Proceedings of the 35th Sapporo Symposium on Partial Differential Equations

Edited by
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PREFACE

This volume is intended as the proceedings of Sapporo Symposium on Partial Differential Equations, held on August 23 through August 25 in 2010 at Faculty of Science, Hokkaido University.

Sapporo Symposium on PDE has been held annually to present the latest developments on PDE with a broad spectrum of interests not limited to the methods of a particular school. Professor Taira Shirota started the symposium more than 30 years ago. Professor Kôji Kubota and Professor Rentaro Agemi made a large contribution to its organization for many years.

We always thank their significant contribution to the progress of the Sapporo Symposium on PDE.

T. Ozawa, Y. Giga, T. Sakajo, S. Jimbo,
H. Takaoka, K. Tsutaya, Y. Tonegawa,
and G. Nakamura

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（第３５回偏微分方程式論札幌シンポジウム）

組織委員: 小澤 徹, 儀我 美一, 坂上 貴之, 神保 秀一,
高岡 秀夫, 中村 玄, 利根川 吉廣, 津田谷 公利

Organizers: T. Ozawa, Y. Giga, T. Sakajo, S. Jimbo,
H. Takaoka, G. Nakamura, Y. Tonegawa, K. Tsutaya

Period (期間) August 23 , 2010 - August 25 , 2010

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北海道大学 理学部５号館大講義室 (203 号室)

URL http://www.math.sci.hokudai.ac.jp/sympo/sapporo/program.html

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GLOBAL WELLPOSEDNESS OF THE 3-D FULL WATER WAVE PROBLEM

SIJUE WU

The mathematical problem of n-dimensional water wave concerns the motion of the interface separating an inviscid, incompressible, irrotational fluid, under the influence of gravity, from a region of zero density (i.e. air) in n-dimensional space. It is assumed that the fluid region is below the air region. Assume that the density of the fluid is 1, the gravitational field is $-\mathbf{k}$, where $\mathbf{k}$ is the unit vector pointing in the upward vertical direction, and at time $t \geq 0$, the free interface is $\Sigma(t)$, and the fluid occupies region $\Omega(t)$. When surface tension is zero, the motion of the fluid is described by

$$\begin{cases} \mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = -\mathbf{k} - \nabla P \qquad \text{on } \Omega(t),\ t \geq 0, \\ \operatorname{div} \mathbf{v} = 0, \qquad \operatorname{curl} \mathbf{v} = 0, \qquad \text{on } \Omega(t),\ t \geq 0, \\ P = 0, \qquad \text{on } \Sigma(t) \\ (1, \mathbf{v}) \text{ is tangent to the free surface } (t, \Sigma(t)), \end{cases} \tag{0.1}$$

where $\mathbf{v}$ is the fluid velocity, P is the fluid pressure. It is well-known that when surface tension is neglected, the water wave motion can be subject to the Taylor instability [19, 2]. Assume that the free interface $\Sigma(t)$ is described by $\xi = \xi(\alpha, t)$, where $\alpha \in R^{n-1}$ is the Lagrangian coordinate, i.e. $\xi_t(\alpha, t) = \mathbf{v}(z(\alpha, t), t)$ is the fluid velocity on the interface, $\xi_{tt}(\alpha, t) = (\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v})(z(\alpha, t), t)$ is the acceleration. Let $\mathbf{n}$ be the unit normal pointing out of $\Omega(t)$. The Taylor sign condition relating to Taylor instability is

$$-\frac{\partial P}{\partial \mathbf{n}} = (\xi_{tt} + \mathbf{k}) \cdot \mathbf{n} \geq c_0 > 0, \tag{0.2}$$

point-wisely on the interface for some positive constant c_0. In previous works [20, 21], we showed that the Taylor sign condition (0.2) always holds for the n-dimensional infinite depth water wave problem (0.1), $n \geq 2$, as long as the interface is non-self-intersecting; and the initial value problem of the water wave system (0.1) is uniquely solvable **locally** in time in Sobolev spaces for arbitrary given data. Earlier work includes Nalimov [16], and Yosihara [24] on local existence and uniqueness for small data in 2D. We mention the following recent work on local wellposedness [1, 3, 4, 11, 14, 15, 17, 18, 25]. However the global in time behavior of the solutions remained open until 2008.

The main content of this extended abstract is from the introduction of [23].

In [22], we showed that for the 2D full water wave problem (0.1) ($n = 2$), the quantity $\Theta = (I - \mathfrak{H})y$, under an appropriate coordinate change $k = k(\alpha, t)$, satisfy an equation of the type

$$\partial_t^2 \Theta - i\partial_\alpha \Theta = G \tag{0.3}$$

with G consisting of nonlinear terms of only cubic and higher orders. Here $\mathfrak{H}$ is the Hilbert transform related to the water region $\Omega(t)$, y is the height function for the interface $\Sigma(t) : (x(\alpha, t), y(\alpha, t))$. Using this favorable structure, and the L^∞ time decay rate for the 2D water wave $1/t^{1/2}$, we showed that the full water wave equation (0.1) in two space dimensions has a unique smooth solution for a time period $[0, e^{c/\epsilon}]$ for initial data $\epsilon\Phi$, where Φ is arbitrary, c depends only on Φ, and ϵ is sufficiently small.

Briefly, the structural advantage of (0.3) can be explained as the following. We know the water wave equation (0.1) is equivalent to an equation on the interface of the form

$$\partial_t^2 u + |D|u = \text{nonlinear terms} \tag{0.4}$$

where the nonlinear terms contain quadratic nonlinearity. For given smooth data, the free equation $\partial_t^2 u + |D|u = 0$ has a unique solution globally in time, with L^∞ norm decays at the rate $1/t^{\frac{n-1}{2}}$. However the nonlinear interaction can cause blow-up at finite time. The weaker the nonlinear interaction, the longer the solution stays smooth. For small data, quadratic interactions are in general stronger than the cubic and higher order interactions. In (0.3) there is no quadratic terms, using it we were able to prove a longer time existence of classical solutions for small initial data in 2D.

Naturally, we would like to know if the 3D water wave equation also posses such special structures. We find that indeed this is the case. A natural setting for 3D to utilize the ideas of 2D is the Clifford analysis. However deriving such equations (0.3) in 3D in the Clifford Algebra framework is not straightforward due to the non-availability of the Riemann mapping, the non-commutativity of the Clifford numbers, and the fact that the multiplication of two Clifford analytic functions is not necessarily analytic. Nevertheless we have overcome these difficulties.

Let $\Sigma(t) : \xi = (x(\alpha, \beta, t), y(\alpha, \beta, t), z(\alpha, \beta, t))$ be the interface in Lagrangian coordinates $(\alpha, \beta) \in \mathbb{R}^2$, and let $\mathfrak{H}$ be the Hilbert transform associated to the water region $\Omega(t)$, $N = \xi_\alpha \times \xi_\beta$ be the outward normal. In this work, we show that the quantity $\theta = (I - \mathfrak{H})z$ satisfies such equation

$$\partial_t^2 \theta - \mathfrak{a} N \times \nabla\theta = G \tag{0.5}$$

where G is a nonlinearity of cubic and higher orders in nature. We also find a coordinate change k that transforms (0.5) into an equation consisting of a linear part plus only cubic

and higher order nonlinear terms.[1] As a consequence of this special structure and the faster L^∞ time decay rate $1/t$ in 3D we prove the global in time wellposedness of the full water wave equation (0.1) in 3D.

In fact we obtain better results in 3D than in 2D in terms of the initial data set. We show that if the steepness of the initial interface and the velocity along the initial interface (and finitely many of their derivatives) are sufficiently small, then the solution of the 3D full water wave equation (0.1) remains smooth for all time and decays at a L^∞ rate of $1/t$. No smallness assumptions are made to the height of the initial interface and the velocity field in the fluid domain. In particular, this means that the amplitude of the initial interface can be arbitrary large, the initial kinetic energy $\frac{1}{2}\|\mathbf{v}\|^2_{L^2(\Omega(0))}$ can be infinite. This certainly makes sense physically. We note that the almost global wellposedness result we obtained for 2D water wave [22] requires the initial amplitude of the interface and the initial kinetic energy $\frac{1}{2}\|\mathbf{v}\|^2_{L^2(\Omega(0))}$ being small. One may view 2D water wave as a special case of 3D where the wave is constant in one direction. In 2D there is one less direction for the wave to disperse and the L^∞ time decay rate is a slower $1/t^{1/2}$. Technically our proof of the almost global wellposedness result in 2D [22] used to the full extend the decay rate and required the smallness in the amplitude and kinetic energy since we needed to control the derivatives in the full range. One may think the assumption on the smallness in amplitude and kinetic energy is to compensate the lack of decay in one direction. However this is merely a technical reason. In 3D assuming the wave tends to zero at spatial infinity, we have a faster L^∞ time decay rate $1/t$. This allows us a less elaborate proof and a global wellposedness result with less assumptions on the initial data.

0.1. **Notations and Clifford analysis.** We study the 3D water wave problem in the setting of the Clifford Algebra $\mathcal{C}(V_2)$, i.e. the algebra of quaternions. We refer to [9] for an in depth discussion of Clifford analysis.

