M_{a\varphi+b}(f)$$

for each $a, b \in \mathbb{R}$ with $a \neq 0$.

1. Main results

In this section, we first give a new interpretation of Jensen's inequality by φ–mean. Next, as an application, we consider some geometric properties of φ–means of a real-valued measurable function f on Ω.

The first result asserts that a φ–mean function : $\nabla_\varphi \longrightarrow M_\varphi(f)$ is well-defined and order-preserving, and this assertion simultaneously gives a new interpretation of Jensen's inequality. However, this assertion also teaches us that a simple inequality yields a complicated inequality.

Theorem 1. *Suppose that f is non-constant and $\varphi, \psi \in C_{sm,f}(I)$. Then*

(i) *If $\nabla_\varphi \leq \nabla_\psi$ holds, then $M_\varphi(f) \leq M_\psi(f)$.*
(ii) *If $\nabla_\varphi < \nabla_\psi$ holds, then $M_\varphi(f) < M_\psi(f)$.*

The next result asserts that there is a strictly monotone increasing φ–mean (continuous) path between two comparable φ–means.

Theorem 2. *Suppose that f is non-constant and $\varphi, \psi \in C_{sm,f}(I)$ with $\nabla_\varphi < \nabla_\psi$.*

(i) *If $\varphi, \psi \in C^+_{sm}(I)$ (or $C^-_{sm}(I)$), then a function : $s \longrightarrow M_{(1-s)\varphi+s\psi}(f)$ is strictly monotone increasing on $[0, 1]$.*
(ii) *If $\varphi, \psi - \varphi \in C^+_{sm}(I)$ (resp. $C^-_{sm}(I)$) and $\psi(x) - \varphi(x) \geq 0$ (resp. ≤ 0) for all $x \in I$, then a function : $s \longrightarrow M_{(1-s)\varphi+s\psi}(f)$ is strictly monotone increasing and continuous on $[0, 1]$.*

The next result asserts that the φ–mean function is strictly concave (or convex) on a suitable convex subset of $C_{sm,f}(I)$.

Theorem 3. *Suppose that f is non-constant and $\varphi, \psi \in C_{sm,f}(I)$ with $\nabla_\varphi < \nabla_\psi$. Then*

(i) *If* $\varphi, \psi - \varphi \in C^+_{sm}(I)$ *(resp.* $C^-_{sm}(I)$*) and* ψ *is convex (resp. concave) on* I*, then*

$$(1-t)M_\varphi(f) + tM_\psi(f) < M_{(1-t)\varphi+t\psi}(f)$$

holds for all $t \in (0,1)$.

(ii) *If* $\psi, \varphi - \psi \in C^-_{sm}(I)$ *(resp.* $C^+_{sm}(I)$*) and* ψ *is convex (resp. concave) on* I*, then*

$$(1-t)M_\varphi(f) + tM_\psi(f) > M_{(1-t)\varphi+t\psi}(f)$$

holds for all $t \in (0,1)$.

Remark. It seems that Theorem 3 is slightly related to [4, 5] which discuss a comparison between a convex linear combination of the arithmetic and geometric means and the generalized logarithmic mean.

The following result describes a certain boundedness of φ–means.

Theorem 4. *Suppose that* f *is non-constant and* $\varphi, \psi \in C_{sm,f}(I)$ *with* $\nabla_\varphi < \nabla_\psi$.

(i) *If* $\varphi, \psi - \varphi \in C^+_{sm}(I)$ *(or* $C^-_{sm}(I)$*), then a function :* $s \longrightarrow M_{(1-s)\varphi+s\psi}(f)$ *is strictly monotone increasing on* $[0,\infty)$ *and*

$$\lim_{s\to\infty} M_{(1-s)\varphi+s\psi}(f) = M_{\psi-\varphi}(f).$$

(ii) *If* $\varphi, \psi - \varphi \in C^+_{sm}(I)$ *(resp.* $C^-_{sm}(I)$*) and* $\psi(x) - \varphi(x) \geq 0$ *(resp.* ≤ 0*) for all* $x \in I$*, then a function :* $s \longrightarrow M_{(1-s)\varphi+s\psi}(f)$ *is strictly monotone increasing and continuous on* $[0,\infty)$.

2. C^2–CONDITIONS

In this section, we see that main conditions which appear in the preceding section can be realized by conditions of C^2-functions.

For each real-valued measurable function f on Ω, let $C^2_{sm*,f}(I)$ be the set of all C^2–functions φ in $C_{sm,f}(I)$ with no stationary points, that is, $\varphi'(t) \neq 0$ for all $t \in I$.

Corollary 1. *Suppose that* f *is non-constant and* $\varphi, \psi \in C^2_{sm*,f}(I)$. *If* $\frac{\varphi''(x)}{\varphi'(x)} < \frac{\psi''(x)}{\psi'(x)}$ *and* $\varphi'(x)\psi'(x) > 0$ *for all* $x \in I^\circ$*, then a function :* $s \longrightarrow M_{(1-s)\varphi+s\psi}(f)$ *is strictly increasing on* $[0,1]$.

Corollary 2. *Suppose that* f *is non-constant and that* $\varphi, \psi \in C^2_{sm*,f}(I)$ *is such that* $\frac{\varphi''(x)}{\varphi'(x)} < \frac{\psi''(x)}{\psi'(x)}$ *for for all* $x \in I^\circ$. *Then*

(i) *Either if* $0 < \varphi' < \psi'$ *and* $\psi'' \geq 0$ *on* I° *or if* $\psi' < \varphi' < 0$ *and* $\psi'' \leq 0$ *on* I°*, then*

$$(1-t)M_\varphi(f) + tM_\psi(f) < M_{(1-t)\varphi+t\psi}(f)$$

holds for all $t \in (0,1)$.

(ii) *Either if* $\varphi' < \psi' < 0$ *and* $\psi'' \geq 0$ *on* I° *or if* $0 < \psi' < \varphi'$ *and* $\psi'' \leq 0$ *on* I°, *then*

$$(1-t)M_\varphi(f) + tM_\psi(f) > M_{(1-t)\varphi+t\psi}(f)$$

holds for all $t \in (0,1)$.

3. Remarks

(i) Let $I = \mathbb{R}^+$. Put $\varphi(x) = \frac{1}{x}$ and $\psi(x) = x$ for each $x \in I$. Of course, these functions belong to $C_{sm}(I)$. The harmonic-arithmetic mean inequality asserts that $\nabla_\varphi < \nabla\psi$. Take a non-constant positive measurable function f on a probability space (Ω, μ) such that $\varphi \circ f$ and $\psi \circ f$ are in $L^1(\Omega, \mu)$. Then we have from Theorem 1–(ii) that $M_\varphi(f) < M_\psi(f)$. Observe that this inequality means

$$1 < \left(\int \frac{1}{f} d\mu\right)\left(\int f d\mu\right).$$

This is a special case of Jensen's inequality (or Schwarz's inequality). We note that if $0 < m \leq f \leq M$, then $(\int \frac{1}{f} d\mu)(\int f d\mu) \leq \frac{(m+M)^2}{4mM}$. The right side of this inequality is called a Kantorovich constant (cf. [3, 7, 8).

(ii) A similar consideration for the geometric-arithmetic mean inequality yields that

$$\int \log f d\mu < \log \int f d\mu.$$

This is also a special case of Jensen's inequality. We note that if $0 < m \leq f \leq M$, then $\log \int f d\mu - \int \log f d\mu \leq h^{\frac{1}{h-1}} \left(e \log h^{\frac{1}{h-1}}\right)^{-1}$, where $h = \frac{M}{m}$. The right side of this inequality is called Specht's ratio (cf. [2]).

(iii) A similar consideration for the harmonic-geometric mean inequality yields that

$$-\log \int \frac{1}{f} d\mu < \int \log f d\mu.$$

We note that this inequality is just one of (ii) replacing f by $\frac{1}{f}$ and multiplying -1 the both sides of resulting inequality in (ii).

Acknowledgement. 講演終了後，畏友藤井正俊さんから、相加平均の記号は通常 ∇ である事や、3節の (i) で Kantorovich 定数を出すなら、(ii) では Specht's ratio を出すべきであるとのご指摘を賜りました。この場をお借りしてお礼申し上げます。

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Fundamental error estimate inequalities for the Tikhonov regularization using reproducing kernels

Dedicated to the Memory of Wolfgang Walter

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Abstract

First of all, we will be concentrated in some particular but very important inequalities. Namely, for a real-valued absolutely continuous function on $[0, 1]$, satisfying $f(0) = 0$ and $\int_0^1 f'(x)^2 dx < 1$, we have, by using the theory of reproducing kernels

$$\int_0^1 \left(\frac{f(x)}{1-f(x)}\right)'^2 (1-x)^2 dx \leq \frac{\int_0^1 f'^2(x)\, dx}{1-\int_0^1 f'^2(x)\, dx}.$$

A. Yamada gave a direct proof for this inequality with a generalization and, as an application, he unified the famous Opial inequality and its generalizations.

Meanwhile, we gave some explicit representations of the solutions of nonlinear simultaneous equations and of the explicit functions in the implicit function theory by using singular integrals. In addition, we derived estimate inequalities for the consequent regularizations of singular integrals.

Our main purpose in this paper is to introduce our method of constructing approximate and numerical solutions of bounded linear operator equations on reproducing kernel Hilbert spaces by using the Tikhonov regularization. In view of this, for the error estimates of the solutions, we will need the inequalities for the approximate solutions. As a typical example, we shall present our new numerical and real inversion formulas of the Laplace transform whose problems are famous as typical ill-posed and difficult ones. In fact, for this matter, a software realizing the corresponding formulas in computers is now included in a present request for international patent. Here, we will be able to see a great computer power of H. Fujiwara with infinite precision algorithms in connection with the error estimates.

1 Yamada's results

Let H_K denote a Hilbert space admitting a reproducing kernel K on a set E. For all $f \in H_K$ and for a very general transform ϕ of f, there exists a naturally determined function Φ satisfying

$$\|\phi(f)\|^2_{H(\Phi(K))} \leq \Phi(\|f\|^2_{H_K}). \tag{1.1}$$

Here, $H(\Phi(K))$ is the reproducing kernel Hilbert space which is determined by the positive definite quadratic function $\Phi(K)$ (cf. [16, 17, 18, 19]).

We are considering a very general nonlinear transform $\phi(f)$. As an application of (1.1), we derived the identification method for the nonlinear system $\phi(f)$ in [23].

As a typical example of (1.1), in the framework of [17, 18, 19] we have that for a real-valued absolutely continuous function on $[0, 1]$, satisfying $f(0) = 0$ and $\int_0^1 f'(x)^2 dx < 1$, it holds

$$\int_0^1 \left(\frac{f(x)}{1 - f(x)} \right)'^2 (1-x)^2 dx \leq \frac{\int_0^1 f'^2(x)\, dx}{1 - \int_0^1 f'^2(x)\, dx},$$

for the nonlinear transform $f + f^2 + f^3 + \cdots$. We would like to call the reader's attention to [15, Appendix] and [18], where some essays on this inequality and mathematics in general can be found.

Meanwhile, we know the Opial inequality ([14]): For $f \in \mathrm{AC}[0,a]$ (i.e., an absolutely continuous function on $[0,a]$), with $f(0) = 0$, we have

$$\int_0^a |f(x) f'(x)|\, dx \leq \frac{a}{2} \int_0^a |f'(x)|^2\, dx.$$

Since this starting result proved in 1960 by Opial, a wide variety of generalizations and extensions was introduced in the last half-century. We are particularly interested in the generalization provided by A. Yamada (see [22]), which he managed to derived by a direct proof.

Furthermore, by some specialization, he was able to give a full generalization (cf. [1, 2, 3, 9, 10, 13]) of the Opial inequality with the equality statement.

2 The implicit function theorem

Let us now turn to the *Implicit Function Theorem.* For a simplification of the statement, we shall assume some global properties: On a smooth bounded domain $U \subset \mathbb{R}^{n+k}$ surrounded by a finite number of C^1 class and simple closed surfaces, for k functions

$$f_i(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+k}), \quad i = 1, 2, \ldots, k,$$

we assume that for some point on U it holds

$$f_i(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+k}) = 0$$

and on U we have

$$\det \frac{\partial(f_1, f_2, \ldots, f_k)}{\partial(x_{n+1}, x_{n+2}, \ldots, x_{n+k})}(x) > 0.$$

Then, we assume globally that there exist k functions of C^1 class, $g_j(x_1, x_2, \ldots, x_n)$ for $j = 1, 2, \ldots, k$, on $U \cap \mathbb{R}^n$, satisfying the properties:

$$f_i(x_1, x_2, \ldots, x_n, g_1, g_2, \ldots, g_k) = 0, \quad i = 1, 2, \ldots, k,$$

and

$$x_{n+j} = g_j(x_1, x_2, \ldots, x_n), \quad j = 1, 2, \ldots, k.$$

We were able to represent the functions g_j explicitly, in terms of the implicit functions $\{f_i\}$, by using singular integrals in the sense of Cauchy's principal value in [4], and by using some explicit representations of the solutions of nonlinear simultaneous equations (cf. [24]). We shall state here the results for the simplest cases.

Theorem 2.1 *For a C^1 class function $f(x_1, x_2)$ on a domain U in $\mathbb{R}^2$, we assume that for a point $x^0 = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix}$ it holds*

$$f(x_1^0, x_2^0) = 0$$
$$\frac{\partial f}{\partial x_2}(x_1^0, x_2^0) \neq 0.$$

Then, there exist a neighbourhood $U_1 \times U_2$ ($\subset U$) around the point x^0 and an explicit function $g: U_1 \to U_2$ determined by the implicit function $f = 0$ as $f(x_1, g(x_1)) = 0$ and, furthermore, it is represented as follows:

$$g(x_1^*) = \frac{1}{2\pi}\left(\int_{\partial(U_1\times U_2)} x_2 d\theta - \int_{U_1\times U_2} dx_2 \wedge d\theta\right), \quad \theta = \mathrm{Arctan}\,\frac{f(x_1,x_2)}{x_1 - x_1^*},$$

for any $x_1^* \in U_1$.

Corollary 2.2 *(Representations of the inverse functions). On an interval $[a,b]$, for a C^1 class function f satisfying $f'(x) > 0$, its inverse function $f^{-1}(y^*)$ on $[f(a), f(b)]$ is represented as follows:*

$$f^{-1}(y^*) = \frac{1}{2\pi}\left(\int_{\partial([a,b]\times[f(a),f(b)])} x d\theta_1 - \int_{[a,b]\times[f(a),f(b)]} dx \wedge d\theta_1\right),$$

$$\theta_1 = -\,\mathrm{Arctan}\,\frac{y - f(x)}{y - y^*},$$

for any $y^* \in [f(a), f(b)]$.