Let $\{1, e_1, e_2, e_3\}$ be the basis of $\mathcal{C}(V_2)$ satisfying

$$e_i^2 = -1, \quad e_i e_j = -e_j e_i, \quad i, j = 1, 2, 3,\ i \neq j, \qquad e_3 = e_1 e_2. \tag{0.6}$$

An element $\sigma \in \mathcal{C}(V_2)$ has a unique representation $\sigma = \sigma_0 + \sum_{i=1}^3 \sigma_i e_i$, with $\sigma_i \in \mathbb{R}$ for $0 \leq i \leq 3$. We call σ_0 the real part of σ and denote it by $\operatorname{Re}\sigma$ and $\sum_{i=1}^3 \sigma_i e_i$ the vector part of σ. We call σ_i the e_i component of σ. We denote $\overline{\sigma} = e_3 \sigma e_3$, $|\sigma|^2 = \sum_{i=0}^3 \sigma_i^2$. If not otherwise specified, we always assume in such an expression $\sigma = \sigma_0 + \sum_{i=1}^3 \sigma_i e_i$ that $\sigma_i \in \mathbb{R}$, for $0 \leq i \leq 3$. We define $\sigma \cdot \xi = \sum_{j=0}^3 \sigma_j \xi_j$. We call $\sigma \in \mathcal{C}(V_2)$ a vector if $\operatorname{Re}\sigma = 0$. We

[1] We will explain more precisely the meaning of these statements in subsection 0.2.

identify a point or vector $\xi = (x, y, z) \in \mathbb{R}^3$ with its $\mathcal{C}(V_2)$ counterpart $\xi = xe_1 + ye_2 + ze_3$. For vectors $\xi,\ \eta \in \mathcal{C}(V_2)$, we know

$$\xi\eta = -\xi \cdot \eta + \xi \times \eta, \tag{0.7}$$

where $\xi \cdot \eta$ is the dot product, $\xi \times \eta$ the cross product. For vectors $\xi,\ \zeta,\ \eta,\ \xi(\zeta \times \eta)$ is obtained by first finding the cross product $\zeta \times \eta$, then regard it as a Clifford vector and calculating its multiplication with ξ by the rule (0.6). We write $\mathcal{D} = \partial_x e_1 + \partial_y e_2 + \partial_z e_3$, $\nabla = (\partial_x, \partial_y, \partial_z)$. At times we also use the notation $\xi = (\xi_1, \xi_2, \xi_3)$ to indicate a point in $\mathbb{R}^3$. In this case $\nabla = (\partial_{\xi_1}, \partial_{\xi_2}, \partial_{\xi_3})$, $\mathcal{D} = \partial_{\xi_1} e_1 + \partial_{\xi_2} e_2 + \partial_{\xi_3} e_3$.

Let Ω be an unbounded[2] C^2 domain in $\mathbb{R}^3$, $\Sigma = \partial\Omega$ be its boundary and Ω^c be its complement. A $\mathcal{C}(V_2)$ valued function F is Clifford analytic in Ω if $\mathcal{D}F = 0$ in Ω. Let

$$\Gamma(\xi) = -\frac{1}{\omega_3}\frac{1}{|\xi|}, \qquad K(\xi) = -2\mathcal{D}\Gamma(\xi) = -\frac{2}{\omega_3}\frac{\xi}{|\xi|^3}, \qquad \text{for } \xi = \sum_1^3 \xi_i e_i, \tag{0.8}$$

where ω_3 is the surface area of the unit sphere in $\mathbb{R}^3$. Let $\xi = \xi(\alpha, \beta)$, $(\alpha, \beta) \in \mathbb{R}^2$ be a parameterization of Σ with $N = \xi_\alpha \times \xi_\beta$ pointing out of Ω. The Hilbert transform associated to the parameterization $\xi = \xi(\alpha, \beta)$, $(\alpha, \beta) \in \mathbb{R}^2$ is defined by

$$\mathfrak{H}_\Sigma f(\alpha, \beta) = p.v. \iint_{\mathbb{R}^2} K(\xi(\alpha', \beta') - \xi(\alpha, \beta))\, (\xi'_{\alpha'} \times \xi'_{\beta'})\, f(\alpha', \beta')\, d\alpha' d\beta'. \tag{0.9}$$

We know a $\mathcal{C}(V_2)$ valued function F that decays at infinity is Clifford analytic in Ω if and only if its trace on Σ: $f(\alpha, \beta) = F(\xi(\alpha, \beta))$ satisfies

$$f = \mathfrak{H}_\Sigma f. \tag{0.10}$$

We know $\mathfrak{H}_\Sigma^2 = I$ in L^2. We use the convention $\mathfrak{H}_\Sigma 1 = 0$. We abbreviate

$$\begin{aligned}\mathfrak{H}_\Sigma f(\alpha, \beta) &= \iint K(\xi(\alpha', \beta') - \xi(\alpha, \beta))\, (\xi'_{\alpha'} \times \xi'_{\beta'})\, f(\alpha', \beta')\, d\alpha' d\beta' \\ &= \iint K(\xi' - \xi)\, (\xi'_{\alpha'} \times \xi'_{\beta'})\, f'\, d\alpha' d\beta' = \iint K\, N'\, f'\, d\alpha' d\beta'.\end{aligned}$$

Assume that for each $t \in [0, T]$, $\Omega(t)$ is a C^2 domain with boundary $\Sigma(t)$. Let $\Sigma(t) : \xi = \xi(\alpha, \beta, t)$, $(\alpha, \beta) \in \mathbb{R}^2$; $\xi \in C^2(\mathbb{R}^2 \times [0, T])$, $N = \xi_\alpha \times \xi_\beta$. We know $N \times \nabla = \xi_\beta \partial_\alpha - \xi_\alpha \partial_\beta$. Denote $[A, B] = AB - BA$. We have

Lemma 0.1. *1. Let $f = f(\alpha, \beta)$, $(\alpha, \beta) \in \mathbb{R}^2$ be a real valued smooth function decays fast at infinity. We have*

$$\iint K(\xi(\alpha', \beta') - \xi(\alpha, \beta)) \cdot (N' \times \nabla' f)(\alpha', \beta')\, d\alpha' d\beta' = 0. \tag{0.11}$$

2. For any function $f = \sum_1^3 f_i e_i$ satisfying $f = \mathfrak{H}_\Sigma f$ or $f = -\mathfrak{H}_\Sigma f$, we have

$$\xi_\beta \cdot \partial_\alpha f - \xi_\alpha \cdot \partial_\beta f = 0. \tag{0.12}$$

[2]Similar definitions and results exist for bounded domains, see [9]. For the purpose of this paper, we discuss only for unbounded domain Ω.

Lemma 0.2. *Let $f \in C^1(\mathbb{R}^2\times[0,T])$ be a $\mathcal{C}(V_2)$ valued function vanishing at spatial infinity, and $\mathfrak{a}$ be real valued. Then*

$$[\partial_t, \mathfrak{H}_{\Sigma(t)}]f = \iint K(\xi'-\xi)\,(\xi_t-\xi_t')\times(\xi_{\beta'}'\partial_{\alpha'}-\xi_{\alpha'}'\partial_{\beta'})f'\,d\alpha' d\beta'. \tag{0.13}$$

$$[\partial_\alpha, \mathfrak{H}_{\Sigma(t)}]f = \iint K(\xi'-\xi)\,(\xi_\alpha-\xi_{\alpha'}')\times(\xi_{\beta'}'\partial_{\alpha'}-\xi_{\alpha'}'\partial_{\beta'})f'\,d\alpha' d\beta' \tag{0.14}$$

$$[\partial_\beta, \mathfrak{H}_{\Sigma(t)}]f = \iint K(\xi'-\xi)\,(\xi_\beta-\xi_{\beta'}')\times(\xi_{\beta'}'\partial_{\alpha'}-\xi_{\alpha'}'\partial_{\beta'})f'\,d\alpha' d\beta' \tag{0.15}$$

$$[\mathfrak{a}N\times\nabla, \mathfrak{H}_{\Sigma(t)}]f = \iint K(\xi'-\xi)\,(\mathfrak{a}N-\mathfrak{a}'N')\times(\xi_{\beta'}'\partial_{\alpha'}-\xi_{\alpha'}'\partial_{\beta'})f'\,d\alpha' d\beta' \tag{0.16}$$

$$\begin{aligned}[\partial_t^2, \mathfrak{H}_{\Sigma(t)}]f = &\iint K(\xi'-\xi)\,(\xi_{tt}-\xi_{tt}')\times(\xi_{\beta'}'\partial_{\alpha'}-\xi_{\alpha'}'\partial_{\beta'})f'\,d\alpha' d\beta'\\ &+\iint K(\xi'-\xi)\,(\xi_t-\xi_t')\times(\xi_{t\beta'}'\partial_{\alpha'}-\xi_{t\alpha'}'\partial_{\beta'})f'\,d\alpha' d\beta'\\ &+\iint \partial_t K(\xi'-\xi)\,(\xi_t-\xi_t')\times(\xi_{\beta'}'\partial_{\alpha'}-\xi_{\alpha'}'\partial_{\beta'})f'\,d\alpha' d\beta'\\ &+2\iint K(\xi'-\xi)\,(\xi_t-\xi_t')\times(\xi_{\beta'}'\partial_{\alpha'}-\xi_{\alpha'}'\partial_{\beta'})f_t'\,d\alpha' d\beta'\end{aligned} \tag{0.17}$$

The proof of Lemmas 0.1, 0.2 can be found in [23].

0.2. **The main equations and main results.** We now discuss the 3D water wave. Let $\Sigma(t): \xi(\alpha,\beta,t) = x(\alpha,\beta,t)e_1+y(\alpha,\beta,t)e_2+z(\alpha,\beta,t)e_3$, $(\alpha,\beta)\in\mathbb{R}^2$ be the parameterization of the interface at time t in Lagrangian coordinates (α,β) with $N=\xi_\alpha\times\xi_\beta=(N_1,N_2,N_3)$ pointing out of the fluid domain $\Omega(t)$. Let $\mathfrak{H}=\mathfrak{H}_{\Sigma(t)}$, and

$$\mathfrak{a} = -\frac{1}{|N|}\frac{\partial P}{\partial \mathbf{n}}.$$

We know from [21] that $\mathfrak{a}>0$ and equation (0.1) is equivalent to the following nonlinear system defined on the interface $\Sigma(t)$:

$$\xi_{tt}+e_3 = \mathfrak{a}N \tag{0.18}$$

$$\xi_t = \mathfrak{H}\xi_t \tag{0.19}$$

Motivated by [22], we would like to know whether in 3-D, the quantity $\pi = (I-\mathfrak{H})ze_3$ under an appropriate coordinate change satisfies an equation with nonlinearities containing no quadratic terms. We first derive the equation for π in Lagrangian coordinates.

Proposition 0.3. *We have*

$$\begin{aligned}(\partial_t^2-\mathfrak{a}N\times\nabla)\pi = &\iint K(\xi'-\xi)\,(\xi_t-\xi_t')\times(\xi_{\beta'}'\partial_{\alpha'}-\xi_{\alpha'}'\partial_{\beta'})\overline{\xi_t'}\,d\alpha' d\beta'\\ &-\iint K(\xi'-\xi)\,(\xi_t-\xi_t')\times(\xi_{t\beta'}'\partial_{\alpha'}-\xi_{t\alpha'}'\partial_{\beta'})z'\,d\alpha' d\beta' e_3\\ &-\iint \partial_t K(\xi'-\xi)\,(\xi_t-\xi_t')\times(\xi_{\beta'}'\partial_{\alpha'}-\xi_{\alpha'}'\partial_{\beta'})z'\,d\alpha' d\beta' e_3\end{aligned} \tag{0.20}$$

Proof. Notice from (0.18)

$$(\partial_t^2 - \mathfrak{a}N \times \nabla) z e_3 = z_{tt} e_3 + \mathfrak{a} N_1 e_1 + \mathfrak{a} N_2 e_2 = \xi_{tt} \tag{0.21}$$

and from (0.19) that

$$(I - \mathfrak{H}) \xi_{tt} = [\partial_t, \mathfrak{H}] \xi_t \tag{0.22}$$

(0.20) is an easy consequence of (0.13), (0.16) and (0.17) and (0.18), (0.21), (0.22):

$$\begin{aligned}
(\partial_t^2 - \mathfrak{a}N \times \nabla)\pi &= (I - \mathfrak{H})(\partial_t^2 - \mathfrak{a}N \times \nabla) z e_3 - [\partial_t^2 - \mathfrak{a}N \times \nabla, \mathfrak{H}] z e_3 \\
&= [\partial_t, \mathfrak{H}] \xi_t - [\partial_t^2 - \mathfrak{a}N \times \nabla, \mathfrak{H}] z e_3 \\
&= \iint K(\xi' - \xi)\,(\xi_t - \xi_t') \times (\xi'_{\beta'} \partial_{\alpha'} - \xi'_{\alpha'} \partial_{\beta'}) \overline{\xi_t'}\, d\alpha' d\beta' \\
&- \iint K(\xi' - \xi)\,(\xi_t - \xi_t') \times (\xi'_{t\beta'} \partial_{\alpha'} - \xi'_{t\alpha'} \partial_{\beta'}) z'\, d\alpha' d\beta' e_3 \\
&- \iint \partial_t K(\xi' - \xi)\,(\xi_t - \xi_t') \times (\xi'_{\beta'} \partial_{\alpha'} - \xi'_{\alpha'} \partial_{\beta'}) z'\, d\alpha' d\beta' e_3
\end{aligned}$$

□

We see that the second and third terms in the right hand side of (0.20) are consisting of terms of cubic and higher orders, while the first term contains quadratic terms. Unlike the 2D case, multiplications of Clifford analytic functions are not necessarily analytic, so we cannot reduce the first term at the right hand side of equation (0.20) into a cubic form. However we notice that $\overline{\xi}_t = x_t e_1 + y_t e_2 - z_t e_3$ is almost analytic in the air region $\Omega(t)^c$, and this implies that the first term is almost analytic in the fluid domain $\Omega(t)$, or in other words, is almost of the type $(I + \mathfrak{H})Q$ in nature, with Q a quadratic term. Notice that the left hand side of (0.20) is almost analytic in the air region, or of the type $(I - \mathfrak{H})$. The orthogonality of the projections $(I - \mathfrak{H})$ and $(I + \mathfrak{H})$ allows us to reduce the first term into cubic in energy estimates.

Notice that the left hand side of (0.20) still contains quadratic terms and (0.20) is invariant under a change of coordinates. We now want to see if in 3D, there is a coordinate change k, such that under which the left hand side of (0.20) becomes a linear part plus only cubic and higher order terms. In 2D, such a coordinate change exists (see (2.18) in [22]). However it is defined by the Riemann mapping. Although there is no Riemann mapping in 3D, we realize that the Riemann mapping used in 2-D is just a holomorphic function in the fluid region with its imaginary part equal to zero on $\Sigma(t)$. This motivates us to define

$$k = k(\alpha, \beta, t) = \xi(\alpha, \beta, t) - (I + \mathfrak{H}) z(\alpha, \beta, t) e_3 + \mathfrak{K} z(\alpha, \beta, t) e_3 \tag{0.23}$$

Here $\mathfrak{K} = \operatorname{Re} \mathfrak{H}$:

$$\mathfrak{K} f(\alpha, \beta, t) = - \iint K(\xi(\alpha', \beta', t) - \xi(\alpha, \beta, t)) \cdot N' f(\alpha', \beta', t)\, d\alpha'\, d\beta' \tag{0.24}$$

is the double layered potential operator. It is clear that the e_3 component of k as defined in (0.23) is zero. In fact, the real part of k is also zero. This is because

$$\begin{aligned}&\iint K(\xi'-\xi)\times(\xi'_{\alpha'}\times\xi'_{\beta'})z'e_3\,d\alpha'\,d\beta' = \iint(\xi'_{\alpha'}\xi'_{\beta'}\cdot K-\xi'_{\beta'}\xi'_{\alpha'}\cdot K)z'e_3\,d\alpha'\,d\beta'\\&= -2\iint(\xi'_{\alpha'}\partial_{\beta'}\Gamma(\xi'-\xi)-\xi'_{\beta'}\partial_{\alpha'}\Gamma(\xi'-\xi))z'e_3\,d\alpha'\,d\beta'\\&= 2\iint\Gamma(\xi'-\xi)(\xi'_{\alpha'}z_{\beta'}-\xi'_{\beta'}z_{\alpha'})e_3\,d\alpha'\,d\beta' = 2\iint\Gamma(\xi'-\xi)(N'_1e_1+N'_2e_2)\,d\alpha'\,d\beta'\end{aligned}$$

So

$$\mathfrak{H}ze_3 = \mathfrak{K}ze_3 + 2\iint\Gamma(\xi'-\xi)(N'_1e_1+N'_2e_2)\,d\alpha'\,d\beta' \tag{0.25}$$

This shows that the mapping k defined in (0.23) has only the e_1 and e_2 components $k = (k_1,k_2) = k_1e_1+k_2e_2$. If $\Sigma(t)$ is a graph of small steepness, i.e. if z_α and z_β are small, then the Jacobian of $k = k(\cdot,t)$: $J(k) = J(k(t)) = \partial_\alpha k_1\partial_\beta k_2 - \partial_\alpha k_2\partial_\beta k_1 > 0$ and $k(\cdot,t):\mathbb{R}^2\to\mathbb{R}^2$ defines a valid coordinate change (see [23]).

Denote $\nabla_\perp = (\partial_\alpha,\partial_\beta)$, $U_gf(\alpha,\beta,t) = f(g(\alpha,\beta,t),t) = f\circ g(\alpha,\beta,t)$. Assume that $k = k(\cdot,t):\mathbb{R}^2\to\mathbb{R}^2$ defined in (0.23) is a diffeomorphism satisfying $J(k(t))>0$. Let k^{-1} be such that $k\circ k^{-1}(\alpha,\beta,t) = \alpha e_1+\beta e_2$. Define

$$\zeta = \xi\circ k^{-1} = \mathfrak{x}e_1+\mathfrak{y}e_2+\mathfrak{z}e_3,\quad u = \xi_t\circ k^{-1},\quad\text{and}\quad w = \xi_{tt}\circ k^{-1}. \tag{0.26}$$

Let

$$b = k_t\circ k^{-1}\quad A\circ ke_3 = \mathfrak{a}J(k)e_3 = \mathfrak{a}k_\alpha\times k_\beta,\quad\text{and}\quad \mathcal{N} = \zeta_\alpha\times\zeta_\beta. \tag{0.27}$$

By a simple application of the chain rule, we have

$$U_k^{-1}\partial_tU_k = \partial_t + b\cdot\nabla_\perp,\quad\text{and}\quad U_k^{-1}(\mathfrak{a}N\times\nabla)U_k = A\mathcal{N}\times\nabla = A(\zeta_\beta\partial_\alpha-\zeta_\alpha\partial_\beta), \tag{0.28}$$

and $U_k^{-1}\mathfrak{H}U_k = \mathcal{H}$, with

$$\mathcal{H}f(\alpha,\beta,t) = \iint K(\zeta(\alpha',\beta',t)-\zeta(\alpha,\beta,t))(\zeta'_{\alpha'}\times\zeta'_{\beta'})f(\alpha',\beta',t)\,d\alpha'\,d\beta' \tag{0.29}$$

Let $\chi = \pi\circ k^{-1}$. Applying coordinate change U_k^{-1} to equation (0.20). We get

$$\begin{aligned}((\partial_t+b\cdot\nabla_\perp)^2 - A\mathcal{N}\times\nabla)\chi &= \iint K(\zeta'-\zeta)\,(u-u')\times(\zeta'_{\beta'}\partial_{\alpha'}-\zeta'_{\alpha'}\partial_{\beta'})\overline{u'}\,d\alpha'd\beta'\\&\quad-\iint K(\zeta'-\zeta)\,(u-u')\times(u'_{\beta'}\partial_{\alpha'}-u'_{\alpha'}\partial_{\beta'})\mathfrak{z}'\,d\alpha'd\beta'e_3\\&\quad-\iint((u'-u)\cdot\nabla)K(\zeta'-\zeta)\,(u-u')\times(\zeta'_{\beta'}\partial_{\alpha'}-\zeta'_{\alpha'}\partial_{\beta'})\mathfrak{z}'\,d\alpha'd\beta'e_3\end{aligned} \tag{0.30}$$

We show in the following proposition that b, $A-1$ are consisting of only quadratic and higher order terms. Let $\mathcal{K} = \operatorname{Re}\mathcal{H} = U_k^{-1}\mathfrak{K}U_k$, $P = \alpha e_1+\beta e_2$, and

$$\Lambda^* = (I+\mathfrak{H})ze_3,\quad \Lambda = (I+\mathfrak{H})ze_3-\mathfrak{K}ze_3,\quad \lambda^* = (I+\mathcal{H})\mathfrak{z}e_3,\quad \lambda = \lambda^*-\mathcal{K}\mathfrak{z}e_3 \tag{0.31}$$

Therefore

$$\zeta = P+\lambda. \tag{0.32}$$

Let the velocity $u = u_1 e_1 + u_2 e_2 + u_3 e_3$.