3 Singular integral estimates

We gave various error estimates for the regularizations for the singular integrals appearing in the above representations. For example, for the singularity

$$\frac{1}{(|x-y|)^\alpha},$$

we consider the regularization

$$\frac{1}{(|x-y|+\delta)^\alpha}$$

for a small δ and then analyse their error estimates. For the regularized integrals, their numerical calculations are done easily by using computers.

4 Best approximations

Let L be any bounded linear operator from a reproducing kernel Hilbert space H_K into a Hilbert space $\mathcal{H}$. Then, the following problem is a classical and fundamental problem known as the best approximate mean square norm problem: For any member $\mathbf{d}$ of $\mathcal{H}$, we would like to find

$$\inf_{f\in H_K} \|Lf - \mathbf{d}\|_{\mathcal{H}}.$$

It is clear that we are considering operator equations, generalized solutions and corresponding generalized inverses within the framework of $f \in H_K$ and $\mathbf{d} \in \mathcal{H}$, having in mind

$$Lf = \mathbf{d}. \tag{4.1}$$

However, this problem has a complicated structure, specially in the infinite dimension Hilbert spaces case, leading in fact to the consideration of generalized inverses (in the Moore-Penrose sense). Following our theory (cf. [19]), we can realize its complicated structure. Anyway, the problem turns to be well-posed within the reproducing kernels theory framework.

However, the result is involved and so **the Moore-Penrose generalized inverse $f_{\mathbf{d}}$ is not good**, when the data contain error or noises in some practical cases. So, we shall introduce the idea of the Tikhonov regularization in the present framework.

5 The Tikhonov regularization

We shall give some practical and more concrete representation in the extremal functions involved in the Tikhonov reguralization by using the theory of reproducing kernels.

Theorem 5.1 *Let* $L : H_K \to \mathcal{H}$ *be a bounded linear operator, and define the inner product*

$$\langle f_1, f_2 \rangle_{H_{K_\lambda}} = \alpha \langle f_1, f_2 \rangle_{H_K} + \langle Lf_1, Lf_2 \rangle_{\mathcal{H}}$$

for $f_1, f_2 \in H_K$. *Then* $(H_K, \langle \cdot, \cdot \rangle_{H_{K_\alpha}})$ *is a reproducing kernel Hilbert space whose reproducing kernel is given by*

$$K_\alpha(p,q) = [(\alpha + L^* L)^{-1} K_q](p).$$

Here, $K_\alpha(p,q)$ *is the solution* $\tilde{K}_\alpha(p,q)$ *of the functional equation*

$$\tilde{K}_\alpha(p,q) + \frac{1}{\alpha}(L\tilde{K}_q, LK_p)_{\mathcal{H}} = \frac{1}{\alpha} K(p,q), \tag{5.1}$$

that is corresponding to the Fredholm integral equation of the second kind for many concrete cases. Moreover, we are using

$$\tilde{K}_q = \tilde{K}_\alpha(\cdot, q) \in H_K \quad \text{for} \quad q \in E, \qquad K_p = K(\cdot, p) \quad \text{for} \quad p \in E.$$

We shall, furthermore, need error estimates, when $\mathbf{d}$ contains error or noises. For this fundamental problem, we obtain the following conclusion.

Theorem 5.2 *Under the same assumption as Theorem 5.1,*

$$f \in H_K \mapsto \left\{ \alpha \|f \,:\, H_K\|^2 + \|Lf - \mathbf{d} \,:\, \mathcal{H}\|^2 \right\} \in \mathbb{R}$$

attains the minimum and the minimum is attained only at $f_{\mathbf{d},\alpha} \in H_K$ *such that*

$$(f_{\mathbf{d},\alpha})(p) = \langle \mathbf{d}, LK_\alpha(\cdot, p) \rangle_{\mathcal{H}}.$$

Furthermore, $(f_{\mathbf{d},\alpha})(p)$ *satisfies*

$$|(f_{\mathbf{d},\alpha})(p)| \leq \sqrt{\frac{K(p,p)}{2\alpha}} \|\mathbf{d}\|_{\mathcal{H}}. \tag{5.2}$$

6 Real and numerical inversion formula of the Laplace transform

We shall consider the inversion formula of the Laplace transform

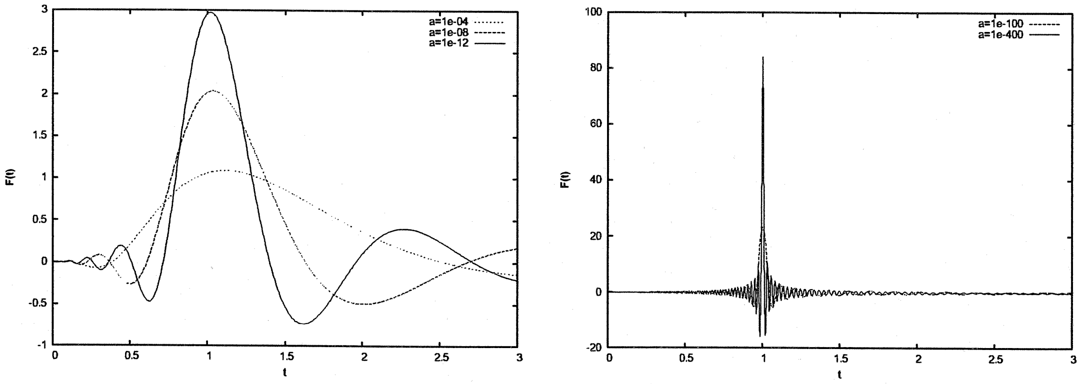
$$(\mathcal{L}F)(p) = f(p) = \int_0^\infty e^{-pt} F(t) dt, \quad p > 0$$

for some natural function spaces. For more general functions, we shall apply their transforms suitably in order to apply the results (cf. [20]). We shall consider, in general, the complex inversion formulas, because the images of the Laplace transform are analytic functions. However, we are requested to use only real and discrete data to obtain the inversion formula. This is the **real inversion formula** of the Laplace transform, and we must represent the analytic function of the image in terms of the data on the positive real line. This problem is a very famous difficult one.

In some case, we must solve the **Fredholm integral equation of the second kind:**

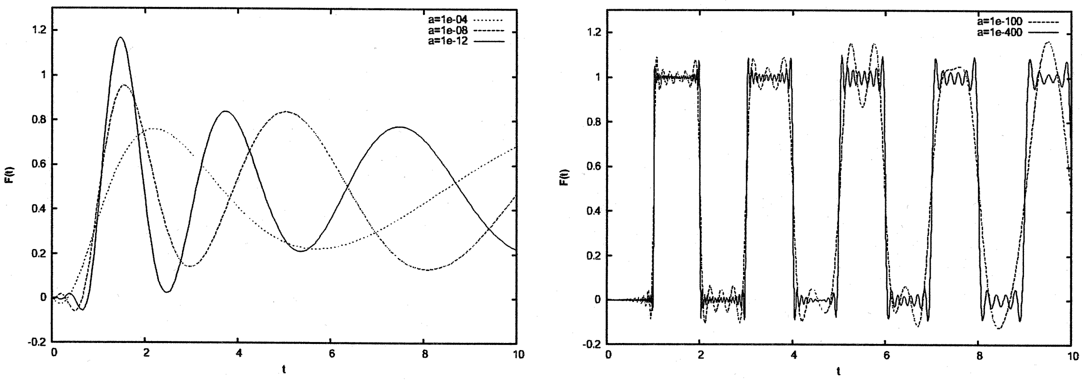
$$\alpha H_\alpha(\xi, t) + \int_0^\infty \frac{H_\alpha(p,t)}{(p+\xi+1)^2} dp = -\frac{e^{-t\xi} e^{-t}}{\xi+1} \left(t + \frac{1}{\xi+1} \right) + \frac{1}{(\xi+1)^2}. \tag{6.1}$$

By solving this integral equation, H. Fujiwara (cf. [6, 7, 8]) derived a very reasonable numerical inversion formula for the integral transform and he expanded very good algorithms for numerical



(a) $\alpha \geq 10^{-12}$ (b) $\alpha = 10^{-100}, 10^{-400}$

Figure 1: Numerical results for the delta function δ_1



(a) $\alpha \geq 10^{-12}$ (b) $\alpha = 10^{-100}, 10^{-400}$

Figure 2: Numerical results for a square wave function

and real inversion formulas of the Laplace transform. Figure 1 is an example for $\mathcal{L}F(p) = \exp(-p)$ for which $F(t) = \delta_1(t)$ in the distribution sense, and Figure 2 is for

$$\mathcal{L}F(p) = \frac{e^{-p}}{p(1 + e^{-p})}$$

for which $F(t)$ is a square wave function.

In both figures, (a) is computed with large regularization parameters $\alpha \geq 10^{-12}$, and (b) is computed with small regularization parameters $\alpha = 10^{-100}, 10^{-400}$. At this moment, theoretically we shall use the whole data of the output - in fact, 6000 data. Surprisingly enough, Fujiwara gave the solutions with $\alpha = 10^{-400}$ and **600 digits precision**. The core of the above mentioned and corresponding patent is **10 GB** data for the solutions.

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NON-α-NORMAL FUNCTIONS WITH GOOD INTEGRABILITY

Rauno Aulaskari, Shamil Makhmutov and Jouni Rättyä

ABSTRACT. Blaschke products are used to construct concrete examples of analytic functions with good integrability and bad behavior of spherical derivative. These examples are used to show that none of the classes $M_p^\#$, $0 < p < \infty$, is contained in the α-normal class $\mathcal{N}^\alpha$ when $0 < \alpha < 2$. This implies that $M_p^\#$ is in a sense a much larger class than $Q_p^\#$.

Let $\mathcal{M}(\mathbb{D})$ denote the class of all meromorphic functions in the unit disc $\mathbb{D} = \{z : |z| < 1\}$. A function $f \in \mathcal{M}(\mathbb{D})$ is called normal if

$$\|f\|_{\mathcal{N}} = \sup_{z\in\mathbb{D}} f^\#(z)(1-|z|^2) < \infty,$$

where $f^\#(z) = |f'(z)|/(1+|f(z)|^2)$ is the spherical derivative of f at z. The class of normal functions is denoted by $\mathcal{N}$. For a given sequence $\{z_n\}_{n=1}^\infty$ of points in $\mathbb{D}$ for which $\sum_{n=1}^\infty (1-|z_n|^2)$ converges (with the convention $z_n/|z_n| = 1$ for $z_n = 0$), the Blaschke product associated with the sequence $\{z_n\}_{n=1}^\infty$ is defined as

$$B(z) = \prod_{n=1}^\infty \frac{|z_n|}{z_n} \frac{z_n - z}{1-\overline{z}_n z}.$$

Allen and Belna [1] showed that the analytic function $f_s(z) = B(z)/(1-z)^s$, where $B(z)$ is the Blaschke product associated with the sequence $\{1-e^{-n}\}_{n=1}^\infty$, is not a normal function if $0 < s < \frac{1}{2}$, but satisfies the integrability condition

$$\int_{\mathbb{D}} |f_s'(z)|\, dA(z) < \infty.$$

It is well-known that if f is analytic in $\mathbb{D}$ and satisfies

$$\int_{\mathbb{D}} |f'(z)|^2\, dA(z) < \infty,$$

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that is, f belongs to the Dirichlet space (analytic functions in $\mathbb{D}$ with bounded area of image counting multiplicities), then $f \in \mathcal{N}$. Concerning the normality, the question arose if

$$\int_{\mathbb{D}} |f'(z)|^p \, dA(z) < \infty, \quad 1 < p < 2, \tag{1}$$

implies $f \in \mathcal{N}$. Yamashita [5] showed that this is not the case since the function

$$f(z) = B(z) \log \frac{1}{1-z}, \tag{2}$$

where B is a Blaschke product associated with an exponential sequence $\{z_n\}_{n=1}^{\infty}$ whose limit is 1, is not normal but satisfies (1) for all $1 < p < 2$. Recall that a sequence $\{z_n\}_{n=1}^{\infty}$ is exponential if

$$1 - |z_{n+1}| \leq \beta(1 - |z_n|), \quad n \in \mathbb{N}, \tag{3}$$

for some $\beta \in (0,1)$. It is well known that every such sequence $\{z_n\}_{n=1}^{\infty}$ satisfies

$$\prod_{k \neq n} \left| \frac{z_n - z_k}{1 - \overline{z}_n z_k} \right| \geq \delta, \quad n \in \mathbb{N}, \tag{4}$$

for some $\delta = \delta(\beta) > 0$, and is therefore an interpolating sequence (uniformly separated sequence).

The basic idea in this note is to find a function f that satisfies (1) (or another integrability condition) but the behavior of $f^{\#}$ is worse than the behavior of the spherical derivative of a non-normal function necessarily is. To make this precise, for $0 < \alpha < \infty$, a function $f \in \mathcal{M}(\mathbb{D})$ is called α-normal if

$$\sup_{z \in \mathbb{D}} f^{\#}(z)(1 - |z|^2)^{\alpha} < \infty.$$

The class of all α-normal functions is denoted by $\mathcal{N}^{\alpha}$.

Theorem 1. *Let B be the Blaschke product associated with an exponential sequence $\{z_n\}_{n=1}^{\infty}$ whose limit is 1. Let $1 \leq \alpha < \infty$, $0 < p < 2$ and*

$$f_s(z) = \frac{B(z)}{(1-z)^s}, \quad 0 < s < \infty. \tag{5}$$

Then $f_s \notin \mathcal{N}^{\alpha}$ for all $s > \alpha - 1$, but

$$\int_{\mathbb{D}} |f_s'(z)|^p \, dA(z) < \infty$$

for all $s \in (0, 2/p - 1)$.

It is easy to see that $f_{\alpha-1} \in \mathcal{N}^{\alpha}$. Moreover, the following result proves the sharpness of Theorem 1.

Theorem 2. *Let B be the Blaschke product associated with an exponential sequence $\{z_n\}_{n=1}^{\infty}$ such that $|z_n - \frac{1}{2}| = \frac{1}{2}$, $\Im z_n > 0$ and $\lim_{n\to\infty} z_n = 1$. Let $0 < p < 1$. Then*

$$f_{\frac{2}{p}-1}(z) = \frac{B(z)}{(1-z)^{\frac{2}{p}-1}}$$

satisfies

$$\int_{\mathbb{D}} |f'_{\frac{2}{p}-1}(z)|^p \, dA(z) = \infty.$$

The following result is of the same nature as Theorem 1.

Theorem 3. *Let B be the Blaschke product associated with an exponential sequence $\{z_n\}_{n=1}^{\infty}$ whose limit is 1. Let $1 \le \alpha < \infty$ and*

$$f(z) = \log \frac{1}{1-z} \frac{B(z)}{(1-z)^{\alpha-1}}.$$

Then $f \notin \mathcal{N}^{\alpha}$, but

$$\int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^{2\alpha-2+\varepsilon} \, dA(z) < \infty \tag{6}$$

for all $\varepsilon > 0$.

Wulan [4] showed that the function f, defined in (2), satisfies

$$f \notin \bigcup_{0<p<\infty} Q_p^{\#} \quad \text{but} \quad f \in \bigcap_{0<p<\infty} M_p^{\#}, \tag{7}$$

where

$$Q_p^{\#} = \left\{ f \in \mathcal{M}(\mathbb{D}) : \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} (f^{\#}(z))^2 g^p(z,a) \, dA(z) < \infty \right\}$$

and

$$M_p^{\#} = \left\{ f \in \mathcal{M}(\mathbb{D}) : \|f\|_{M_p^{\#}}^2 = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} (f^{\#}(z))^2 (1-|\varphi_a(z)|^2)^p \, dA(z) < \infty \right\}.$$

Here $\varphi_a(z) = (a-z)/(1-\bar{a}z)$ is a Möbius transformation and $g(z,a) = -\log|\varphi_a(z)|$ is a Green's function of $\mathbb{D}$.

Keeping (7) in mind, we will show that the function f_s, defined in (5), belongs to $M_p^{\#}$ for certain values of s.

Theorem 4. *Let B be the Blaschke product associated with an exponential sequence $\{z_n\}_{n=1}^{\infty}$ whose limit is 1. Then the function f_s, defined in (5), satisfies*

$$f_s \in \bigcap_{0<p<\infty} M_p^{\#}$$

for all $0 < s \le 1$.

Theorems 1 and 4 have the following immediate consequence.

Corollary 5.

$$\bigcap_{0<p<\infty} M_p^{\#} \not\subset \bigcup_{0<\alpha<2} \mathcal{N}^{\alpha}.$$

Using [2, Theorem 3.3.3], with $\alpha = 2 - 2/q$, we see that if $f \in \mathcal{M}(\mathbb{D})$ satisfies

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} (f^{\#}(z))^q (1-|z|^2)^{2q-4}(1-|\varphi_a(z)|^2)^p \, dA(z) < \infty,$$

for some $2 < q < \infty$ and $0 \le p < \infty$, then $f \in \mathcal{N}^2$. In view of this fact and Corollary 5 it is natural to ask the following questions.

Question 1. For which values of p the class $M_p^{\#}$ is contained in $\mathcal{N}^2$?

Question 2. Is the class

$$\mathcal{B}^{\#} = \left\{ f \in \mathcal{M}(\mathbb{D}) : \sup_{a\in\mathbb{D}} \int_{D(a,r)} (f^{\#}(z))^2 \, dA(z) < \infty \right\}$$

contained in $\mathcal{N}^2$?

Recall that $M_p^{\#} = \mathcal{B}^{\#}$ for all $1 < p < \infty$, see [3, 4].

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ON UNBOUNDED WEIGHTED SHIFTS ON DIRECTED TREES

IL BONG JUNG

ABSTRACT. A new class of (not necessarily bounded) operators related to (mainly infinite) directed trees is introduced and investigated. Operators in question are to be considered as a generalization of classical weighted shifts, on the one hand, and of weighted adjacency operators, on the other; they are called weighted shifts on directed trees. The basic properties of such operators are studied. Hyponormality, cohyponormality, and subnormality are entirely characterized in terms of their weights.

1. Notation and definitions

This was presented at the international conference: The Seminar on Function Spaces, which was held at Hokkaido University in Japan on December 22-25, 2010. And the results of this note are contained in **[2]** and **[5]**, that is the joint work with P. Budzyński, Z. Jabłoński and J. Stochel. Also, the results in this note will be appeared partially in our papers which will be published in some other journals.

Let A be an operator in a complex Hilbert space $\mathcal{H}$. Denote by $\mathcal{D}(A)$, $\mathcal{R}(A)$, A^* and $\bar{A}$ the domain, the range, the adjoint and the closure of A. Set $\mathcal{D}^\infty(A) = \bigcap_{n=0}^\infty \mathcal{D}(A^n)$; members of $\mathcal{D}^\infty(A)$ are called $\mathcal{C}^\infty$*-vectors* of A. A linear subspace $\mathcal{E}$ of $\mathcal{D}(A)$ is said to be a *core* of A if the graph of A is contained in the closure of the graph of the restriction $A|_{\mathcal{E}}$ of A to $\mathcal{E}$. If A is closed, then $\mathcal{E}$ is a core of A if and only if $A = \overline{A|_{\mathcal{E}}}$. Given a closed densely defined operator A in $\mathcal{H}$, we denote by $|A|$ the square root of the positive selfadjoint operator A^*A. A densely defined operator S in $\mathcal{H}$ is said to be *subnormal* if there exists a complex Hilbert space $\mathcal{K}$ and a normal operator N in $\mathcal{K}$ such that $\mathcal{H} \subseteq \mathcal{K}$ and $Sh = Nh$ for all $h \in \mathcal{D}(S)$. A densely defined operator S in $\mathcal{H}$ is said to be *hyponormal* if $\mathcal{D}(S) \subseteq \mathcal{D}(S^*)$ and $\|S^*f\| \leqslant \|Sf\|$ for all $f \in \mathcal{D}(S)$. In what follows, $\boldsymbol{B}(\mathcal{H})$ stands for the C^*-algebra of all bounded operators A in $\mathcal{H}$ such that $\mathcal{D}(A) = \mathcal{H}$. We write $\operatorname{lin} \mathcal{F}$ for the linear span of a subset $\mathcal{F}$ of $\mathcal{H}$.

Let $\mathscr{T} = (V, E)$ be a directed graph (i.e., V is the set of all vertexes of $\mathscr{T}$ and E is the set of all edges of $\mathscr{T}$). If for a given vertex $u \in V$, there exists a unique vertex $v \in V$ such that $(v, u) \in E$, then we say that u has a parent v and write $\mathsf{par}(u)$ for v. Since the correspondence $u \mapsto \mathsf{par}(u)$ is a partial function (read: a

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relation) in V, we can compose it with itself k-times ($k \in \mathbb{N}$); the result is denoted by par^k. We adhere to the convention that par^0 is the identity mapping on V. A vertex v of $\mathscr{T}$ is called a *root* of $\mathscr{T}$, or briefly $v \in \mathsf{Root}(\mathscr{T})$, if there is no vertex u of $\mathscr{T}$ such that (u, v) is an edge of $\mathscr{T}$. Note that if $\mathscr{T}$ is connected and each vertex $v \in V^\circ := V \setminus \mathsf{Root}(\mathscr{T})$ has a parent, then the set $\mathsf{Root}(\mathscr{T})$ has at most one element. If $\mathsf{Root}(\mathscr{T})$ is a one-point set, then its unique element is denoted by root. We say that a directed graph $\mathscr{T}$ is a *directed tree* if $\mathscr{T}$ is connected, has no circuits and each vertex $v \in V^\circ$ has a parent $\mathsf{par}(v)$. In what follows, given a directed tree $\mathscr{T}$, we tacitly assume that V and E stand for the sets of vertexes and edges of $\mathscr{T}$, respectively.

Let $\mathscr{T}$ be a directed tree. Set $\mathsf{Chi}(u) = \{v \in V : (u, v) \in E\}$, $u \in V$. A member of $\mathsf{Chi}(u)$ is called a *child* (or *successor*) of u. We say that $\mathscr{T}$ is *leafless* if $V = V'$, where $V' := \{u \in V : \mathsf{Chi}(u) \neq \varnothing\}$. It is clear that every leafless directed tree is infinite. A vertex $u \in V$ is called a *branching vertex* of $\mathscr{T}$ if $\operatorname{card}(\mathsf{Chi}(u)) \geqslant 2$.

2. Fundamental properties

Denote by $\ell^2(V)$ the Hilbert space of all square summable complex functions on V with the standard inner product

$$\langle f, g \rangle = \sum_{u \in V} f(u)\overline{g(u)}, \quad f, g \in \ell^2(V).$$

For $u \in V$, we define $e_u \in \ell^2(V)$ by

$$e_u(v) = \begin{cases} 1 & \text{if } u = v, \\ 0 & \text{otherwise.} \end{cases}$$

The set $\{e_u\}_{u \in V}$ is an orthonormal basis of $\ell^2(V)$; we call it the *canonical orthogonal basis* of $\ell^2(V)$. Set $\mathscr{E}_V = \operatorname{lin}\{e_u : u \in V\}$.

Given $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ} \subseteq \mathbb{C}$, we define the operator $S_{\boldsymbol{\lambda}}$ in $\ell^2(V)$ by

$$\mathcal{D}(S_{\boldsymbol{\lambda}}) = \{f \in \ell^2(V) : \Lambda_{\mathscr{T}} f \in \ell^2(V)\},$$
$$S_{\boldsymbol{\lambda}} f = \Lambda_{\mathscr{T}} f, \quad f \in \mathcal{D}(S_{\boldsymbol{\lambda}}),$$

where $\Lambda_{\mathscr{T}}$ is the mapping defined on functions $f : V \to \mathbb{C}$ via

$$(\Lambda_{\mathscr{T}} f)(v) = \begin{cases} \lambda_v \cdot f\big(\mathsf{par}(v)\big) & \text{if } v \in V^\circ, \\ 0 & \text{if } v = \mathsf{root}. \end{cases} \tag{2.1}$$

The operator $S_{\boldsymbol{\lambda}}$ is called a *weighted shift* on the directed tree $\mathscr{T}$ with weights $\{\lambda_v\}_{v \in V^\circ}$.

We now give some basic properties of weighted shifts on directed trees.

PROPOSITION 2.1. *Let $S_{\boldsymbol{\lambda}}$ be a weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ}$. Then the following assertions hold:*

(i) *$S_{\boldsymbol{\lambda}}$ is closed,*

(ii) *e_u is in $\mathcal{D}(S_{\boldsymbol{\lambda}})$ if and only if $\sum_{v \in \mathsf{Chi}(u)} |\lambda_v|^2 < \infty$; if $e_u \in \mathcal{D}(S_{\boldsymbol{\lambda}})$, then*

$$S_{\boldsymbol{\lambda}} e_u = \sum_{v \in \mathsf{Chi}(u)} \lambda_v e_v \quad \text{and} \quad \|S_{\boldsymbol{\lambda}} e_u\|^2 = \sum_{v \in \mathsf{Chi}(u)} |\lambda_v|^2 \tag{2.2}$$

,

(iii) *$S_{\boldsymbol{\lambda}}$ is densely defined if and only if $\mathscr{E}_V \subseteq \mathcal{D}(S_{\boldsymbol{\lambda}})$,*

(iv) *if $S_{\boldsymbol{\lambda}}$ is densely defined, then $\mathscr{E}_V$ is a core of $S_{\boldsymbol{\lambda}}$,*

(v) $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V))$ *if and only if* $\alpha_{\boldsymbol\lambda} := \sup_{u\in V}\sum_{v\in\mathsf{Chi}(u)}|\lambda_v|^2 < \infty$; *moreover, if* $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V))$, *then* $\|S_{\boldsymbol\lambda}\|^2 = \alpha_{\boldsymbol\lambda}$,

(vi) *if* $S_{\boldsymbol\lambda}$ *is densely defined, then* $\mathscr{E}_V \subseteq \mathcal{D}(S_{\boldsymbol\lambda}^*)$ *and*

$$S_{\boldsymbol\lambda}^* e_u = \begin{cases} \overline{\lambda_u} e_{\mathsf{par}(u)} & \text{if } u \in V^\circ, \\ 0 & \text{if } u = \mathsf{root}, \end{cases} \qquad u \in V \tag{2.3}$$

,

(vii) $S_{\boldsymbol\lambda}$ *is injective if and only if* $\mathscr{T}$ *is leafless and* $\sum_{v\in\mathsf{Chi}(u)}|\lambda_v|^2 > 0$ *for every* $u \in V$,

(viii) *if* $S_{\boldsymbol\lambda}$ *is densely defined and* $\lambda_v \neq 0$ *for all* $v \in V^\circ$, *then* V *is at most countable.*

3. Hyponormality

To discuss the hyponormality, we characterize the circumstances under which the inclusion $\mathcal{D}(S_{\boldsymbol\lambda}) \subseteq \mathcal{D}(S_{\boldsymbol\lambda}^*)$ holds.

THEOREM 3.1. *If* $S_{\boldsymbol\lambda}$ *is a densely defined weighted shift on a directed tree* $\mathscr{T}$ *with weights* $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ}$, *then the following conditions are equivalent:*

(i) $\mathcal{D}(S_{\boldsymbol\lambda}) \subseteq \mathcal{D}(S_{\boldsymbol\lambda}^*)$,

(ii) *there exists* $c > 0$ *such that*

$$\sum_{v\in\mathsf{Chi}(u)} \frac{|\lambda_v|^2}{1+\|S_{\boldsymbol\lambda} e_v\|^2} \leqslant c, \quad u \in V. \tag{3.1}$$

The circumstances under which the inclusion $\mathcal{D}(S_{\boldsymbol\lambda}^*) \subseteq \mathcal{D}(S_{\boldsymbol\lambda})$ holds are more elaborate and require much more effort to be accomplished. For this reason, we attach to a densely defined weighted shift $S_{\boldsymbol\lambda}$ on a directed tree $\mathscr{T}$ the diagonal operators M_u in $\ell^2(\mathsf{Chi}(u))$, $u \in V'$, given by M_u

$$\begin{aligned} \mathcal{D}(M_u) &= \{g \in \ell^2(\mathsf{Chi}(u))\colon \sum_{v\in\mathsf{Chi}(u)} \|S_{\boldsymbol\lambda} e_v\|^2 |g(v)|^2 < \infty\}, \\ (M_u g)(v) &= \|S_{\boldsymbol\lambda} e_v\| g(v), \quad v \in \mathsf{Chi}(u),\ g \in \mathcal{D}(M_u). \end{aligned} \tag{3.2}$$

If $u \in V'$ is such that the function $\boldsymbol\lambda^u\colon \mathsf{Chi}(u) \ni v \to \lambda_v \in \mathbb{C}$ belongs to $\mathcal{D}(M_u)$, then we define the operator T_u in $\ell^2(\mathsf{Chi}(u))$ by T_u

$$T_u = M_u^2 - \frac{1}{1+\|S_{\boldsymbol\lambda} e_u\|^2} M_u(\boldsymbol\lambda^u) \otimes M_u(\boldsymbol\lambda^u), \quad u \in V'. \tag{3.3}$$

THEOREM 3.2. *If* $S_{\boldsymbol\lambda}$ *is a densely defined weighted shift on a directed tree* $\mathscr{T}$ *with weights* $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ}$, *then the following two conditions are equivalent:*

(i) $\mathcal{D}(S_{\boldsymbol\lambda}^*) \subseteq \mathcal{D}(S_{\boldsymbol\lambda})$,

(ii) $T_u \in \boldsymbol{B}(\ell^2(\mathsf{Chi}(u)))$ *for all* $u \in V'$, *and*

$$\sup_{u\in V'} \|T_u\| < \infty. \tag{3.4}$$

As pointed out below, hyponormal weighted shifts on a directed tree with nonzero weights must be injective.