Proposition 0.4. *Let $b = k_t \circ k^{-1}$ and $A \circ k = \mathfrak{a} J(k)$. We have*

$$b = \frac{1}{2}(\mathcal{H} - \overline{\mathcal{H}})\overline{u} - [\partial_t + b \cdot \nabla_\perp, \mathcal{H}]\mathfrak{z} e_3 + [\partial_t + b \cdot \nabla_\perp, \mathcal{K}]\mathfrak{z} e_3 + \mathcal{K} u_3 e_3 \tag{0.33}$$

$$\begin{aligned}(A-1)e_3 = {} & \frac{1}{2}(-\mathcal{H} + \overline{\mathcal{H}})\overline{w} + \frac{1}{2}([\partial_t + b \cdot \nabla_\perp, \mathcal{H}]u - \overline{[\partial_t + b \cdot \nabla_\perp, \mathcal{H}]u}) \\ & + [A\mathcal{N} \times \nabla, \mathcal{H}]\mathfrak{z} e_3 - A\zeta_\beta \times (\partial_\alpha \mathcal{K}\mathfrak{z} e_3) + A\zeta_\alpha \times (\partial_\beta \mathcal{K}\mathfrak{z} e_3) + A\partial_\alpha \lambda \times \partial_\beta \lambda\end{aligned} \tag{0.34}$$

Here $\overline{\mathcal{H}} f = e_3 \mathcal{H}(e_3 f) = \iint e_3 K \mathcal{N}' e_3 f'$.

Proof. Taking derivative to t to (0.23), we get

$$\begin{aligned}k_t &= \xi_t - \partial_t(I + \mathfrak{H}) z e_3 + \partial_t \mathfrak{K} z e_3 \\ &= \xi_t - z_t e_3 - \mathfrak{H} z_t e_3 - [\partial_t, \mathfrak{H}] z e_3 + \partial_t \mathfrak{K} z e_3\end{aligned} \tag{0.35}$$

Now

$$\xi_t - z_t e_3 - \mathfrak{H} z_t e_3 = \frac{1}{2}(\xi_t + \overline{\xi}_t) - \frac{1}{2}\mathfrak{H}(\xi_t - \overline{\xi}_t) = \frac{1}{2}\overline{\xi}_t + \frac{1}{2}\mathfrak{H}\overline{\xi}_t = \frac{1}{2}(\mathfrak{H} - \overline{\mathfrak{H}})\overline{\xi}_t \tag{0.36}$$

Combining (0.35), (0.36) we get

$$k_t = \frac{1}{2}(\mathfrak{H} - \overline{\mathfrak{H}})\overline{\xi}_t - [\partial_t, \mathfrak{H}] z e_3 + [\partial_t, \mathfrak{K}] z e_3 + \mathfrak{K} z_t e_3 \tag{0.37}$$

Making the change of coordinate U_k^{-1}, we get (0.33).

Notice that $A \circ k e_3 = \mathfrak{a} k_\alpha \times k_\beta$. From the definition $k = \xi - \Lambda^* + \mathfrak{K} z e_3 = \xi - \Lambda$, we get

$$\begin{aligned}k_\alpha \times k_\beta = {} & \xi_\alpha \times \xi_\beta + \xi_\beta \times \partial_\alpha \Lambda^* - \xi_\alpha \times \partial_\beta \Lambda^* \\ & - \xi_\beta \times (\partial_\alpha \mathfrak{K} z e_3) + \xi_\alpha \times (\partial_\beta \mathfrak{K} z e_3) + \partial_\alpha \Lambda \times \partial_\beta \Lambda\end{aligned}$$

Using (0.25) and (0.12), we have

$$\xi_\beta \times \partial_\alpha \Lambda^* - \xi_\alpha \times \partial_\beta \Lambda^* = \xi_\beta \partial_\alpha \Lambda^* - \xi_\alpha \partial_\beta \Lambda^* = (N \times \nabla)\Lambda^*$$

From (0.18), and the fact that $\mathfrak{a} N \times \nabla z e_3 = -\mathfrak{a} N_1 e_1 - \mathfrak{a} N_2 e_2$, we obtain

$$\begin{aligned}\mathfrak{a}\xi_\alpha \times \xi_\beta + \mathfrak{a}(N \times \nabla)\Lambda^* &= \xi_{tt} + e_3 + (I + \mathfrak{H})(\mathfrak{a} N \times \nabla) z e_3 + [\mathfrak{a} N \times \nabla, \mathfrak{H}] z e_3 \\ &= \xi_{tt} + e_3 - \frac{1}{2}(I + \mathfrak{H})(\xi_{tt} + \overline{\xi}_{tt}) + [\mathfrak{a} N \times \nabla, \mathfrak{H}] z e_3\end{aligned}$$

and furthermore from (0.19),

$$\begin{aligned}&\xi_{tt} - \frac{1}{2}(I + \mathfrak{H})(\xi_{tt} + \overline{\xi}_{tt}) = \frac{1}{2}(\xi_{tt} - \mathfrak{H}\xi_{tt}) - \frac{1}{2}(\overline{\xi}_{tt} + \mathfrak{H}\overline{\xi}_{tt}) \\ &= \frac{1}{2}[\partial_t, \mathfrak{H}]\xi_t - \frac{1}{2}(\overline{\xi}_{tt} - \overline{\mathfrak{H}\xi_{tt}}) - \frac{1}{2}(\mathfrak{H}\overline{\xi}_{tt} + \overline{\mathfrak{H}\xi_{tt}}) = \frac{1}{2}[\partial_t, \mathfrak{H}]\xi_t - \frac{1}{2}\overline{[\partial_t, \mathfrak{H}]\xi_t} + \frac{1}{2}(\overline{\mathfrak{H}} - \mathfrak{H})\overline{\xi}_{tt}\end{aligned}$$

Combining the above calculations and make the change of coordinates U_k^{-1}, we obtain (0.34).

□

From Proposition 0.4, we see that b and $A-1$ are consisting of terms of quadratic and higher orders. Therefore the left hand side of equation (0.30) is

$$(\partial_t^2 - e_2\partial_\alpha + e_1\partial_\beta)\chi - \partial_\beta\lambda\partial_\alpha\chi + \partial_\alpha\lambda\partial_\beta\chi + \text{cubic and higher order terms}$$

The quadratic term $\partial_\beta\lambda\partial_\alpha\chi - \partial_\alpha\lambda\partial_\beta\chi$ is new in 3D. We notice that this is one of the null forms studied in [13] and we find that it is also null for our equation and can be written as the factor $1/t$ times a quadratic expression involving some "invariant vector fields" for $\partial_t^2 - e_2\partial_\alpha + e_1\partial_\beta$. Therefore this term is cubic in nature and equation (0.30) is of the type "linear + cubic and higher order perturbations".

We prove the global in time wellposedness of (0.1) by applying the method of invariant vector fields to (0.30). We note that it is more natural to treat $(\partial_t + b\cdot\nabla_\perp)^2 - A\mathcal{N}\times\nabla$ as the main operator for the water wave equation than treating it as a perturbation of the linear operator $\partial_t^2 - e_2\partial_\alpha + e_1\partial_\beta$. We obtain a uniform bound for all time of a properly constructed energy that involves invariant vector fields of $\partial_t^2 - e_2\partial_\alpha + e_1\partial_\beta$ by combining energy estimates for the equation (0.30) and a generalized Sobolev inequality that gives a $L^2 \to L^\infty$ estimate with the decay rate $1/t$. We point out that not only does the projection $(I-\mathfrak{H})$ give us the quantity $(I-\mathfrak{H})ze_3$, but it is also used in various ways to project away "quadratic noises" in the course of deriving the energy estimates. The global in time existence follows from a local well-posedness result, the uniform boundedness of the energy and a continuity argument. We state our main theorem.

Let $|D| = \sqrt{-\partial_\alpha^2 - \partial_\beta^2}$, $H^s(\mathbb{R}^2) = \{f \,|\, (I+|D|)^s f \in L^2(\mathbb{R}^2)\}$, with $\|f\|_{H^s} = \|f\|_{H^s(\mathbb{R}^2)} = \|(I+|D|)^s f\|_{L^2(\mathbb{R}^2)}$.

Let $s \geq 27$, $\max\{[\frac{s}{2}]+1, 17\} \leq l \leq s-10$. Assume that initially

$$\xi(\alpha,\beta,0) = \xi^0 = (\alpha,\beta,z^0(\alpha,\beta)), \quad \xi_t(\alpha,\beta,0) = \mathfrak{u}^0(\alpha,\beta), \quad \xi_{tt}(\alpha,\beta,0) = \mathfrak{w}^0(\alpha,\beta), \quad (0.38)$$

and the data in (0.38) satisfy the compatibility condition (5.29)-(5.30) of [21]. Let $\Gamma = \partial_\alpha,\ \partial_\beta,\ \alpha\partial_\alpha + \beta\partial_\beta,\ \alpha\partial_\beta - \beta\partial_\alpha$. Assume that

$$\sum_{\substack{|j|\leq s-1\\ \partial=\partial_\alpha,\partial_\beta}} \|\Gamma^j|D|^{1/2}z^0\|_{L^2(\mathbb{R}^2)} + \|\Gamma^j\partial z^0\|_{H^{1/2}(\mathbb{R}^2)} + \|\Gamma^j\mathfrak{u}^0\|_{H^{3/2}(\mathbb{R}^2)} + \|\Gamma^j\mathfrak{w}^0\|_{H^1(\mathbb{R}^2)} < \infty \quad (0.39)$$

Let

$$\epsilon = \sum_{\substack{|j|\leq l+3\\ \partial=\partial_\alpha,\partial_\beta}} \|\Gamma^j|D|^{1/2}z^0\|_{L^2(\mathbb{R}^2)} + \|\Gamma^j\partial z^0\|_{L^2(\mathbb{R}^2)} + \|\Gamma^j\mathfrak{u}^0\|_{H^{1/2}(\mathbb{R}^2)} + \|\Gamma^j\mathfrak{w}^0\|_{L^2(\mathbb{R}^2)}. \quad (0.40)$$

Theorem 0.5 (Main Theorem)**.** *There exists $\epsilon_0 > 0$, such that for $0 \leq \epsilon \leq \epsilon_0$, the initial value problem* (0.18)-(0.19)-(0.38) *has a unique classical solution globally in time. For each time $0 \leq t < \infty$, the interface is a graph, the solution has the same regularity as the initial*

data and remains small. Moreover the L^∞ norm of the steepness, the acceleration of the interface, and the derivative of the velocity on the interface decay at the rate $\frac{1}{t}$.