PROPOSITION 3.3. *Let* $\mathscr{T}$ *be a directed tree with* $V^\circ \neq \varnothing$. *If* $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V))$ *is a hyponormal weighted shift on* $\mathscr{T}$ *whose all weights are nonzero, then* $\mathscr{T}$ *is leafless. In particular,* $S_{\boldsymbol\lambda}$ *is injective and* $\operatorname{card}(V) = \aleph_0$.

We now characterize the hyponormality of weighted shifts on directed trees in terms of weights. Given a directed tree $\mathscr{T}$ and $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ} \subseteq \mathbb{C}$, we define

$$\mathsf{Chi}^+_{\boldsymbol{\lambda}}(u) = \{v \in \mathsf{Chi}(u) \colon \sum_{w \in \mathsf{Chi}(v)} |\lambda_w|^2 > 0\}, \quad u \in V.$$

THEOREM 3.4. *Let $S_{\boldsymbol{\lambda}} \in \boldsymbol{B}(\ell^2(V))$ be a weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ}$. Then the following assertions are equivalent:*

(i) *$S_{\boldsymbol{\lambda}}$ is hyponormal,*

(ii) *for every $u \in V$, it holds that if $v \in \mathsf{Chi}(u)$ and $\|S_{\boldsymbol{\lambda}} e_v\| = 0$, then $\lambda_v = 0$, and $\sum_{v \in \mathsf{Chi}^+_{\boldsymbol{\lambda}}(u)} \frac{|\lambda_v|^2}{\|S_{\boldsymbol{\lambda}} e_v\|^2} \leqslant 1$.*

REMARK 3.5. The notion of hyponormality can be extended to the case of unbounded operators. It is known that hyponormal operators are closable and their closures are hyponormal as well (see [**10, 6, 7, 8, 11**] for elements of the theory of unbounded hyponormal operators). A close inspection of the proof reveals that *Theorem* 3.4 *remains true for densely defined weighted shifts on directed trees.* Note also that if $S_{\boldsymbol{\lambda}}$ is a densely defined weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ}$, then the conditions (3.4) and (3.4) imply (3.1) with $c = 1$.

Recall that an operator $A \in \boldsymbol{B}(\mathcal{H})$ is said to be *cohyponormal* if its adjoint A^* is hyponormal. The question of cohyponormality of weighted shifts on directed trees is more delicate than hyponormality.

THEOREM 3.6. *Let $S_{\boldsymbol{\lambda}} \in \boldsymbol{B}(\ell^2(V))$ be a nonzero weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ}$. Then $S_{\boldsymbol{\lambda}}$ is cohyponormal if and only if the tree $\mathscr{T}$ is rootless and one of the following two disjunctive conditions holds:*

(i) *there exists a sequence $\{u_n\}_{n=-\infty}^{\infty} \subseteq V$ such that*

$$0 < |\lambda_{u_n}| \leqslant |\lambda_{u_{n-1}}| \text{ and } u_{n-1} = \mathsf{par}(u_n) \tag{3.5}$$

for all $n \in \mathbb{Z}$, and $\lambda_v = 0$ for all $v \in V \setminus \{u_n \colon n \in \mathbb{Z}\}$,

(ii) *there exist a sequence $\{u_n\}_{n=-\infty}^{0} \subseteq V$ such that*

$$0 < \sum_{v \in \mathsf{Chi}(u_0)} |\lambda_v|^2 \leqslant |\lambda_{u_0}|^2,\ 0 < |\lambda_{u_n}| \leqslant |\lambda_{u_{n-1}}| \text{ and } u_{n-1} = \mathsf{par}(u_n) \tag{3.6}$$

for all integers $n \leqslant 0$, and $\lambda_v = 0$ for all $v \in V \setminus (\{u_n \colon n \leqslant 0\} \cup \mathsf{Chi}(u_0))$.

REMARK 3.7. A closed densely defined operator A in a complex Hilbert space $\mathcal{H}$ is said to be *cohyponormal* if A^* is hyponormal (cf. Remark 3.5), i.e., $\mathcal{D}(A^*) \subseteq \mathcal{D}(A)$ and $\|Af\| \leqslant \|A^* f\|$ for all $f \in \mathcal{D}(A^*)$. A thorough inspection of proofs shows that Theorem *3.6* remains true for densely defined weighted shifts on directed trees.

4. Subnormality

We now show that within this class of operators subnormality is completely characterized by the existence of a consistent system of probability measures.

THEOREM 4.1. *Let $S_{\boldsymbol{\lambda}}$ be a densely defined weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ}$ such that $\mathscr{E}_V \subseteq \mathscr{Q}(S_{\boldsymbol{\lambda}})$. Then the following conditions are equivalent:*

(i) *$S_{\boldsymbol{\lambda}}$ is subnormal,*

(ii) *$\{\|S^n_{\boldsymbol{\lambda}} e_u\|^2\}_{n=0}^{\infty}$ is a Stieltjes moment sequence for every $u \in V$,*

(iii) *there exist a system* $\{\mu_v\}_{v\in V}$ *of Borel probability measures on* $\mathbb{R}_+ = [0,\infty)$ *and a system* $\{\varepsilon_v\}_{v\in V}$ *of nonnegative real numbers which satisfy the positive Borel measure* μ_u *on* $\mathbb{R}_+$ *defined by*

$$\mu_u(\sigma) = \sum_{v\in\mathsf{Chi}(u)} |\lambda_v|^2 \int_\sigma \frac{1}{s}\,\mathrm{d}\,\mu_v(s) + \varepsilon_u\delta_0(\sigma), \quad \sigma\in\mathfrak{B}(\mathbb{R}_+), \tag{4.1}$$

where $\mathfrak{B}(\mathbb{R}_+)$ *for the* σ*-algebra of all Borel subsets of* $\mathbb{R}_+$, *with*

$$\varepsilon_u = \Big(1 - \sum_{v\in\mathsf{Chi}(u)} |\lambda_v|^2 \int_0^\infty \frac{1}{s}\,\mathrm{d}\,\mu_v(s)\Big) \tag{4.2}$$

is a representing measure of $\{\|S_{\boldsymbol\lambda}^n e_u\|^2\}_{n=0}^\infty$ *for every* $u\in V$.

We begin by characterizing directed trees admitting densely defined weighted shifts $S_{\boldsymbol\lambda}$ with nonzero weights such that $\mathcal{D}(S_{\boldsymbol\lambda}^2) = \{0\}$. It turns out that such pathological weighted shifts exist. This never happens for classical (unilateral or bilateral) weighted shifts S because if such an operator is densely defined, then we always have the inclusion $\mathscr{E}_V \subseteq \mathcal{D}^\infty(S)$.

PROPOSITION 4.2. *Let* $\mathscr{T}$ *be a directed tree. Then the following assertions are equivalent*:

(i) *there exists a family* $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ}$ *of nonzero complex numbers such that* $\overline{\mathcal{D}(S_{\boldsymbol\lambda})} = \ell^2(V)$ *and* $\mathcal{D}(S_{\boldsymbol\lambda}^2) = \{0\}$,

(ii) $\mathrm{card}(\mathsf{Chi}(u)) = \aleph_0$ *for every* $u\in V$.

Slightly modifying the above proof Proposition 4.2, we obtain a version of Proposition 4.2 for weighted shifts with arbitrary weights.

PROPOSITION 4.3. *Let* $\mathscr{T}$ *be a directed tree. Then the following assertions are equivalent*:

(i) *there exists a family* $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ} \subseteq \mathbb{C}$ *such that* $\overline{\mathcal{D}(S_{\boldsymbol\lambda})} = \ell^2(V)$ *and* $\mathcal{D}(S_{\boldsymbol\lambda}^2) = \{0\}$,

(ii) *the set* $\mathsf{Chi}(u)$ *is infinite for every* $u\in V$.

5. Normality and quasinormality

In this subsection we show that formally normal weighted shifts on directed trees are always bounded and normal. Recall that a densely defined operator N in a complex Hilbert space $\mathcal{H}$ is said to be *formally normal* if $\mathcal{D}(N)\subseteq\mathcal{D}(N^*)$ and $\|N^*h\| = \|Nh\|$ for all $h\in\mathcal{D}(N)$ (cf. [**3**]).

PROPOSITION 5.1. *If* $S_{\boldsymbol\lambda}$ *is a nonzero weighted shift on a directed tree* $\mathscr{T}$ *with weights* $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ}$, *then the following three conditions are equivalent*:

(i) $S_{\boldsymbol\lambda}$ *is formally normal,*

(ii) *there exists a sequence* $\{u_n\}_{n=-\infty}^\infty \subseteq V$ *such that*

$$u_{n-1} = \mathsf{par}(u_n) \text{ and } |\lambda_{u_{n-1}}| = |\lambda_{u_n}| \text{ for all } n\in\mathbb{Z},$$

and $\lambda_v = 0$ *for all* $v\in V\setminus\{u_n\colon n\in\mathbb{Z}\}$,

(iii) $S_{\boldsymbol\lambda}\in\boldsymbol{B}(\ell^2(V))$ *and* $S_{\boldsymbol\lambda}$ *is normal.*

Following [**12**] (for the case of bounded operators see [**1**]) we say that a closed densely defined operator A in a complex Hilbert space $\mathcal{H}$ is *quasinormal* if A commutes with the spectral measure E of $|A|$, i.e., $E(\sigma)A \subseteq AE(\sigma)$ for all $\sigma \in \mathfrak{B}(\mathbb{R}_+)$. By [**12**, Proposition 1], a closed densely defined operator A in $\mathcal{H}$ is quasinormal if and only if $U|A| \subseteq |A|U$, where $A = U|A|$ is the polar decomposition of A. It turns out that quasinormal operators are always subnormal (see [**12**, Theorem 2] for the general case; the bounded case can be deduced from [**1**, Theorem 1]). The reverse implication does not hold in general. For more information on quasinormal operators we refer the reader to [**1, 4**] (bounded operators) and [**12, 9**] (unbounded operators).

The characterization of quasinormal weighted shifts on directed trees now discussed below.

PROPOSITION 5.2. *Let $S_{\boldsymbol\lambda}$ be a densely defined weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v \in V^\circ}$. Then the following conditions are equivalent:*

(i) *$S_{\boldsymbol\lambda}$ is quasinormal,*

(ii) *$\|S_{\boldsymbol\lambda} e_u\| = \|S_{\boldsymbol\lambda} e_v\|$ for all $u \in V$ and $v \in \mathsf{Chi}(u)$ such that $\lambda_v \neq 0$.*

Moreover, if $V^\circ \neq \varnothing$ and $\lambda_v \neq 0$ for all $v \in V^\circ$, then $S_{\boldsymbol\lambda}$ is quasinormal if and only if $\|S_{\boldsymbol\lambda}\|^{-1} S_{\boldsymbol\lambda}$ is an isometry.

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ON HAHN BANACH THEOREM IN A PARTIALLY ORDERED VECTOR SPACE AND ITS APPLICATIONS

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ABSTRACT. In this paper, using the Bourbaki-Kneser fixed point theorem, we give a new proof of the Hahn-Banach theorem in case where the range space is a partially ordered vector space.

1. INTRODUCTION

The Hahn-Banach theorem is one of the most fundamental theorems in the functional analysis theory. This theorem is known well in the case where the range space is the real number system as follows:

Let p be a sublinear mapping from a vector space X to the real number system R, Y a vector subspace of X and q a linear mapping from Y to R such that $q \leq p$ on Y. Then q can be extended to a linear mapping g defined on the whole space X to R such that $g \leq p$.

It is known that this theorem establishes in case where the range space is a Dedekind complete Riesz space. The Hahn-Banach theorem is proved often using the Zorn lemma. On the other hand, Hirano, Komiya, and Takahashi [4] showed the Hahn-Banach theorem by using the Markov-Kakutani fixed point theorem [5] in the case where the range space is the real number system.

In this paper, using the Bourbaki-Kneser fixed point theorem, we give a new proof of the Hahn-Banach theorem and the Mazur-Orlicz theorem in the case where the range space is a Dedekind complete partially ordered vector space (Theorem 4, Theorem 5). We also give a new proof of the separation theorem in a Cartesian product of the vector space and Dedekind complete partially ordered vector space (Theorem 6) [2, 3, 8].

2. PRELIMINARIES

Let R be the set of real numbers, N the set of natural numbers, I an indexed set, $(E, \leq)$ a partially ordered set and F a subset of E. The set F is called a *chain* if any two elements are comparable, that is, $x \leq y$ or $y \leq x$ for any x, $y \in F$. An element $x \in E$ is called a *lower bound* of F if $x \leq y$ for any $y \in F$. An element $x \in E$ is called the *minimum* of F if x is a lower bound of F and $x \in F$. If there exists a lower bound of F, then F is said to be *bounded from below*. An element $x \in E$ is called an *upper bound* of F if $y \leq x$ for any $y \in F$. An element $x \in E$ is called the *maximum* of F if x is an upper bound and $x \in F$. If there exists an upper bound of

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F, then F is said to be *bounded from above.* If the set of all lower bounds of F has the maximum, then the maximum is called an *infimum* of F and denoted by $\inf F$. If the set of all upper bounds of F has the minimum, then the minimum is called a *supremum* of F and denoted by $\sup F$. A partially ordered set E is said to be *complete* if every nonempty chain of E has an infimum; E is said to be *Dedekind complete* if every nonempty subset of E which is bounded from below has an infimum. A mapping f from E to E is called *decreasing* if $f(x) \leq x$ for every $x \in E$.

In a complete partially ordered set, the following theorem is obtained [1, 6, 7].

Theorem 1 (Bourbaki-Kneser). *Let E be a complete partially ordered set. Let f be a decreasing mapping from E to E. Then f has a fixed point.*

A partially ordered set E is called a partially ordered vector space if E is a vector space and $x + z \leq y + z$ and $\alpha x \leq \alpha y$ hold whenever $x, y, z \in E$, $x \leq y$, and α is a nonnegative real number. If a partially ordered vector space E is a lattice, that is, any two elements have a supremum and an infimum, then E is called a *Riesz space.*

Let X be a vector space and E a partially ordered vector space. A mapping f from X to E is said to be *concave* if

$$f(tx + (1-t)y) \geq tf(x) + (1-t)f(y)$$

for any $x, y \in X$ and $t \in [0, 1]$. A mapping f from X to E is called *sublinear* if the following conditions are satisfied.