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ON DE GIORGI'S CONJECTURE AND BEYOND

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1. Introduction

The Allen-Cahn equation in $\mathbb{R}^N$ is the semilinear elliptic problem

$$\Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^N . \tag{1.1}$$

Originally formulated in the description of bi-phase separation in fluids [5] and ordering in binary alloys [4], Equation (1.1) has received extensive mathematical study. It is a prototype for the modeling of phase transition phenomena in a variety of contexts.

Introducing a small positive parameter ε and writing $v(x) := u(\varepsilon^{-1}x)$, we get the scaled version of (1.1),

$$\varepsilon^2\Delta v + v - v^3 = 0 \quad \text{in } \mathbb{R}^N . \tag{1.2}$$

On every bounded domain $\Omega \subset \mathbb{R}^N$, (1.1) is the Euler-Lagrange equation for the action functional

$$J_\varepsilon(v) = \int_\Omega \frac{\varepsilon}{2}|\nabla v|^2 + \frac{1}{4\varepsilon}(1-v^2)^2.$$

We observe that the constant functions $v = \pm 1$ minimize J_ε. They are idealized as two *stable phases* of a material in Ω. It is of interest to analyze configurations in which the two phases coexist. These states are represented by stationary points of J_ε, or solutions v_ε of Equation (1.2), that take values close to $+1$ in a subregion of Ω of and -1 in its complement. The theory of Γ-convergence developed in the 70s and 80s, showed a deep connection between this problem and the theory of minimal surfaces, see Modica, Mortola, Kohn, Sternberg, [32, 35, 36, 37, 49]. In fact, it is known that for a family u_ε of local minimizers of u_ε with uniformly bounded energy must converge, up to subsequences, in L^1-sense to a function of the form $\chi_E - \chi_{E^c}$ where χ denotes characteristic function, and ∂E has minimal perimeter. Thus the interface between the stable *phases* $u = 1$ and $u = -1$, represented by the sets $[u_\varepsilon = \lambda]$ with $|\lambda| < 1$ approach a minimal hypersurface, see Caffarelli and Córdoba [9, 10], Hutchinson and Tonegawa [31], Röger and Tonegawa [43] for stronger convergence and uniform regularity results on these level surfaces.

1.1. Formal asymptotic behavior of v_ε.

Let us argue formally to obtain an idea on how a solution v_ε of Equation (1.2) with uniformly bounded energy should look like near a limiting interface Γ. Let us assume that Γ is a smooth hypersurface and let ν designate a choice of its unit normal. Points δ-close to Γ can be uniquely represented as

$$x = y + z\nu(y), \quad y \in \Gamma,\ |z| < \delta. \tag{1.3}$$

A well known formula for the Laplacian in these coordinates reads as follows:

$$\Delta_x = \partial_{zz} + \Delta_{\Gamma^z} - H_{\Gamma_z}\partial_z. \tag{1.4}$$

Here

$$\Gamma^z := \{y + z\nu(y) \ / \ y \in \Gamma\}.$$

Δ_{Γ^z} is the Laplace-Beltrami operator on Γ^z acting on functions of the variable y, and H_{Γ^z} is its mean curvature. Let $k_1, \ldots, k_N$ denote the principal curvatures of Γ. Then we have the validity of the expression

$$H_{\Gamma^z} = \sum_{i=1}^{N} \frac{k_i}{1 - zk_i}. \tag{1.5}$$

It is reasonable to assume that the solution is a smooth function of the variables (y, ζ), where $\zeta = \varepsilon^{-1} z$, and the equation for $v_\varepsilon(y, \zeta)$ reads

$$\begin{aligned} &\varepsilon^2 \Delta_{\Gamma^{\varepsilon\zeta}} v_\varepsilon - \varepsilon H_{\Gamma^{\varepsilon\zeta}}(y)\, \partial_\zeta v_\varepsilon + \\ &\qquad \partial_\zeta^2 v_\varepsilon + v_\varepsilon - v_\varepsilon^3 = 0, \quad y \in \Gamma, \quad |\zeta| < \delta\varepsilon^{-1}. \end{aligned} \tag{1.6}$$

We shall make two assumptions:

1. The zero-level set of v_ε lies within a $O(\varepsilon^2)$-neighborhood of Γ, that is in the region $|\zeta| = O(\varepsilon)$ and $\partial_\zeta v_\varepsilon > 0$ along this nodal set, and

2. $v_\varepsilon(y, \zeta)$ can be expanded in powers of ε as

$$v_\varepsilon(y, \zeta) = v_0(y, \zeta) + \varepsilon v_1(y, \zeta) + \varepsilon^2 v_2(y, \zeta) + \cdots, \tag{1.7}$$

where v_j are smooth and uniformly bounded together with their derivatives. We observe also that

$$\int_\Gamma \int_{-\delta/\varepsilon}^{\delta/\varepsilon} [\, \frac{1}{2} |\partial_\zeta v_\varepsilon|^2 + \frac{1}{4}(1 - v_\varepsilon^2)^2 \,]\, d\zeta\, d\sigma(y) \ \le \ J_\varepsilon(v_\varepsilon) \le C. \tag{1.8}$$

Substituting Expression (1.7) in Equation (1.6), using the first assumption, and letting $\varepsilon \to 0$, we get

$$\begin{aligned} \partial_\zeta^2 v_0 + v_0 - v_0^3 = 0, \quad (y, \zeta) \in \Gamma \times \mathbb{R}, \\ v_0(0, y) = 0, \quad \partial_\zeta(0, y) \ge 0, \quad y \in \Gamma, \end{aligned} \tag{1.9}$$

while from (1.8) we get

$$\int_{\mathbb{R}} [\, \frac{1}{2} |\partial_\zeta v_0|^2 + \frac{1}{4}(1 - v_0^2)^2 \,]\, d\zeta \ < \ +\infty. \tag{1.10}$$

Conditions (1.10) and (1.9) force $v_0(y, \zeta) = w(\zeta)$ where w is the unique solution of the ordinary differential equation

$$w'' + w - w^3 = 0, \quad w(0) = 0, \quad w(\pm\infty) = \pm 1,$$

which is given explicitly by

$$w(\zeta) \ := \ \tanh(\zeta/\sqrt{2}). \tag{1.11}$$

On the other hand, substitution yields that $v_1(y, \zeta)$ satisfies

$$\partial_\zeta^2 v_1 + (1 - 3w(\zeta)^2) v_1 \ = \ H_\Gamma(y)\, w'(\zeta), \quad \zeta \in (-\infty, \infty). \tag{1.12}$$

Testing this equation against $w'(\zeta)$ and integrating by parts in ζ we get the relation

$$H_\Gamma(y) = 0 \quad \text{for all} \quad y \in \Gamma$$

which tells us precisely that Γ must be a minimal surface, as expected. Hence, we get $v_1 = -h_0(y)w'(\zeta)$ for a certain function $h_0(y)$. As a conclusion, from (1.7) and a Taylor expansion, we can write

$$v_\varepsilon(y,\zeta) = w(\zeta - \varepsilon h_0(y)) + \varepsilon^2 v_2 + \cdots$$

It is convenient to write this expansion in terms of the variable $t = \zeta - \varepsilon h_0(y)$ in the form

$$v_\varepsilon(y,\zeta) = w(t) + \varepsilon^2 v_2(t,y) + \varepsilon^3 v_3(t,y) + \cdots \tag{1.13}$$

Using expression (1.5) and the fact that Γ is a minimal surface, we expand

$$H_{\Gamma_{\varepsilon\zeta}}(y) = \varepsilon^2 \zeta\, |A_\Gamma(y)|^2 + \varepsilon^3 \zeta^2\, H_3(y) + \cdots$$

where

$$|A_\Gamma|^2 = \sum_{i=1}^{8} k_i^2, \quad H_3 = \sum_{i=1}^{8} k_i^3.$$

Thus setting $t = \zeta - \varepsilon h_0(y)$ and using (1.13), we compute

$$\begin{aligned} 0 = \Delta v_\varepsilon + v_\varepsilon + v_\varepsilon^3 = {} & [\partial_t^2 + (1 - 3w(t)^2)]\,(\varepsilon^2 v_2 + \varepsilon^3 v_3) \\ -w'(t)\,[\varepsilon^3 \Delta_\Gamma h_0 \, & + \varepsilon^3 H_3\, t^2 + \varepsilon^2 |A_\Gamma|^2\,(t + \varepsilon h_0)\,] + O(\varepsilon^4). \end{aligned}$$

And then letting $\varepsilon \to 0$ we arrive to the equations

$$\partial_t^2 v_2 + (1 - 3w^2) v_2 = |A_\Gamma|^2\, t w', \tag{1.14}$$

$$\partial_t^2 v_3 + (1 - 3w^2) v_3 = [\Delta_\Gamma h_0 + |A_\Gamma|^2 h_0 + H_3\, t^2]\, w'. \tag{1.15}$$

Equation (1.14) has a bounded solution since $\int_{\mathbb{R}} t w'(t)^2\, dt = 0$. Instead the bounded solvability of (1.15) is obtained if and only if h_0 solves the following elliptic equation in Γ

$$\mathcal{J}_\Gamma[h_0](y) \,:=\, \Delta_\Gamma h_0 + |A_\Gamma|^2 h_0 \,=\, c_0 \sum_{i=1}^{8} k_i^3 \quad \text{in } \Gamma, \tag{1.16}$$

where $c_0 = -\int_{\mathbb{R}} t^2 w'^2\, dt / \int_{\mathbb{R}} w'^2\, dt$. $\mathcal{J}_\Gamma$ is by definition the *Jacobi operator* of the minimal surface Γ.

This talk deals with the problem of constructing entire solutions of Equation (1.2), that exhibit the asymptotic behavior described above, around a given, fixed minimal hypersurface Γ that splits the space $\mathbb{R}^N$ into two components, and for which the coordinates (1.3) are defined for some uniform $\delta > 0$. A key element for such a construction is precisely the question of solvability of Equation (1.16), that determines at main order the deviation of the nodal set of the solution from Γ.