(S1) For any $x, y \in X$, $p(x + y) \leq p(x) + p(y)$.
(S2) For any $x \in X$ and $\alpha \geq 0$ in R, $p(\alpha x) = \alpha p(x)$.

3. The Hahn-Banach Theorem

Lemma 2. *Let p be a sublinear mapping from a vector space X to a Dedekind complete partially ordered vector space E, K a nonempty convex subset of X and q a concave mapping from K to E such that $q \leq p$ on K. For any $x \in X$, let*

$$f(x) = \inf\{p(x + ty) - tq(y) \mid t \in [0, \infty) \text{ and } y \in K\}.$$

Then f is sublinear such that $f \leq p$. Moreover if g is a linear mapping from X to E, then $g \leq f$ is equivalent to $g \leq p$ on X and $q \leq g$ on K.

Proof. For any $x \in X$,

$$\{p(x + ty) - tq(y) \mid t \in [0, \infty) \text{ and } y \in K\}$$

is bounded from below. Indeed, since

$$p(x + ty) - tq(y) \geq p(ty) - p(-x) - tq(y) \geq -p(-x),$$

it is established. Since E is Dedekind complete, f is well-defined and we have $f(x) \geq -p(-x)$. By definition of f, we have $f(x) \leq p(x)$ and $f(\alpha x) = \alpha f(x)$ for any $\alpha \geq 0$. Thus (S2) is established. Let $x_1, x_2 \in X$. For any $y_1, y_2 \in K$

and $s, t > 0$, we have

$$\begin{aligned} p(x_1 + sy_1) - sq(y_1) + p(x_2 + ty_2) - tq(y_2) \\ &\geq p(x_1 + x_2 + (s+t)w) - (s+t)q(w) \\ &\geq f(x_1 + x_2), \end{aligned}$$

where $w = \frac{1}{s+t}(sy_1 + ty_2) \in K$. For $p(x_1 + sy_1) - sq(y_1)$, take infimum with respect to $s > 0$ and $y_1 \in K$, we have

$$f(x_1) + p(x_2 + ty_2) - tq(y_2) \geq f(x_1 + x_2)$$

and for $p(x_2 + ty_2) - tq(y_2)$, also take infimum with respect to $t > 0$ and $y_2 \in K$, we have

$$f(x_1) + f(x_2) \geq f(x_1 + x_2).$$

Thus (S1) is established. Suppose that g is a linear mapping from X to E. If $g \leq f$, then we have $g \leq p$. Moreover for any $y \in K$, since

$$-g(y) = g(-y) \leq f(-y) \leq p(-y + y) - q(y) = -q(y),$$

we have $g \geq q$ on K. To prove the converse, suppose that $g \leq p$ on X and $q \leq g$ on K. For any $x \in X$, $y \in K$ and $t > 0$, we have

$$g(x) = g(x + ty) - tg(y) \leq p(x + ty) - tq(y).$$

This implies that $g \leq f$. □

The above lemma is proved in case where the range space is a Dedekind complete Riesz space, see [9, Lemma 1.5.1].

By Theorem 1 and Lemma 2, we can give a following lemma.

Lemma 3. *Let f be a sublinear mapping from a vector space X to a Dedekind complete partially ordered vector space E. Then there exists a linear mapping g from X to E such that $g \leq f$.*

Proof. Let E^X be the set of mappings of X into E. Define $f \leq g$ for $f, g \in E^X$ by $f(x) \leq g(x)$ for all $x \in X$. Then $(E^X, \leq)$ is a partially ordered vector space. Put $f^*(x) = -f(-x)$ for any $x \in X$. Let

$$Y = \{h \in E^X \mid h \text{ is sublinear, } f^* \leq h \leq f\}.$$

Then Y is an ordered set. Since E is Dedekind complete, E^X is also so. Consider an arbitrary chain $Z \subset Y$. Since E^X is Dedekind complete and Z is bounded from below, there exists a $g = \inf Z$ in E^X. Then g is sublinear. Since Y is bounded from below, it holds that $g \in Y$. Thus Y is complete. Let $K = \{y\}$. Then h is also a concave mapping from K to E. We define a decreasing operator S by

$$Sh(x) = \inf\{h(x + ty) - th(y) \mid t \in [0, \infty), y \in K\}$$

for any $h \in Y$. By Lemma 2, Sh is sublinear and S is a mapping from Y to Y. Thus by Theorem 1, we have a fixed point $g \in Y$. Then for any $x \in X$, we have $g(x) \leq g(x + y) - g(y)$ and

$$g(x) + g(y) \leq g(x + y) \leq g(x) + g(y).$$

Since

$$0 = g(0) = g(-\alpha x + \alpha x) = g(-\alpha x) + \alpha g(x)$$

for any $\alpha > 0$ and $x \in X$, we have $g(-\alpha x) = -\alpha g(x)$. Thus $g(\alpha x) = \alpha g(x)$ for any $\alpha \in R$ and $x \in X$. Therefore, g is linear. □

By Lemma 3, we have the following Hahn-Banach Theorem in a partially ordered vector space.

Theorem 4. *Let p be a sublinear mapping from a vector space X to a Dedekind complete ordered vector space E, Y a vector subspace of X and q a linear mapping from Y to E such that $q \leq p$ on Y. Then q can be extended to a linear mapping g defined on the whole space X such that $g \leq p$.*

Proof. By Lemma 2, there exists a sublinear mapping f such that $f \leq p$. By Lemma 3, there exists a linear mapping g such that $g \leq f$. Then putting $K = Y$ in Lemma 2, we have $g \leq p$ on X and $q \leq g$ on Y. Since q is linear, for any $y \in Y$, we have

$$g(-y) \leq f(-y) \leq p(-y + y) - q(y) = -q(y) = q(-y).$$

Then we have $g \leq q$ on Y. Thus $q = g$ on Y. Therefore, the assertion holds. □

Moreover we obtain the Mazur-Orlicz theorem in a Dedekind complete partially ordered vector space.

Theorem 5. *Let p be a sublinear mapping from a vector space X to a Dedekind complete partially ordered vector space E. Let $\{x_j \mid j \in I\}$ be a family of elements of X and $\{y_j \mid j \in I\}$ a family of elements of E. Then the following* (1) *and* (2) *are equivalent.*

(1) *There exists a linear mapping f from X to E such that $f(x) \leq p(x)$ for any $x \in X$ and $y_j \leq f(x_j)$ for any $j \in I$.*

(2) *For any $n \in N$, $\alpha_1, \alpha_2, \ldots, \alpha_n \geq 0$ and $j_1, j_2, \ldots, j_n \in I$, we have*

$$\sum_{i=1}^{n} \alpha_i y_{j_i} \leq p\left(\sum_{i=1}^{n} \alpha_i x_{j_i}\right).$$

Proof. The assertion from (1) to (2) is clear. For any $x \in X$, by (2), we have

$$-p(-x) \leq p\left(x + \sum_{i=1}^{n} \alpha_i x_{j_i}\right) - \sum_{i=1}^{n} \alpha_i y_{j_i}.$$

Put

$$p_0(x) = \inf\left\{ p\left(x + \sum_{i=1}^{n} \alpha_i x_{j_i}\right) - \sum_{i=1}^{n} \alpha_i y_{j_i} \;\middle|\; n \in N, \alpha_i \geq 0 \text{ and } j_i \in I \right\}.$$

Since E is Dedekind complete, p_0 is well-defined and p_0 is sublinear. Thus by Lemma 3, there exists a linear mapping f from X to E such that $f(x) \leq p_0(x)$ for any $x \in X$. Since $p_0(-x_j) \leq -y_j$, we have

$$y_j \leq -p_0(-x_j) \leq f(x_j).$$

Since $p_0(x) \leq p(x)$, we have $f(x) \leq p(x)$. Thus the assertion holds. □

4. The separation theorem

Let A be a nonempty subset of X and $L(A)$ denotes the affine manifold spanned by A. We define

$$Int(A) = \left\{ x \in X \;\middle|\; \begin{array}{l} \text{For any } x' \in L(A) \text{ there exists } \varepsilon > 0 \text{ such that} \\ x + \lambda(x' - x) \in A \text{ for any } \lambda \in [0, \varepsilon) \end{array} \right\}.$$

If $L(A) = X$, then we write $I(A)$ insted of $Int(A)$. Let f be a linear mapping from X to E, g a linear mapping from E to E and u_0 a point in E. Then $H = \{(x, y) \in X \times E \mid f(x) + g(y) = u_0\}$ is empty or an affine manifold in $X \times E$. Let A, B be nonempty subsets of $X \times E$. A subset $A \subset X \times E$ is cone if $\lambda > 0$ implies $\lambda A \subset A$. It is said that an affine manifold H separates A and B if $H_- = \{(x, y) \in X \times E \mid f(x) + g(y) \leq u_0\} \supset A$ and $H_+ = \{(x, y) \in X \times E \mid f(x) + g(y) \geq u_0\} \supset B$ hold. The operator P_X defined by $P_X(x, y) = x$ for any $(x, y) \in X \times E$ is called the projection of $X \times E$ onto X. We define $P_X(A) = \{x \in X \mid \text{ there exists } y \in E \text{ such that } (x, y) \in A\}$. The subset $C(A) = \{\lambda z \in X \times E \mid \lambda \geq 0, z \in A\}$ is called the cone spanned by A.

By Lemma 3, we obtain the separation theorem in a Cartesian product of the vector space and the Dedekind complete partially ordered vector space.

Theorem 6. *Let A and B be subsets of $X \times E$ such that $C(A-B)$ is convex cone, $P_X(A-B)$ satisfies the following* (i) *and* (ii) :
(i) $0 \in I(P_X(A-B))$,
(ii) *if $(x, y_1) \in A$ and $(x, y_2) \in B$, then $y_1 \geq y_2$ holds.*
Then there exists a linear mapping f from X to E and a $y_0 \in E$ such that the affine manifold $H = \{(x, y) \in X \times E \mid f(x) - y = y_0\}$ separates A and B.

Proof. By assumption (i) and the definition of $I(P_X(A-B))$, for any $x \in X$ there exists $\varepsilon > 0$ and for any $\lambda \in [0, \varepsilon)$, there exists $y \in E$ such that $(\lambda x, y) \in A - B$. Then there exist $x_1, x_2 \in X$ and $y_1, y_2 \in E$ such that

$$(\lambda x, y) = (x_1 - x_2, y_1 - y_2) = (x_1, y_1) - (x_2, y_2) \in A - B.$$

Define

$$E_x = \{y \in E \mid (x, y) \in C(A - B)\}, \text{ for any } x \in X.$$

Since $\lambda^{-1}(y_1 - y_2) \in E_x$ for any $\lambda \in (0, \varepsilon)$, we have $E_x \neq \emptyset$. Let $y \in E_0$ and $y \neq 0$, then there exists $\lambda > 0$, $(x_1, y_1) \in A$ and $(x_2, y_2) \in B$ such that

$$(0, y) = \lambda\{(x_1, y_1) - (x_2, y_2)\}$$

and $x_1 = x_2$. By assumption (ii), we have $y = \lambda(y_1 - y_2) \geq 0$. We define $E_+ = \{y \in E \mid y \geq 0\}$. Then we have $y \in E_+$. Since $C(A - B)$ is convex cone, we have $E_x + E_{x'} \subset E_{x+x'}$ for any $x, x' \in X$. We prove that for every $x \in X$ the subset E_x possesses a lower bound in E. Since E_x is nonempty, for any $x \in X$, there exists $y' \in E$ with $-y' \in E_{-x}$. Then we have

$$y - y' \in E_x + E_{-x} \subset E_0 \subset E_+$$

for any $y \in E_x$. This implies $y' \leq y$ for any $y \in E_x$. Since E is Dedekind complete, operator $p(x) = \inf\{y \mid y \in E_x\}$ is well defined. Then $p(x)$ is sublinear. By Lemma 3, there exists a linear mapping f from X to E such that $f(x) \leq p(x)$ for all $x \in X$. Then for any $(x_1, y_1) \in A$, $(x_2, y_2) \in B$, take $x = x_1 - x_2$, we have

$$f(x_1 - x_2) \leq p(x_1 - x_2) \leq y_1 - y_2.$$

Therefore,

$$f(x_1) - y_1 \leq f(x_2) - y_2.$$

Since E is Dedekind complete, there exists a $y_0 \in E$ such that

$$f(x_1) - y_1 \leq y_0 \leq f(x_2) - y_2$$

for any $(x_1, y_1) \in A$, $(x_2, y_2) \in B$. Thus the affine manifold H separates A and B. □

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WANDERING SUBSPACES AND THE BEURLING TYPE THEOREM

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ABSTRACT. An elementary proof of the Aleman, Richter and Sundberg theorem concerning with invariant subspaces of the Bergman space is given.

1. INTRODUCTION

Let $\mathbb{D}$ be the open unit disk in the complex plane, and let dA denote the normalized Lebesgue measure on $\mathbb{D}$. The Bergman space L_a^2 is a Hilbert space consisting of square integrable analytic functions on $\mathbb{D}$. We denote by $\mathcal{B}$ the multiplication operator by the coordinate function z on L_a^2, $\mathcal{B}f = zf$, which is called the Bergman shift. On the Hardy space H^2 on $\mathbb{D}$, we denote by T_z the multiplication operator by z, the unilateral shift. For an operator T on a Hilbert space H and an invariant subspace M of T, the subspace $M \ominus TM$ is called a wandering subspace of M. We say that the Beurling type theorem holds for T if $[M \ominus TM] = M$ for all invariant subspaces M of T, where $[M \ominus TM]$ is the smallest invariant subspace of T containing $M \ominus TM$. The well known Beurling theorem [3] says that for all invariant subspaces M of the unilateral shift T_z, their wandering subspaces have dimension 1, and the Beurling type theorem holds for T_z. On the other hand, the situation of the Bergman shift is a little bit different. There are studies of the dimension of wandering subspaces of invariant subspaces of $\mathcal{B}$, and it is known that the dimension ranges from 1 to ∞, see [2, 5, 7]. In 1996, Aleman, Richter, and Sundberg [1] gave a big progress in the study of invariant subspaces of $\mathcal{B}$. They proved the Beurling type theorem for the Bergman shift. This result reveals the inside of the structure of invariant subspaces of the Bergman space and becomes a fundamental theorem in the function theory on L_a^2 [4, 6]. Later, different proofs of the the Beurling type theorem are given in [8, 9, 10, 11]. In [10], Shimorin proved the following theorem.

Shimorin's Theorem. *Let T be a bounded linear operator on a Hilbert space H. If T satisfies the following conditions*

(a) $\|Tx + y\|^2 \leq 2(\|x\|^2 + \|Ty\|^2), \quad x, y \in H,$
(b) $\bigcap\{T^n H : n \geq 0\} = \{0\},$
then $H = [H \ominus TH]$.