To put the above in terms of the original problem (1.1), we consider a fixed minimal surface $\Gamma \in \mathbb{R}^N$ together with its image by a dilation:

$$\Gamma_\varepsilon \,:=\, \varepsilon^{-1}\Gamma.$$

We want to find an entire solution u_ε to problem (1.1) such that for a function h_ε defined on Γ with

$$\sup_{\varepsilon > 0} ||h_\varepsilon||_{L^\infty(\Gamma)} \,<\, +\infty, \tag{1.17}$$

we have

$$u_\varepsilon(x) = w(\zeta - \varepsilon h_\varepsilon(\varepsilon y)) + O(\varepsilon^2), \tag{1.18}$$

uniformly for

$$x = y + \zeta\nu(\varepsilon y), \quad |\zeta| \le \frac{\delta}{\varepsilon}, \quad y \in \Gamma_\varepsilon,$$

while

$$|u_\varepsilon(x)| \to 1 \quad \text{as dist}\,(x, \Gamma_\varepsilon) \to +\infty. \tag{1.19}$$

In what remains of this talk we shall answer affirmatively this question for some important examples of minimal surfaces. One of them is a non-hyperplanar minimal graph in $\mathbb{R}^9$. In this case the solution of (1.1) is a counterexample to a famous conjecture due to Ennio De Giorgi [15]. As another example, in $\mathbb{R}^3$ we find entire solutions of (1.1) with finite Morse index. Our results suggest extensions of De Giorgi's conjecture for solutions of (1.1) which parallel known classification results for minimal surfaces.

2. From Bernstein's to De Giorgi's conjecture

Ennio De Giorgi [15] formulated in 1978 the following celebrated conjecture concerning entire solutions of equation (1.1).

De Giorgi's Conjecture: *Let u be a bounded solution of equation* (1.1) *such that $\partial_{x_N} u > 0$. Then the level sets $[u = \lambda]$ are all hyperplanes,* **at least** *for dimension $N \le 8$.*

Equivalently, u must depend only on one Euclidean variable so that it must have the form $u(x) = w((x-p)\cdot\nu)$ for some $p \in \mathbb{R}^N$ and some ν with $|\nu| = 1$ and $\nu_N > 0$.

The condition $\partial_{x_N} u > 0$ implies that the level sets of u are all graphs of functions of the first $N-1$ variables. As we have discussed in the previous section, level sets of non-constant solutions are closely connected to minimal hypersurfaces. De Giorgi's conjecture is in fact a parallel to the following classical statement.

Bernstein's conjecture: *A minimal hypersurface in $\mathbb{R}^N$, which is also the graph of a smooth entire function of $N-1$ variables, must be a hyperplane.*

In other words, if Γ is an *entire minimal graph*, namely

$$\Gamma = \{(x', x_N) \mid x' \in \mathbb{R}^{N-1},\ x_N = F(x')\} \tag{2.20}$$

where F solves the minimal surface equation

$$H_\Gamma \equiv \nabla \cdot \left(\frac{\nabla F}{\sqrt{1 + |\nabla F|^2}} \right) = 0 \quad \text{in } \mathbb{R}^{N-1}, \tag{2.21}$$

then Γ must be a hyperplane, hence F must be a linear affine function.

Bernstein's conjecture is known to be true up to dimension $N = 8$, see Simons [48] and references therein, while it is *false* for $N \ge 9$, as proven by Bombieri, De Giorgi and Giusti [7], who found a nontrivial solution to Equation (2.21). To explain the idea of their construction, let us write $x' \in \mathbb{R}^8$ as $x' = (\mathbf{u}, \mathbf{v}) \in \mathbb{R}^4 \times \mathbb{R}^4$ and consider the set

$$T := \{(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^8 \mid |\mathbf{v}| > |\mathbf{u}| > 0\}. \tag{2.22}$$

The set $\{|\mathbf{u}| = |\mathbf{v}|\} \in \mathbb{R}^8$ is Simons' minimal cone [48]. The solution found in [7] is radially symmetric in both variables, namely $F = F(|\mathbf{u}|, |\mathbf{v}|)$. In addition, F is positive in T and it vanishes along Simons' cone. Moreover, it satisfies

$$F(|\mathbf{u}|, |\mathbf{v}|) = -F(|\mathbf{v}|, |\mathbf{u}|)\ . \tag{2.23}$$

Let us write $(|\mathbf{u}|, |\mathbf{v}|) = (r\cos\theta, r\sin\theta)$. In [17] it is found that there is a function $g(\theta)$ with

$$g(\theta) > 0, \quad \text{in } (\pi/4, \pi/2), \quad g'(\pi/2) = 0 = g(\pi/4), \quad g'(\pi/4) > 0,$$

such that for some $\sigma > 0$,

$$F(|\mathbf{u}|, |\mathbf{v}|) \ = \ g(\theta)\, r^3 + O(r^{-\sigma}) \quad \text{in } T. \tag{2.24}$$

More importantly this asymptotic formula is correct (with obvious adjustments) for the derivatives of F. This nontrivial refinement of the result in [7] relies on a theorem of Simon [47] and a construction of suitable sub/sup-solutions for the mean curvature operator (2.21).

De Giorgi's conjecture has been established for $N = 2$ by Ghoussoub and Gui [24] and for $N = 3$ by Ambrosio and Cabré [6]. Savin [45] proved its validity for $4 \le N \le 8$ under the additional assumption

$$\lim_{x_N \to \pm\infty} u(x', x_N) = \pm 1 \quad \text{for all} \quad x' \in \mathbb{R}^{N-1}. \tag{2.25}$$

The following result shows that De Giorgi's caveat was justified since the conjecture fails for $N \ge 9$.

Theorem 1. *(del Pino-Kowalczyk-Wei* [17]*) Let $N \ge 9$. Then there is an entire minimal graph Γ which is not a hyperplane, such that all $\varepsilon > 0$ sufficiently small there exists a bounded solution $u_\varepsilon(x)$ of equation* (1.1) *that satisfies properties* (1.17)-(1.19)*. Besides, $\partial_{x_N} u_\varepsilon > 0$ and u_ε satisfies condition* (2.25)*.*

A counterexample to De Giorgi's conjecture in dimension $N \ge 9$ was believed to exist for a long time. Partial progress in this direction was made by Jerison and Monneau [30] and by Cabré and Terra [8]. See also the survey article by Farina and Valdinoci [22].

2.1. **Outline of the proof.** To begin with we observe that a counterexample in dimension $N = 9$ automatically provides one in all dimensions. Thus in what follows we will assume $N = 9$. For a small $\varepsilon > 0$ we look for a solution u_ε of the form (near Γ_ε),

$$u_\varepsilon(x) = w(\zeta - \varepsilon h(\varepsilon y)) + \phi(\zeta - \varepsilon h(\varepsilon y), y), \quad x = y + \zeta\nu(\varepsilon y), \tag{2.26}$$

where $y \in \Gamma_\varepsilon$, ν is a unit normal to Γ with $\nu_N > 0$, h is a function defined on Γ, which is left as a parameter to be adjusted. Setting $r(y', y_9) = |y'|$, we assume a priori in h that

$$\|(1 + r^2) D_\Gamma h\|_{L^\infty(\Gamma)} + \|(1 + r) h\|_{L^\infty(\Gamma)} \ \le \ M \tag{2.27}$$

for some large, fixed number M, also with a uniform control on $(1 + r^3) D^2_\Gamma h$. In addition, because of (2.23) it is natural to assume that u_ε and h satisfy similar symmetries, consistent with those of the minimal graph, namely:

$$u_\varepsilon(\mathbf{u}, \mathbf{v}, x_9) = -u_\varepsilon(P\mathbf{v}, Q\mathbf{u}, -x_9), \quad h(\mathbf{u}, \mathbf{v}) = -h(P\mathbf{v}, Q\mathbf{u}), \tag{2.28}$$

where P and Q are orthogonal transformations of $\mathbb{R}^4$. Most of our argument does not in fact depend on (2.28) and the significance of this assumption becomes apparent only at the end of the construction.

Letting $f(u) = u - u^3$ and using Expression (1.4) for the Laplacian, the equation becomes

$$\begin{aligned} S(u_\varepsilon) := \Delta u_\varepsilon + f(u_\varepsilon) \; = \\ \Delta_{\Gamma_\varepsilon^\zeta} u_\varepsilon - \varepsilon H_{\Gamma_\varepsilon^\zeta}(\varepsilon y)\, \partial_\zeta u_\varepsilon \; + \\ \partial_\zeta^2 u_\varepsilon + f(u_\varepsilon) \; = 0, \quad y \in \Gamma_\varepsilon, \; |\zeta| < \delta/\varepsilon. \end{aligned} \tag{2.29}$$

Letting $t = \zeta - \varepsilon h(\varepsilon y)$, we look for u_ε of the form

$$u_\varepsilon(t, y) = w(t) + \phi(t, y)$$

for a small function ϕ. The equation in terms of ϕ becomes

$$\partial_t^2 \phi + \Delta_{\Gamma_\varepsilon} \phi + B\phi + f'(w(t))\phi + N(\phi) + E \; = \; 0 \tag{2.30}$$

where B is a small linear second order differential operator, and

$$E = S(w(t)), \quad N(\phi) = f(w + \phi) - f(w) - f'(w)\phi \approx f''(w)\phi^2.$$

While the expression (2.30) makes sense only for $|t| < \delta\varepsilon^{-1}$, it turns out that the equation in the entire space can be reduced to one similar to (2.30) in entire $\mathbb{R} \times \Gamma_\varepsilon$, where E and the undefined coefficients in B are just cut-off far away, while the operator N is slightly modified by the addition of a small nonlinear, nonlocal operator of ϕ. Rather than solving this problem directly we carry out an infinite dimensional form of Lyapunov-Schmidt reduction, considering a projected version of it,

$$\begin{aligned} \partial_t^2 \phi + \Delta_{\Gamma_\varepsilon} \phi + B\phi + f'(w(t))\phi + N(\phi) + E \; = \\ c(y) w'(t) \quad \text{in } \mathbb{R} \times \Gamma_\varepsilon, \\ \int_{\mathbb{R}} \phi(t, y) w'(t)\, dt \; = \; 0 \quad \text{for all} \quad y \in \Gamma_\varepsilon. \end{aligned} \tag{2.31}$$