If T satisfies conditions (a) and (b), then $T|_M : M \to M$ also satisfies conditions (a) and (b). Hence by Shimorin's theorem, the Beurling type theorem holds for T. As an application of this theorem, Shimorin gave a simpler proof of the Aleman, Richter, and Sundberg theorem. In [11], Sun and Zheng gave another proof of this theorem. Their idea was to lift up the Bergman shift as the compression of a commuting pair of isometries on the subspace of the Hardy space over the bidisk. Sun and Zheng's idea has two aspects. One is to show some identities in the Bergman space. Another one is a technique how to prove the Beurling type theorem.

Here we give an elementary proof of the Aleman, Richter, and Sundberg theorem using some basic function theory in L_a^2 and elementary techniques in functional analysis. Our idea of the proof comes from the one given by Sun and Zheng [11] essentially. Our proof is just rewriting their proof in the most elementary way. In Section 2, we prove the following theorem.

Theorem 1.1. *Let T be a bounded linear operator on a Hilbert space H. If T satisfies the following conditions*
(i) $\|Tx\|^2 + \|T^{*2}Tx\|^2 \leq 2\|T^*Tx\|^2, \quad x \in H,$
(ii) *T is bounded below, that is, there is $c > 0$ satisfying $\|Tx\| \geq c\|x\|$ for every $x \in H$,*
(iii) $\|T\| \leq 1,$
(iv) *$\|T^{*k}x\| \to 0$ as $k \to \infty$ for every $x \in H$,*
then $H = [H \ominus TH]$.

We also give some identities in the Bergman space L_a^2, and as application of Theorem 1.1 we prove the Aleman, Richter, and Sundberg theorem. In Section 3, we shall discuss on a relationship between the conditions in Shimorin's theorem and ones in Theorem 1.1. We show that condition (iii) follows from conditions (i) and (ii), and that conditions (i), (ii), and (iv) are equivalent to Shimorin's conditions (a) and (b).

2. The Beurling type theorem

Proof of Theorem 1.1. Let $N = H \ominus [H \ominus TH]$. It is sufficient to show that $N = \{0\}$. To do this, let $x \in N$. Since $x \perp T^k(H \ominus TH)$, $T^{*k}x \perp H \ominus TH$ for every $k \geq 0$. By condition (ii), TH is closed.

Hence $T^{*k}x \in TH$ and there is $y_k \in H$ such that $Ty_k = T^{*k}x$. By condition (iii), we have $\|T^{*(k+1)}x\| \le \|T^{*k}x\|$. Let

$$r_k = \|T^{*k}x\|^2 - \|T^{*(k+1)}x\|^2.$$

Then we have

$$\begin{aligned} r_k - r_{k+1} &= \|T^{*k}x\|^2 + \|T^{*(k+2)}x\|^2 - 2\|T^{*(k+1)}x\|^2 \\ &= \|Ty_k\|^2 + \|T^{*2}Ty_k\|^2 - 2\|T^*Ty_k\|^2 \\ &\le 0 \qquad \text{by (i).} \end{aligned}$$

Thus we get $0 \le r_k \le r_{k+1}$. By condition (iv), we have

$$r_k = \|T^{*k}x\|^2 - \|T^{*(k+1)}x\|^2 \to 0 \quad \text{as } k \to \infty.$$

Hence $r_k = 0$ for every $k \ge 0$. Thus we get $\|x\|^2 = \|T^{*k}x\|^2$ for every $k \ge 0$. By condition (iv) again, we get $x = 0$, so $N = \{0\}$. This completes the proof. □

As an application of Theorem 1.1 we give a simple and elementary proof of Aleman, Richter, and Sundberg theorem in the Bergman space L_a^2. To do this, we need some identities in the Bergman space. It is known that $\|z^n\| = 1/\sqrt{n+1}$ and $\{\sqrt{n+1}z^n\}_{n\ge 0}$ is an orthonormal basis of L_a^2. So for each $f(z)$ in L_a^2, we may write

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1} < \infty.$$

We have $\mathcal{B}^*1 = 0$ and

$$\mathcal{B}^* z^n = \frac{n}{n+1} z^{n-1}, \quad n \ge 1.$$

By this fact, we have the following.

Lemma 2.1. (1) $\mathcal{B}f = \sum_{n=0}^{\infty} a_n z^{n+1}$.

(2) $\mathcal{B}^*\mathcal{B}f = \sum_{n=0}^{\infty} \frac{n+1}{n+2} a_n z^n$.

(3) $\mathcal{B}^{*2}\mathcal{B}f = \sum_{n=1}^{\infty} \frac{n}{n+2} a_n z^{n-1}$.

(4) $\mathcal{B}\mathcal{B}^{*2}\mathcal{B}f = \sum_{n=1}^{\infty} \frac{n}{n+2} a_n z^n$.

Since

$$\|\mathcal{B}f\|^2 = \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+2} = \sum_{n=0}^{\infty} \frac{n+1}{n+2} \frac{|a_n|^2}{n+1} \ge \frac{1}{2}\|f\|^2,$$

$\|\mathcal{B}\| \leq 1$ and $\mathcal{B}$ is bounded below. By Lemma 2.1, one easily checks the following identities in the Bergman space.

Lemma 2.2. (1) $\mathcal{B}\mathcal{B}^{*2}\mathcal{B}f = 2\mathcal{B}^*\mathcal{B}f - f$.
(2) $\|\mathcal{B}f\|^2 + \|\mathcal{B}^{*2}\mathcal{B}f\|^2 = 2\|\mathcal{B}^*\mathcal{B}f\|^2$.

Let $M \subset L_a^2$ be an invariant subspace of $\mathcal{B}$. Let $M^\perp$ be the orthogonal complement of M in L_a^2. We denote by P_M and $P_{M^\perp}$ the orthogonal projections from L_a^2 onto M and $M^\perp$, respectively. For $f \in L_a^2$, we have $f = P_M f \oplus P_{M^\perp} f$ and $\|f\|^2 = \|P_M f\|^2 + \|P_{M^\perp} f\|^2$. Let $\mathcal{B}|_M$ be the restriction operator on M. For $f, h \in M$, we have $\langle (\mathcal{B}|_M)^* f, h\rangle = \langle f, \mathcal{B}h\rangle = \langle P_M \mathcal{B}^* f, h\rangle$, so $(\mathcal{B}|_M)^* = P_M \mathcal{B}^*$. We use this fact frequently. The following follows from Lemma 2.2 (1).

Lemma 2.3. *Let M be an invariant subspace of $\mathcal{B}$. Then for each $f \in M$,*

$$\|\mathcal{B}f\|^2 + \|\mathcal{B}^{*2}\mathcal{B}f\|^2 = 2\|(\mathcal{B}|_M)^*\mathcal{B}f\|^2 + \frac{1}{2}\|P_{M^\perp}\mathcal{B}\mathcal{B}^{*2}\mathcal{B}f\|^2.$$

The following theorem is just a rewriting of an identity given in the proof of Theorem 3.1 in [11]. Using Lemma 2.3, we can give a simpler proof.

Theorem 2.4. *Let M be an invariant subspace of $\mathcal{B}$. Then for each $f \in M$,*

$$\begin{aligned}\|\mathcal{B}f\|^2 &+ \|(\mathcal{B}|_M)^{*2}\mathcal{B}f\|^2 - 2\|(\mathcal{B}|_M)^*\mathcal{B}f\|^2 \\ &= \frac{1}{2}\|P_{M^\perp}\mathcal{B}P_{M^\perp}\mathcal{B}^{*2}\mathcal{B}f\|^2 - \|P_{M^\perp}\mathcal{B}^{*2}\mathcal{B}f\|^2.\end{aligned}$$

Since $\|P_{M^\perp}\mathcal{B}\| \leq 1$, we have the following corollary.

Corollary 2.5. *Let M be an invariant subspace of $\mathcal{B}$. Then for each $f \in M$,*

$$\|\mathcal{B}f\|^2 + \|(\mathcal{B}|_M)^{*2}\mathcal{B}f\|^2 + \frac{1}{2}\|P_{M^\perp}\mathcal{B}^{*2}\mathcal{B}f\|^2 \leq 2\|(\mathcal{B}|_M)^*\mathcal{B}f\|^2.$$

Corollary 2.6. *The Beurling type theorem holds for $\mathcal{B}$.*

Proof. Let M be an invariant subspace of $\mathcal{B}$. Write $T = \mathcal{B}|_M$ and $H = M$. It is easy to check that T satisfies conditions (ii), (iii), and (iv) in Theorem 1.1. Condition (i) follows from Corollary 2.5. By Theorem 1.1, $[M \ominus TM] = M$. Thus we get the assertion. □

The authors think that this is one of the most simple and elementary proof of Aleman, Richter, and Sundberg theorem.

3. Conditions in Theorem 1.1

We study conditions (i)–(iv) in Theorem 1.1 and conditions (a), (b) in Shimorin's theorem. In [10], Shimorin pointed out the following.

Proposition 3.1. *Let T be a bounded linear operator on a Hilbert space H. Then condition (a) in Shimorin's theorem is equivalent to the conditions that condition (ii), so T^*T is invertible, and $TT^* + (T^*T)^{-1} \leq 2I$.*

Rewriting the above conditions, we have the following.

Proposition 3.2. *Let T be a bounded linear operator on a Hilbert space H. Then conditions (ii) and $TT^* + (T^*T)^{-1} \leq 2I$ are equivalent to conditions (i) and (ii).*

Corollary 3.3. *Let T be a bounded linear operator on a Hilbert space H. If T satisfies conditions (i) and (ii), then for every invariant subspace M of T, we have*

$$\|Tx\|^2 + \|(T|_M)^{*2}Tx\|^2 \leq 2\|(T|_M)^*Tx\|^2, \quad x \in M.$$

Proof. Let M be an invariant subspace of T. By Propositions 3.1 and 3.2, T satisfies condition (a) in Shimorin's theorem. Hence $T|_M$ satisfies condition (a). By Propositions 3.1 and 3.2 again, we get the assertion. □

By Lemma 2.2 (2), $\|\mathcal{B}f\|^2 + \|\mathcal{B}^{*2}\mathcal{B}f\|^2 = 2\|\mathcal{B}^*\mathcal{B}f\|^2$ for every $f \in L^2_a$. By Corollary 3.3, we have

$$\|\mathcal{B}f\|^2 + \|(\mathcal{B}|_M)^{*2}\mathcal{B}f\|^2 \leq 2\|(\mathcal{B}|_M)^*\mathcal{B}f\|^2$$

for every invariant subspace M of $\mathcal{B}$ and $f \in M$. An estimate in Corollary 2.5 is more precise than this one.

In the private communication, S. Richter pointed out the following.

Proposition 3.4. *Let T be a bounded linear operator on a Hilbert space H. If T satisfies conditions (i) and (ii), then $\|T\| \leq 1$.*

By Proposition 3.4, condition (iii) is superfluous in Theorem 1.1.

Corollary 3.5. *Let T be a bounded linear operator on a Hilbert space H. If T satisfies conditions (i), (ii), and (iv), then the Beurling type theorem holds for T.*

Theorem 3.6. *Let T be a bounded linear operator on a Hilbert space H. Then conditions (i), (ii), and (iv) are equivalent to conditions (a) and (b).*

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The Spectra Of Toeplitz Operators On The Bidisc

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Abstract. Let H^2 be the Hardy space on the bidisc. For some special function ϕ in L^∞, we study the spectrum of the Toeplitz operator $\mathbb{T}_\phi$ on H^2. In particular, we describe it for a real valued symbol ϕ.

Let m be the normalized Lebesgue measure on the torus Γ^2. For $1 \leq p \leq \infty$, $L^p = L^p(\Gamma^2, m)$ denotes the Lebesgue space and $H^p = H^p(\Gamma^2, m) = \{f \in L^p : \hat{f}(\ell, n) = 0$ if $\ell < 0$ or $n < 0\}$, that is, H^p denotes the usual Hardy space on Γ^2. Suppose m_z and m_w denote the normalized Lebesgue measures on the circle $\Gamma = \Gamma_z$ and $\Gamma = \Gamma_w$. Then $\Gamma^2 = \Gamma_z \times \Gamma_w$ and $m = m_z \times m_w$. $L^p(\Gamma_z) = L^p(\Gamma_z, m_z)$ and $L^p(\Gamma_w) = L^p(\Gamma_w, m_w)$ denote one variable Lebesgue spaces, and $H^p(\Gamma_z) = H^p(\Gamma_z, m_z)$ and $H^p(\Gamma_w) = H^p(\Gamma_w, m_w)$ denote one variable usual Hardy spaces.

Let P be the orthogonal projection from L^2 onto H^2. For a function ϕ in L^∞, the Toeplitz operator determined by ϕ is

$$\mathbb{T}_\phi f = P(\phi f) \quad (f \in H^2).$$

The Toeplitz operator T_ϕ on $H^2(\Gamma_z)$ or $H^2(\Gamma_w)$ is defined similarly.

Let $\mathbb{H}_w^p$ be the closed linear space of $\bigcup_{n=0}^\infty \bar{z}^n H^p$ and $\mathbb{H}_z^p$ the closed linear space of $\bigcup_{n=0}^\infty \bar{w}^n H^p$. Let $\mathbb{P}^w$ be the orthogonal projection from L^2 onto $\mathbb{H}_w^2$ and $\mathbb{P}^z$ the orthogonal projection from L^2 onto $\mathbb{H}_z^2$. For ϕ in L^∞, $\mathbb{T}_\phi^w$ and $\mathbb{T}_\phi^z$ are Toeplitz operators on $\mathbb{H}_w^2$ and $\mathbb{H}_z^2$, respectively. That is, $\mathbb{T}_\phi^w f = \mathbb{P}^w(\phi f)$ $(f \in \mathbb{H}_w^2)$ and $\mathbb{T}_\phi^z f = \mathbb{P}^z(\phi f)$ $(f \in \mathbb{H}_z^2)$.

In this paper, we study the spectrum of $\sigma(\mathbb{T}_\phi)$ using $\sigma(\mathbb{T}_\phi^w)$ and $\sigma(\mathbb{T}_\phi^z)$. It is known [2] in that

$$\sigma(\mathbb{T}_\phi) \supseteq \sigma(\mathbb{T}_\phi^w) \cup \sigma(\mathbb{T}_\phi^z).$$

We consider a very special symbol, that is, ϕ is real valued or $\phi = \phi_1\phi_2$ where $\phi_1 = \phi_1(z)$ and $\phi_2 = \phi_2(w)$.

In this lecture, Spec$B$ denotes the maximal ideal space of a Banach algebra B and $\hat{f}$ denotes the Gelfand transform of f in B. M_ϕ and $\mathbb{M}_\phi$ denote multiplication operators on $L^2(\Gamma)$ and $L^2(\Gamma^2)$, respectively.

$E \subset [0, 2\pi)$ measurable set, $\quad I \subset [0, 2\pi)$ interval

∘ one variable (A_2)-condition :

$$\frac{1}{m_t(I)} \int_I W dm_t \leq \gamma \left(\frac{1}{m_t(I)} \int_I W^{-1} dm_t \right)^{-1} \qquad {}^\forall I$$

∘ two variable (A_2)-condition for $t = w$ or $t = z$:

$$\frac{1}{m(E\times I)}\int_{E\times I} W dm \leq \gamma \frac{1}{m_z(E)}\int_E dm_z \left(\frac{1}{m_w(I)}\int_I W^{-1} dm_w\right)^{-1} \qquad {}^\forall E,\ {}^\forall I$$

$$\frac{1}{m(I\times E)}\int_{I\times E} W dm \leq \gamma \frac{1}{m_w(E)}\int_E dm_w \left(\frac{1}{m_z(I)}\int_I W^{-1} dm_z\right)^{-1} \qquad {}^\forall E,\ {}^\forall I$$

ONE VARIABLE

Hartman-Wintner Theorem

$\phi \in L^\infty(\Gamma)$. real valued

$\Longrightarrow$

$\sigma(T_\phi) = [\text{ess inf}\phi, \text{ ess sup}\phi]$

Proof

By Brown-Halmos, $\sigma(T_\phi) \subseteq [\text{ess inf}\phi, \text{ ess sup}\phi]$.

It is enough to prove T_ϕ is not invertible when ess inf$\phi < 0$ and ess sup$\phi > 0$.

Suppose T_ϕ is invertible.

$\phi f = 1 + \bar{g}$, ${}^\exists f \in H^2(\Gamma), {}^\exists g \in zH^2(\Gamma)$

$\phi \bar{f} = 1 + g$

$\phi |f|^2 = (1+g)f \in H^1(\Gamma)$

$\phi |f|^2 = \text{constant} \neq 0$

contradiction

Coburn Theorem

(1) $\phi \in L^\infty(\Gamma),\ \phi \not\equiv 0 \quad \Longrightarrow \quad (\text{Ker}T_\phi) \cap (\text{Ker}T_\phi^*) = \{0\}$

(2) ϕ continuous

T_ϕ Fredholm $\Longleftrightarrow$ $|\phi| > 0$ on Γ

Krein-Widom-Devinatz Theorem

ϕ continuous

$\Longrightarrow$

$\sigma(T_\phi) = \sigma(M_\phi) \cup \{\lambda \in \mathbb{C} : \text{ wind } (\phi - \lambda) \neq 0\}$

Proof

- T_ϕ invertible $\Longleftrightarrow$ T_ϕ Fredholm, ind$T_\phi = 0$
- T_ϕ Fredholm $\Longleftrightarrow$ M_ϕ invertible, ind$T_\phi = -$windϕ

TWO VARIABLES

§1. Known result

Lemma 1. (Nakazi, 2001)

If $\mathbb{T}_\phi$ is invertible then for $t = z$ and w $\phi = k_t \dfrac{\overline{h_t}}{h_t}$
where k_t is invertible in $\mathbb{H}_t^\infty$ and h_t is a t-outer function in $\mathbb{H}_t^2$
such that $|h_t|^2$ satisfies two variable (A_2)-condition for t.

Theorem 1. (Nakazi, 2001)
If $\phi \in L^\infty(\Gamma^2)$ then $\sigma(\mathbb{T}_\phi) \supseteq \sigma(\mathbb{T}_\phi^w) \cup \sigma(\mathbb{T}_\phi^z)$

Proof
By Lemma 1, if $\mathbb{T}_\phi$ is invertible then $\mathbb{T}_\phi^t$ is invertible for $t = z$ and w.

§2. Real valued symbol

Lemma 2.
$\sigma(\mathbb{M}_\phi) \subseteq \sigma(\mathbb{T}_\phi) \subseteq$ [ess infϕ, ess supϕ]

Proof
It is well known.

Theorem 2.
If $\phi(z, w)$ is a real valued function then $\sigma(\mathbb{T}_\phi) =$ [ess infϕ, ess supϕ].

Proof
By Lemma 2, it is enough to prove $\mathbb{T}_\phi$ is not invertible when
ess inf$\phi < 0$ and ess sup$\phi > 0$.
Suppose $\mathbb{T}_\phi$ is invertible.
By Lemma 1,

$$\phi = \frac{k_w}{h_w^2} \times |h_w|^2 = \frac{k_z}{h_z^2} \times |h_z|^2,$$

$u(w) = k_w/h_w^2 \in \mathbb{H}_w^1$ real value,
$v(z) = k_z/h_z^2 \in \mathbb{H}_z^1$ real value.

$$\frac{\phi(z,w)}{u(w)} > 0 \ \text{ and } \ \frac{\phi(z,w)}{v(z)} > 0.$$

Put $F = \{w : u(w) > 0\}$ and $F^c = \{w : u(w) < 0\}$ then
$\phi(z, w) > 0$ $((z, w) \in \Gamma_z \times F)$ and $\phi(z, w) < 0$ $((z, w) \in \Gamma_z \times F^c)$.

Put $E = \{z : v(z) > 0\}$ and $E^c = \{z : v(z) < 0\}$ then
$\phi(z,w) > 0$ $((z,w) \in E \times \Gamma_w)$ and $\phi(z,w) < 0$ $((z,w) \in E^c \times \Gamma_w)$.
$E^c \times F \subset \Gamma_z \times F$

- $m_z(E^c) > 0$ and $m_w(F) > 0$
  $\Longrightarrow$ $\phi(z,w) < 0$ and $\phi(z,w) > 0$ on $E^c \times F$
  $\Longrightarrow$ contradiction

- $m_z(E^c) = 0$ or $m_w(F) = 0$
  $m_z(E^c) = 0 \Longrightarrow v(z) > 0$ on Γ_z, $\phi(z,w) > 0$ on Γ^2
  $\Longrightarrow$ contradiction

  $m_w(F) = 0 \Longrightarrow u(w) > 0$ on Γ_w, $\phi(z,w) > 0$ on Γ^2
  $\Longrightarrow$ contradiction

$\mathbb{T}_\phi$ is not invertible.

§3. One variable symbol

$\phi = \phi(z)$: $\phi \in L^\infty(\Gamma_z)$

Lemma 3.
one-variable (A_2)-condition $\Longrightarrow$ two variable (A_2)-condition

Proof
It is clear by the definitions.

Theorem 3.
$\phi = \phi(z) \Longrightarrow \sigma(\mathbb{T}_\phi) = \sigma(\mathbb{T}_\phi^z) = \sigma(T_\phi)$

Proof
$\sigma(T_\phi) \subseteq \sigma(\mathbb{T}_\phi^z) \subseteq \sigma(\mathbb{T}_\phi)$
$\because$ Theorem 1
$\sigma(T_\phi) \supseteq \sigma(\mathbb{T}_\phi)$
$\because$ T_ϕ invertible $\Longrightarrow \mathbb{T}_\phi H^2 = H^2$
T_ϕ invertible $\Longrightarrow$ $\mathbb{T}_\phi$ one-to-one
$\because$
$\phi(z) = k(z)\overline{h(z)}/h(z)$
$k, k^{-1} \in H^\infty(\Gamma_z)$
$|h|^2$ $\cdots$ one variable (A_2)–condition, $h \in H^2(\Gamma_z)$ outer
(Muckenhoupt-Devinatz-Widom)

We may assume $\phi = \bar{h}/h$.
$0 <^{\exists} \gamma < \infty$
$\int |F|^2 dm \leq \gamma \int |\phi F + \bar{G}|^2 dm \ (F \in H^2, G \in K_0^2)$.
(Lemma 3)

§4. Separating symbol

$\phi(z,w) = \phi_1(z)\phi_2(w) : \phi_1 \in L^\infty(\Gamma_z),\ \phi_2 \in L^\infty(\Gamma_w)$

Theorem 4.
(1) $\phi_1 \in H^\infty(\Gamma_z) \Longrightarrow \sigma(\mathbb{T}_\phi) \supseteq \sigma(\mathbb{T}_{\phi_1}) \times \sigma(\mathbb{M}_{\phi_2})$
(2) $\phi_1 \in H^\infty(\Gamma_z),\ \bar{\phi}_2 \in H^\infty(\Gamma_w) \Longrightarrow$
$\sigma(\mathbb{T}_\phi) \supseteq \{\sigma(\mathbb{T}_{\phi_1}) \times \sigma(\mathbb{M}_{\phi_2})\} \cup \{\sigma(\mathbb{M}_{\phi_1}) \times \sigma(\mathbb{T}_{\phi_2})\}$

Proof
(1) Since $\phi \in \mathbb{H}_z^\infty$, by Theorem 3
$\sigma(\mathbb{T}_\phi^z) = \hat{\phi}_1(\mathrm{Spec}H^\infty(\Gamma_z)) \times \hat{\phi}_2(\mathrm{Spec}L^\infty(\Gamma_z)) = \sigma(\mathbb{T}_{\phi_1}) \times \sigma(\mathbb{M}_{\phi_2})$
Apply Theorem 1.
(2) follows from (1) and Theorem 1

$\phi = \phi(z,w) \in C(\Gamma^2)$ invertible
$\mathrm{wind}_w \phi \cdots$ winding number of $\phi(z,w)$ for fixed w
$\mathrm{wind}_z \phi \cdots$ winding number of $\phi(z,w)$ for fixed z

Lemma 4 (Osher)
$g(z,w) = a(z)b(w) + c(w) \in C(\Gamma^2)$ invertible : $a, b, c \in C(\Gamma)$
$\mathbb{T}_g$ invertible $\Longleftrightarrow$ $\mathrm{wind}_w g \equiv 0$, $\mathrm{wind}_z g \equiv 0$

Theorem 5
$\phi(z,w) = \phi_1(z)\phi_2(w) \ : \ \phi_1, \phi_2 \in C(\Gamma)$
$\Longrightarrow$
$\sigma(\mathbb{T}_\phi) = \sigma(\mathbb{M}_\phi) \cup \{\lambda \in \mathbb{C} : \mathrm{wind}_w(\phi - \lambda) \not\equiv 0 \text{ or } \mathrm{wind}_z(\phi - \lambda) \not\equiv 0\}$

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ON QUADRATICALLY HYPONORMAL WEIGHTED SHIFTS

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Abstract

The distinction between k-hyponormality and weak k-hyponormality via a weighted shift W_α with a weighted sequence α is closely related to the flatness of the weight sequence α. In general this flatness does not hold in the case of quadratical hyponormality. And so the following question rises: describe all quadratically hyponormal weighted shift with first two equal weights ([1]). The recursively generated positively quadratically hyponormal weighted shifts with weight sequence $1, 1, \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ will be discussed.

1. Introduction. This was presented at the international conference: The Seminar on Function Spaces, which was held at Hokkaido University in Japan on December 22-25, 2010, and is the joint work with G. Exner, I.B. Jung, and M.R. Lee. The results in this note will be appeared in some other journal.

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. For $A, B \in \mathcal{L}(\mathcal{H})$, we denote $[A, B] := AB - BA$. An n-tuple $T = (T_1, \cdots, T_n)$ of operators in $\mathcal{L}(\mathcal{H})$ is called *hyponormal* if the operator matrix $([T_j^*, T_i])_{i,j=1}^n$ is positive on the direct sum of n copies of $\mathcal{H}$. For $k \in \mathbb{N}$ and $T \in \mathcal{L}(\mathcal{H})$, T is said to be *k-hyponormal* if $(I, T, \cdots, T^k)$ is hyponormal. An n-tuple $T = (T_1, \cdots, T_n)$ is *weakly hyponormal* if $\lambda_1 T_1 + \cdots + \lambda_n T_n$ is hyponormal for every $\lambda_i \in \mathbb{C}$, $i = 1, \cdots, n$, where $\mathbb{C}$ is the set of complex numbers. An operator T is *weakly k-hyponormal* if $(T, T^2, \cdots, T^n)$ is weakly hyponormal. In particular, weak 2-hyponormality, often referred to as *quadratic hyponormality.*

For (unilateral) weighted shifts W_α with weight sequence $\alpha = \{\alpha_i\}_{i=0}^\infty$, the distinction between k-hyponormality and weak k-hyponormality is closely related to the *flatness* of their weight sequence α. It was shown in [2] that if W_α is a 2-hyponormal weighted shift with $\alpha_n = \alpha_{n+1}$ for some $n \in \mathbb{N}$, then

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$\alpha_1 = \alpha_2 = \cdots$. But in general this flatness does not hold in the case of quadratical hyponormality; for example, if $\alpha : \sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{n+1}{n+2}}$ $(n \geq 2)$, then W_α is quadratically hyponormal. And so the following question rises: describe all quadratically hyponormal weighted shift with first two equal weights (cf. [1]). In [3, Prop. 4.6], [4] and [6], the recursively generated positively quadratically hyponormal weighted shifts with weight sequence $1, (1, \sqrt{x}, \sqrt{y})^\wedge$ were described. And also, in [5], the recursively generated positively quadratically hyponormal weighted shifts with weight sequence $1, 1, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ were described. As a continued study, the following problem which was suggested in [4] remains an open problem.

Problem 1.1 ([4, Prob. 5.3]). Let $\alpha : 1, 1, \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ with $1 \leq x \leq u \leq v \leq w$. Describe $\mathcal{W} := \{x | W_\alpha$ is quadratically hyponormal for some u, v,and $w\}$.

In this note we discuss the above problem.