The error of approximation E has roughly speaking a bound $O(\varepsilon^2 r(\varepsilon y)^{-2} e^{-\sigma|t|})$, and it turns out that one can find a solution $\phi = \Phi(h)$ to problem (2.31) with the same bound. We then get a solution to our original problem if h is such that $c(y) \equiv 0$. Thus the problem is reduced to finding h such that

$$c(y) \int_{\mathbb{R}} w'^2 = \int_{\mathbb{R}} (E + B\Phi(h) + N(\Phi(h)))\, w' dt \equiv 0.$$

A computation similar to that in the formal derivation yields that this problem is equivalent to a small perturbation of Equation (1.16)

$$\mathcal{J}_\Gamma(h) := \Delta_\Gamma h + |A_\Gamma|^2 h \; = \; c_0 \sum_{i=1}^{8} k_i^3 + \mathcal{N}(h) \quad \text{in } \Gamma, \tag{2.32}$$

where $\mathcal{N}(h)$ is a small operator. ¿From an estimate by Simon [47] we know that $k_i = O(r^{-1})$. Hence $H_3 := \sum_{i=1}^8 k_i^3 = O(r^{-3})$. A central point is to show that the unperturbed equation (1.16) has a solution $h = O(r^{-1})$, which justifies a posteriori the assumption (2.27) made originally on h. This step uses the asymptotic expression (2.24). The symmetries of the solution (2.28) allow to reduce the domain of the problem and we end up solving it in the sector T (2.22) with zero Dirichlet

boundary conditions on Simons' cone. From (2.24) we have that $H_3 = O(g(\theta)r^{-3})$ and we get a priori estimates for the equation $\mathcal{J}_\Gamma(h) = O(g(\theta)r^{-3})$ by constructing a positive barrier of size $O(r^{-1})$. The operator $\mathcal{J}_\Gamma$ satisfies maximum principle and existence thus follows. The full nonlinear equation is then solved with the aid of contraction mapping principle. The detailed proof of this theorem is contained in [17].

The program towards the counterexample in [30] and [6] mimics the classical program that lead to the proof of Bernstein's conjecture: the existence of the counterexample is reduced to establishing the minimizing character of a *saddle solution* in $\mathbb{R}^8$ that vanishes on Simon's cone. Our approach of direct construction is actually applicable to build solutions, which may be in principle unstable, associated to general minimal surfaces, as we illustrate in the next section. We should mention that method of infinite dimensional reduction for the Allen Cahn equation in compact settings has precedents with similar flavor in [3], [41], [33], [16]. Using variational approach, local minimizers were built in [32].

3. Generalized De Giorgi Conjecture: Stable Solutions

The assumption of monotonicity in one direction for the solution u in De Giorgi's conjecture implies a form of stability, locally minimizing character for u when compactly supported perturbations are considered in the energy. Indeed, the linearized operator $L = \Delta + (1 - 3u^2)$, satisfies maximum principle since $L(Z) = 0$ for $Z = \partial_{x_N} u > 0$. This implies stability of u, in the sense that its associated quadratic form, namely the second variation of the corresponding energy,

$$\mathcal{Q}(\psi, \psi) := \int_{\mathbb{R}^3} |\nabla \psi|^2 + (3u^2 - 1)\, \psi^2 \tag{3.33}$$

satisfies $\mathcal{Q}(\psi, \psi) > 0$ for all $\psi \neq 0$ smooth and compactly supported. Stability of u is sufficient for De Giorgi's statement to hold in dimension $N = 2$, as observed by Dancer [13] while it remains an open problem for $3 \leq N \leq 8$. The monotonicity assumption actually implies the globally minimizing character of the solution on each compact set, subject to its own boundary conditions, see [1].

Naturally, one would ask the following generalized De Giorgi Conjecture.

Generalized De Giorgi's Conjecture: *Let u be a bounded and* **stable** *solution of equation* (1.1). *Then the level sets $[u = \lambda]$ are all hyperplanes,* **at least** *for dimension $N \leq 7$*

The dimension 7 is again motivated by the study of minimal surface. The generalized De Giorgi's conjecture is in fact a parallel to the following classical statement.

Generalized Bernstein Theorem: *A stable minimal hypersurface must be a hyperplane.*

The stability conjecture for minimal surfaces is known to be true in dimension $N = 3$ by do Carmo and Peng [21], Fischer-Colbrie and Schoen [23], it is *false* for $N \geq 9$, as proven by Bombieri, De Giorgi and Giusti [7], who proved that there is a foliation of Simons's cone in dimension eight or higher. Yau [50] asked whether one can prove that a complete minimal hypersurface in $\mathbb{R}^{n+1}$ $(n \leq 7)$ is a hyperplane. Although much hard work on this problem has been done, it remains still open in dimensions $3 \leq n \leq 7$.

Using the foliation of the Simon's cone, the following theorem shows that the generalized De Giorgi Conjecture is not true in dimension 8 (and hence higher).

Theorem 2. *(Pacard-Wei* [42]*). Let $N = 8$. Then there exists a stable and bounded solution to (1.1) whose level sets approach one of the foliations of the Simons cone.*

4. Finite Morse index solutions in $\mathbb{R}^3$

The *Morse index* $m(u)$ is defined as the maximal dimension of a vector space E of compactly supported functions such that

$$\mathcal{Q}(\psi, \psi) < 0 \quad \text{for all} \quad \psi \in E \setminus \{0\}.$$

In view of the discussion so far, it seems natural to associate complete, embedded minimal surfaces Γ with finite Morse index, and solutions of (1.1). The *Morse index* of the minimal surface Γ, $i(\Gamma)$, has a similar definition relative to the quadratic form for its Jacobi operator $\mathcal{J}_\Gamma := \Delta_\Gamma + |A_\Gamma|^2$. The number $i(\Gamma)$ is the largest dimension for a vector spaced E of compactly supported smooth functions in Γ with

$$\int_\Gamma |\nabla k|^2 \, dV - \int_\Gamma |A|^2 k^2 \, dV < 0 \quad \text{for all} \quad k \in E \setminus \{0\}.$$

We point out that for complete, embedded surfaces in $\mathbb{R}^3$, finite index is equivalent to *finite total curvature,* namely

$$\int_\Gamma |K| \, dV < +\infty$$

where K denotes Gauss curvature of the minimal surface, see §7 of [27] and references therein.

4.1. **Embedded minimal surfaces of finite total curvature.** The theory of embedded, minimal surfaces of finite total curvature in $\mathbb{R}^3$, has reached a notable development in the last 25 years. For more than a century, only two examples of such surfaces were known: the plane and the catenoid. The first nontrivial example was found in 1981 by C. Costa, [11, 12]. The *Costa surface* is a genus one complete and properly embedded minimal surface, which outside a large ball has exactly three components (its *ends*). The upper and the lower end are asymptotic to a catenoid, while the middle end is asymptotic to a plane perpendicular to the axis of the catenoid. The complete proof of embeddedness is due to Hoffman and Meeks [28]. In [29] these authors generalized notably Costa's example by exhibiting a class of three-end, embedded minimal surface, with the same asymptotic behavior the Costa surface far away, but with an array of tunnels connecting the upper and the lower end resulting in a surface with arbitrary genus $\ell \geq 1$. This is known as the Costa-Hoffman-Meeks surface with genus ℓ.

As a special case of the main results of [18] we have the following:

Theorem 3. *(del Pino-Kowalcyzk-Wei* [18]*) Let $\Gamma \subset \mathbb{R}^3$ be either a catenoid or a Costa-Hoffman-Meeks surface with genus $\ell \geq 1$. Then for all sufficiently small $\varepsilon > 0$ there exists a solution u_ε of Problem* (1.1) *with the properties* (1.17)-(1.19)*. In the case of the catenoid, the solution found is radially symmetric in two of its variables and $m(u_\varepsilon) = 1$. For the Costa-Hoffman-Meeks surface with genus $\ell \geq 1$, we have $m(u_\varepsilon) = 2\ell + 3$.*

4.2. **A general case.** In what follows Γ is a complete, embedded minimal surface in $\mathbb{R}^3$ with finite total curvature. Then Γ is orientable and the set $\mathbb{R}^3 \setminus \Gamma$ has exactly two components S_+, S_-, see [27]. In what follows we fix a continuous choice of unit normal field $\nu(y)$, which conventionally we take it to point towards S_+.

For $x = (x', x_3) \in \mathbb{R}^3$, we denote as before, $r = r(x) = |x'|$. It is known that after a suitable rotation of the coordinate axes, outside the infinite cylinder $r < R_0$ with sufficiently large radius R_0, Γ decomposes into a finite number m of unbounded components $\Gamma_1, \dots, \Gamma_m$, its *ends*. From a result in [46], we know that asymptotically each end of Γ_k either resembles a plane or a catenoid. More precisely, Γ_k can be represented as the graph of a function F_k of the first two variables,

$$\Gamma_k = \{\, y \in \mathbb{R}^3 \ / \ r(y) > R_0,\ y_3 = F_k(y') \,\}$$

where F_k is a smooth function which can be expanded as

$$F_k(y') = a_k \log r + b_k + \sum_{i=1}^{3} b_{ik} \frac{y_i}{r^2} + O(r^{-3}) \quad \text{as } r \to +\infty, \tag{4.34}$$

for certain constants a_k, b_k, b_{ik}, and this relation can also be differentiated. Here

$$a_1 \le a_2 \le \ldots \le a_m\ , \qquad \sum_{k=1}^{m} a_k \ = \ 0\,. \tag{4.35}$$

We say that Γ has *non-parallel ends* if all the above inequalities are strict.

Let us consider the Jacobi operator of Γ

$$\mathcal{J}_\Gamma(h) \ := \ \Delta_\Gamma h + |A_\Gamma|^2 h \tag{4.36}$$

where $|A_\Gamma|^2 = k_1^2 + k_2^2 = -2K$. A smooth function $z(y)$ defined on Γ is called a *Jacobi field* if $\mathcal{J}_\Gamma(z) = 0$. Rigid motions of the surface induce naturally some bounded Jacobi fields. For example there 4 obvious Jacobi fields associated, respectively, to translations along coordinates axes and rotation around the x_3-axis:

$$z_1(y) = \nu(y) \cdot e_i, \quad y \in \Gamma, \quad i = 1, 2, 3,$$

$$z_4(y) = (-y_2, y_1, 0) \cdot \nu(y), \quad y \in \Gamma. \tag{4.37}$$

We assume that Γ is *non-degenerate* in the sense that these functions are actually *all* the bounded Jacobi fields, namely

$$\{\, z \in L^\infty(\Gamma) \ / \ \mathcal{J}_\Gamma(z) = 0 \,\} \ = \ \text{span}\,\{\, z_1, z_2, z_3, z_4 \,\}\,. \tag{4.38}$$

This property is known in some important cases, most notably the catenoid and the Costa-Hoffmann-Meeks surface of any order $\ell \ge 1$. See Nayatani [39, 40] and Morabito [38].