2. Preliminaries and Notation. We recall some notation which will be used frequently throughout the paper (cf. [2], [4], [6], [5]). An operator $T \in \mathcal{L}(\mathcal{H})$ is *quadratically hyponormal* if $T + sT^2$ is hyponormal for every $s \in \mathbb{C}$. Let $\{e_k\}_{k=0}^\infty$ be the standard orthonormal basis for ℓ^2, let P_n denote the orthogonal projection onto the subspace generated by $e_0, ..., e_n$, and let W_α be a hyponormal weighted shift with a weight sequence $\alpha = \{\alpha_i\}_{i=0}^\infty$. We denote

$$D(s) := [(W_\alpha + sW_\alpha^2)^*, W_\alpha + sW_\alpha^2], \qquad s \in \mathbb{C}.$$

For $n \geq 0$, let

$$\begin{aligned} D_n(s) &= P_n[(W_\alpha + sW_\alpha^2)^*, W_\alpha + sW_\alpha^2]P_n \\ &= \begin{pmatrix} q_0 & \bar{r}_0 & 0 & \cdots & 0 & 0 \\ r_0 & q_1 & \bar{r}_1 & \cdots & 0 & 0 \\ 0 & r_1 & q_2 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & q_{n-1} & \bar{r}_{n-1} \\ 0 & 0 & 0 & \cdots & r_{n-1} & q_n \end{pmatrix}, \end{aligned}$$

where $q_k := u_k + |s|^2 v_k$; $r_k := s\sqrt{w_k}$; $u_k := \alpha_k^2 - \alpha_{k-1}^2$; $v_k := \alpha_k^2\alpha_{k+1}^2 - \alpha_{k-1}^2\alpha_{k-2}^2$; $w_k := \alpha_k^2(\alpha_{k+1}^2 - \alpha_{k-1}^2)^2$ $(k \geq 0)$, and $\alpha_{-1} = \alpha_{-2} := 0$. Clearly, W_α is quadratically hyponormal if and only if $D_n(s) \geq 0$ for every $s \in \mathbb{C}$ and every $n \geq 0$. To detect this positivity, we consider $d_n(\cdot) := \det(D_n(\cdot))$. By direct computation, we have

$$\begin{aligned} d_0 &= q_0; \quad d_1 = q_0 q_1 - |r_0|^2, \\ d_{n+2} &= q_{n+2} d_{n+1} - |r_{n+1}|^2 d_n \qquad (n \geq 0); \end{aligned}$$

d_n is a polynomial in $t := |s|^2$ of degree $n+1$, with Mclaurin expansion

$$d_n(t) := \sum_{i=0}^{n+1} c(n,i)t^i. \tag{2.1}$$

This gives at once

$$\begin{aligned} c(0,0) &= u_0, \qquad c(0,1) = v_0, \qquad c(1,0) = u_1 u_0, \\ c(1,1) &= u_1 v_0 + u_0 v_1 - w_0, \qquad c(1,2) = v_1 v_0, \\ c(n+2,i) &= u_{n+2} c(n+1,i) + v_{n+2} c(n+1,i-1) - w_{n+1} c(n,i-1). \end{aligned}$$

Observe that $c(n,0) \geq 0$ and $c(n,n+1) \geq 0$ for all $n \geq 0$, and that $d_0(t) = \alpha_0^2(1+t\alpha_1^2) \geq 0$ and

$$d_1(t) = \alpha_0^2[(\alpha_1^2 - \alpha_0^2) + \alpha_1^2(\alpha_2^2 - \alpha_0^2)t + \alpha_1^4\alpha_2^2 t^2)].$$

We also recall [2] that a weighted shift W_α is said to be *recursively generated* if there exist $i \geq 1$ and $\Psi = (\Psi_0, \cdots, \Psi_{i-1}) \in \mathbb{C}^i$ such that

$$\gamma_n = \Psi_{i-1}\gamma_{n-1} + \cdots + \Psi_0\gamma_{n-i} \qquad (n \geq i), \tag{2.2}$$

where $\gamma_n \, (n \geq 0)$ is the moment of W_α, i.e., $\gamma_0 := 1, \gamma_{n+1} := \alpha_n^2\gamma_n (n \geq 0)$. Furthermore, (2.2) is equivalent to

$$\alpha_n^2 = \Psi_{i-1} + \frac{\Psi_{i-2}}{\alpha_{n-1}^2} + \cdots + \frac{\Psi_0}{\alpha_{n-1}^2 \cdots \alpha_{n-i+1}^2} \qquad (n \geq i). \tag{2.3}$$

Given an initial segment $\alpha : \sqrt{\alpha_0}, \sqrt{\alpha_1}, \sqrt{\alpha_2}$ with $0 < \alpha_0 < \alpha_1 < \alpha_2$, let

$$\mathbf{v}_0 = \begin{bmatrix} \gamma_0 \\ \gamma_1 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} \gamma_2 \\ \gamma_3 \end{bmatrix}.$$

The vectors $\mathbf{v}_0$ and $\mathbf{v}_1$ are linearly independent in $\mathbb{R}^2$, so there exists a unique $\Psi = (\Psi_0, \Psi_1) \in \mathbb{R}^2$ such that $\mathbf{v}_2 = \Psi_0 v_0 + \Psi_1 v_1$. In fact,

$$\Psi_0 = -\frac{\alpha_0\alpha_1(\alpha_2 - \alpha_1)}{\alpha_1 - \alpha_0} \quad \text{and} \quad \Psi_1 = \frac{\alpha_1(\alpha_2 - \alpha_0)}{\alpha_1 - \alpha_0}. \tag{2.4}$$

Let $\widehat{\gamma}_n := \gamma_n \; (0 \leq n \leq 1)$ and let $\widehat{\gamma}_n := \Psi_1\widehat{\gamma}_{n-1} + \Psi_0\widehat{\gamma}_{n-2} \quad (n \geq 2)$. Since $\widehat{\gamma}_n > 0 \, (n \geq 0)$ (cf. [2, Proof of Th. 3.5]), we define

$$\widehat{\alpha}_n := \left(\frac{\widehat{\gamma}_{n+1}}{\widehat{\gamma}_n}\right)^{\frac{1}{2}} \qquad (n \geq 0) \tag{2.5}$$

(so that $\widehat{\alpha}_n = \alpha_n$ for $0 \leq n \leq 2$). Hence we obtain the coefficients of a recursively generated weighted shift $W_{\widehat{\alpha}}$ with a weight sequence $\widehat{\alpha} := \{\widehat{\alpha}_n\}_{n=0}^{\infty}$, and $\widehat{\alpha}_n^2 = \Psi_1 + \frac{\Psi_0}{\widehat{\alpha}_{n-1}^2} \; (n \geq 1)$.

3. Results. Let $\alpha := \{\alpha_i\}_{i\in\mathbb{N}_0}$ and let W_α be the unilateral weighted shift with a weight sequence α. Suppose $\alpha_k = \alpha_{k+j}$ for $k, j \in \mathbb{N}$. Then W_α is quadratically hyponormal if and only if $\alpha_1 = \alpha_2 = \cdots$. Hence we assume that $\alpha_k < \alpha_{k+1}$ for all $k \in \mathbb{N}$ to avoid the trivial case throughout this paper.

Let $\alpha : 1, 1, \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ with $1 < x < u < v < w$. For our convenience, we recall that

$$\alpha_n^2 = \Psi_1 + \frac{\Psi_0}{\alpha_{n-1}^2} \quad (n \geq 6),$$

where

$$\Psi_1 = \frac{v(w-u)}{v-u} \quad \text{and} \quad \Psi_0 = -\frac{uv(w-v)}{v-u}.$$

Lemma 3.1. Let $\alpha : 1, 1, \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ with $1 < x < u < v < w$. Then for $n \geq 5$, we have

$$c(n,i) = \begin{cases} v_n \cdots v_4 c(3,4), & i = n+1, \\ u_n c(n-1,n) + v_n \cdots v_5 \eta_2, & i = n, \\ u_n c(n-1,n-1) + v_n \cdots v_5 \eta_3, & i = n-1, \\ u_n \cdots u_{i+2} c(i+1,i), & 3 \leq i \leq n-2, \\ u_n \cdots u_5 c(4,i), & 0 \leq i \leq 2, \end{cases}$$

where $\eta_2 = v_4 c(3,3) - w_3 c(2,3)$ and $\eta_3 = v_4 c(3,2) - w_3 c(2,2)$.

Lemma 3.2. *Under the same notation in the previous section, we have that* $K := \lim_{n\to\infty} \frac{v_n}{u_n}$ *exists, and* $K = \frac{A+\sqrt{B}v(w-u)^2}{C}$,where

$$\begin{aligned} A &= v(w-u)^2 v(w-u) = v^2(w-u)^3; \\ B &= v^2(w-u)^2 - 4uv(w-v)(v-u)), \\ C &= 2u(v-u)^2(w-v). \end{aligned}$$

Lemma 3.3. *Let* $\alpha : 1, 1, \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ *with* $1 < x < u < v < w$. *Then* W_α *is positively quadratically hyponormal if and only if*

(a) $x \leq 2 - \frac{1}{u}$,
(b) $f_1(x) \geq 0$,
(c) $f_2(x) \geq 0$,
(d) $c(5,5) \geq 0$, $c(5,4) \geq 0$, $c(6,5) \geq 0$,
(e) $A_n := u_n u_{n-1} \eta_1 + u_n v_{n-1} \eta_2 + v_n v_{n-1} \eta_3 \geq 0$ $(n \geq 7)$, *and*
(f) *If* $\eta_2 < 0$, *then* $K \leq \frac{\eta_1}{|\eta_2|}$, *where* $\eta_1(x,u,v,w) = v_4 c(3,4)$, $\eta_2(x,u,v,w) = v_4 c(3,3) - w_3 c(2,3)$, *and* $\eta_3(x,u,v,w) = v_4 c(3,2) - w_3 c(2,2)$.

Proposition 3.4. (a) *If* $\tilde{\eta}_3(x,u,v,w) > 0$, *then* $A_n \geq 0$ $(n \geq 7)$ *if and only if the following statements hold:*

(a-i) $K \leq u_2$ *on* $\gamma \leq x$ *and*
(a-ii) $\Theta_1 \geq 0$ *on* $1 < x < \gamma$.
(b) *If* $\tilde{\eta}_3(x,u,v,w) \leq 0$, *then* $A_n \geq 0$ $(n \geq 7) \Longleftrightarrow K \leq u_2$.

Furthermore, we have that $K \leq u_2 \iff$ *one of the following four cases holds:*

(b-i) $\Theta_2 \geq 0, \Theta_3 \geq 0, \Theta_4 \geq 0, \Theta_5 \geq 0$,
(b-ii) $\Theta_2 \geq 0, \Theta_3 \geq 0, \Theta_4 < 0$,
(b-iii) $\Theta_2 < 0, \Theta_3 < 0, \Theta_4 > 0, \Theta_5 < 0$, *or*
(b-iv) $\Theta_2 < 0, \Theta_3 \geq 0$.

By direct computations we have the following Lemma.

Lemma 3.5. (a) $c(4,3) \geq 0 \iff \Theta_6 \geq 0$,
(b) $c(4,4) \geq 0 \iff \Theta_7 \geq 0$,
(c) $c(5,4) \geq 0 \iff \Theta_8 \geq 0$,
(d) $c(5,5) \geq 0 \iff \Theta_9 \geq 0$, *and*
(e) $c(6,5) \geq 0 \iff \Theta_{10} \geq 0$,
where

$$\begin{aligned}
\Theta_6 &= -u^2 + uv - u^3v - vw + 2uvw + ux - 2uvx + 2u^2vx - uvwx, \\
\Theta_7 &= u^2v - uvw - u^2v^2w + (-u + u^2 + v - 2uv + 3uvw + u^2v^2w)x \\
&\quad +(u - 2u^2 - uv + 2u^2v - u^3v - 2uvw)x^2 + u^2x^3, \\
\Theta_8 &= c(5,4)x/(v-u) \\
&= u^3v^2 - u^2v^3 + u^4v^3 + \cdots\cdots + (u^3v - u^2v^2 - u^3w + u^2vw)x^3, \\
\Theta_9 &= c(5,5)x/(v-u) \\
&= u^3v^3 - u^3v^2w - 3u^2v^3w + \cdots\cdots + (u^3v - u^4v - \cdots + u^2vw^2)x^3, \\
\Theta_{10} &= x(-u^2(v-x)^2(-1+x) - u(v-x)^2(-1+ux) \\
&\quad +u(1+uv-2x)(uv-x)(-1+x)x(vw-ux)^2(-1+ux) \\
&\quad +(1+u(-2+x))(-1+x)(-vw+ux)).
\end{aligned}$$

Lemma 3.6. *If* $\eta_2 < 0$, *then* $K \leq \frac{\eta_1}{|\eta_2|} \iff \Theta_{11} \geq 0$ and $\Theta_{12} \geq 0$, *where* $\Theta_{11} := \eta_1^2C^2 - (A^2 + Bv^2(w-u)^4)\eta_2^2$; $\Theta_{12} := \Theta_{11}^2 - (2Av(w-u)^2\eta_2^2\sqrt{B})^2$.

Combining the above statements, we have the following theorem.

Theorem 3.7. *Let* $\alpha : 1, 1, \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ *with* $1 < x < u < v < w$. *Then* W_α *is positively quadratically hyponormal if and only if the following holds:*

(a) $x \leq 2 - \frac{1}{u}$,
(b) $\Theta_k \geq 0, \quad k = 6, 7, 8, 9, 10$,
(c) *If* $\eta_2 < 0$, *then* $\Theta_{11} \geq 0$ *and* $\Theta_{12} \geq 0$, *and*
(d) *Proposition 3.4 holds.*

We close this note with the following example.

Example 3.8. Let $\alpha : 1, 1, \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$. Take $u = 1.1, v = 1.2$, and $w = 1.5$. We denote $D(u,v,w) := u - 2u^2 + 2u^2v - u^2v^2 - vw + uvw$. Then $D(1.1, 1.2, 1.5) = 0.0216 > 0$ and

$$\gamma = \frac{u^2v^2 + vw - 2uvw}{u - 2u^2 + 2u^2v - uvw} = 1.05455.$$

So we apply Proposition 3.4 (a) and obtain the followings:

(i) $x \le 2 - \frac{1}{u} \Longleftrightarrow 1 < x \le 1.09091$,
(ii) $\Theta_1 \ge 0 \Longleftrightarrow 1.00463 \le x \le \gamma$,
(iii) $\Theta_2 \ge 0 \Longleftrightarrow \gamma \le x \le 1.46870$,
(iv) $\Theta_3 \ge 0 \Longleftrightarrow \gamma \le x \le 1.06173$,
(v) $\Theta_4 \ge 0 \Longleftrightarrow \gamma \le x \le 1.07967$,
(vi) $\Theta_5 \ge 0 \Longleftrightarrow \gamma \le x \le 1.05606$,
(vii) $\Theta_6 \ge 0 \Longleftrightarrow 1 < x \le 1.09221$,
(viii) $\Theta_7 \ge 0 \Longleftrightarrow 1.00462 \le x \le 1.18030$,
(ix) $\Theta_8 \ge 0 \Longleftrightarrow 1.00306 \le x \le 1.09629$,
(x) $\Theta_9 \ge 0 \Longleftrightarrow 1.00349 \le x \le 1.17610$,
(xi) $\Theta_{10} \ge 0 \Longleftrightarrow 1.00222 \le x \le 1.09312$, and
(xii) $\eta(x, 1.1, 1.2, 1.5) \ge 0 \Longleftrightarrow 1.02095 \le x \le 1.15695$.

So W_α is positively quadratically hyponormal $\Longleftrightarrow 1.00463 \le x \le \gamma$ or $\gamma \le x \le 1.05606$, which is equivalente to $1.00463 \le x \le 1.05606$.

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