Theorem 4. *(del Pino-Kowalcyzk-Wei* [18]*) Let $N = 3$ and Γ be a minimal surface embedded, complete with finite total curvature and non-parallel ends, which is in addition nondegenerate. Then for all sufficiently small $\varepsilon > 0$ there exists a solution u_ε of Problem* (1.1) *with the properties* (1.17)-(1.19)*. Moreover, we have*

$$m(u_\varepsilon) \ = \ i(\Gamma).$$

Besides, the solution is non-degenerate, in the sense that any bounded solution of

$$\Delta\phi + (1 - 3u_\varepsilon^2)\,\phi = 0 \quad \text{in } \mathbb{R}^3$$

must be a linear combination of the functions Z_i, $i = 1, 2, 3, 4$ *defined as*

$$Z_i = \partial_i u_\varepsilon, \quad i = 1, 2, 3, \quad Z_4 = -x_2 \partial_1 u_\varepsilon + x_1 \partial_2 u_\varepsilon.$$

It is well-known that if Γ is a catenoid then $i(\Gamma) = 1$. Moreover, in the Costa-Hoffmann-Meeks surface it is known that $i(\Gamma) = 2\ell + 3$ where ℓ is the genus of Γ. See [39, 40, 38].

4.3. **Further comments.** In analogy with De Giorgi's conjecture, it seems plausible that qualitative properties of embedded minimal surfaces with finite Morse index should hold for the level sets of finite Morse index solutions of Equation (1.1), provided that these sets are embedded manifolds outside a compact set. As a sample, one may ask if the following two statements are valid:

• *The level sets of any finite Morse index solution* u *of* (1.1) *in* $\mathbb{R}^3$, *such that* $\nabla u \neq 0$ *outside a compact set should have a finite, even number of catenoidal or planar ends with a common axis.*

The above fact does hold for minimal surfaces with finite total curvature and embedded ends as established by Ossermann and Schoen. On the other hand, the above statement should not hold true if the condition $\nabla u \neq 0$ outside a large ball is violated. For instance, let us consider the octant $\{x_1, x_2, x_3 \geq 0\}$. Problem (1.1) in the octant with zero boundary data can be solved by a super-subsolution scheme (similar to that in [14]) yielding a positive solution. Extending by successive odd reflections to the remaining octants, one generates an entire solution (likely to have finite Morse index), whose zero level set does not have the characteristics above: the condition $\nabla u \neq 0$ far away corresponds to *embeddedness of the ends of the level sets.*

An analog of De Giorgi's conjecture for the solutions that follow in complexity the stable ones, namely those with Morse index one, may be the following:

• *A bounded solution* u *of* (1.1) *in* $\mathbb{R}^3$ *with* $i(u) = 1$, *and* $\nabla u \neq 0$ *outside a bounded set, must be axially symmetric, namely radially symmetric in two variables.*

The solution we found, with transition on a dilated catenoid has this property. This statement would be in correspondence with results by Schoen [46] and López and Ros [34]: if $i(\Gamma) = 1$ and Γ has embedded ends, then it must be a catenoid.

5. The case of $\mathbb{R}^2$

5.1. **Solutions with multiply connected nodal set.** The only minimal surface Γ that we can consider in this case is a straight line, to which the planar solution depending on its normal variable can be associated.

A class of solutions to (1.1) with a *finite number of transition lines*, likely to have finite Morse index, has been recently built in [20]. The location and shape of these lines is governed by the *Toda system*, a classical integrable model for particles moving on a line with exponential forces between any two closest neighbors:

$$\frac{\sqrt{2}}{24} f_j'' = e^{-\sqrt{2}(f_j - f_{j-1})} - e^{-\sqrt{2}(f_{j+1} - f_j)}, \quad j = 1, \dots k. \tag{5.39}$$

For definiteness we take $f_0 \equiv -\infty$, $f_{k+1} \equiv +\infty$. It is known that for any given solution there exist numbers $a_j^{\pm}, b_j^{\pm}$ such that

$$f_j(z) = a_j^{\pm}|z| + b_j^{\pm} + O(e^{-|z|}) \quad \text{as } z \to \pm\infty, \tag{5.40}$$

where $a_j^{\pm} < a_{j+1}^{\pm}$, $j = 1, \dots, k-1$ (long-time scattering).

The role of this system in the construction of solutions with multiple transition lines in the Allen-Cahn equation in bounded domains was discovered in [16]. In entire space the following result holds.

Theorem 5. *(del Pino-Kowalczyk-Pacard-Wei* [20]*) Given a solution f of* (5.39) *if we scale*

$$f_{\varepsilon,j}(z) := \sqrt{2}(j - \frac{k+1}{2}) \log \frac{1}{\varepsilon} + f_j(\varepsilon z),$$

then for all small ε there is a solution u_ε with k transitions layers $\Gamma_{\varepsilon,j}$ near the lines $x_2 = f_{\varepsilon,j}(x_1)$. More precisely $\Gamma_{\varepsilon,j}$ are graphs of functions:

$$x_1 = f_{\varepsilon,j}(x_2) + h_{\varepsilon,j}(\varepsilon x_2),$$

where $h_{\varepsilon,j}(z) = O(\varepsilon^\alpha)(|z|+1)$, with some $\alpha > 0$. In addition

$$u_\varepsilon(x_1, x_2) = \sum_{j=1}^{k} (-1)^{j-1} w(x_1 - f_{\varepsilon,j}(x_2) - h_{\varepsilon,j}(\varepsilon x_2)) + \sigma_k + O(\varepsilon^\alpha), \tag{5.41}$$

where $\sigma_k = -\frac{1}{2}(1 + (-1)^k)$.

The transition lines are therefore nearly parallel and asymptotically straight, see (5.40). In particular, if $k = 2$ and f solves the ODE

$$\frac{\sqrt{2}}{24} f''(z) = e^{-2\sqrt{2}f(z)}, \quad f'(0) = 0,$$

and $f_\varepsilon(z) := \sqrt{2} \log \frac{1}{\varepsilon} + f(\varepsilon z)$, then there exists a solution u_ε to (1.1) in $\mathbb{R}^2$ with

$$u_\varepsilon(x_1, x_2) = w(x_1 + f_\varepsilon(x_2)) + w(x_1 - f_\varepsilon(x_2)) - 1 + O(\varepsilon^\alpha). \tag{5.42}$$

In general in the case of even solutions to the Toda system the deficiency functions $h_{\varepsilon,j}(z)$ decay exponentially as $|z| \to \infty$, c.f. [20].

5.2. **Remarks.** The solutions (5.41) show a major difference between the theory of minimal surfaces and the Allen-Cahn equation, as it is the fact that two separate interfaces *interact*, leading to a major deformation in their asymptotic shapes. We believe that these examples should be prototypical of bounded finite Morse index solutions of (1.1). A finite Morse index solution u should be stable outside a bounded set. If we follow a component of its nodal set along a unbounded sequence, translation and a standard compactness argument leads in the limit to a stable solution. Hence from the result in [13] its profile must be one-dimensional and hence its nodal set is a straight line. This makes it plausible that asymptotically the nodal set of u consists of a *finite, even number of straight lines, the ends.* If this is the case, those lines are not distributed in arbitrarily: Gui [25] proved that if $e_1, \dots e_{2k}$ are unit vectors in the direction of the ends of the nodal set of a solution of (1.1) in $\mathbb{R}^2$, then the balancing formula $\sum_{j=1}^{2k} e_j = 0$ holds.

As we have mentioned, another (possibly finite Morse index) solution is known, [14]. This is the so-called saddle solution. It is built by positive barriers with zero boundary data in a quadrant, and then extended by odd reflections to the rest of the plane, so that its nodal set is an infinite cross, hence having 4 straight ends.

An interesting question is whether one can find a 4-end family of solutions (5.42) depending continuously on the parameter $\varepsilon \in (0, \frac{\pi}{2})$ in such a way that when $\varepsilon \searrow 0$ the ends of the nodal set become parallel while when $\varepsilon \nearrow \frac{\pi}{2}$ they become orthogonal, as in the case of the saddle solution. Similarly, a saddle solutions with $2k$ ends with consecutive angles $\frac{\pi}{k}$ has been built in [2]. One may similarly ask whether this solution is in some way connected to the $2k$-end family (5.41).

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STABLE PATTERNS AND SOLUTIONS WITH MORSE INDEX ONE

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Abstract. We study shapes of the stable steady states of a shadow reaction-diffusion system of activator-inhibitor type and of the local minimizers of a variational problem with constraint. We show that these stable patterns are closely related to the solutions of

$$\Delta u + f(u) = 0 \ \text{ in } \ \Omega, \qquad \partial_\nu u = 0 \ \text{ on } \ \partial\Omega$$

with Morse index one. Moreover, we see that shapes of the solutions with Morse index one have a deep relationship with a nonlinear version of the "hot spots" conjecture of J. Rauch. In particular, we show that when the domain is a disk D, each stable pattern has exactly two critical points on $\overline{D}$, they are on the boundary ∂D, and each level set divides the domain into exactly two subdomains. Thus the shape of a stable pattern is like a boundary spike layer.

1. Introduction

In this talk we study shapes of the stable patterns of the two problems: the stationary problem of a shadow reaction-diffusion system

$$\text{(SS)} \qquad u_t = D_u \Delta u + f(u,\xi) \ \text{ in } \ \Omega \times \mathbb{R}_+, \qquad \tau \xi_t = \frac{1}{|\Omega|}\int_\Omega g(u,\xi)dx \ \text{ in } \ \mathbb{R}_+,$$

$$\partial_\nu u = 0 \ \text{ on } \ \partial\Omega \times \mathbb{R}_+$$

satisfying that

$$\text{(1.1)} \qquad \begin{array}{c} f(\,\cdot\,,\,\cdot\,) \text{ and } g(\,\cdot\,,\,\cdot\,) \text{ are of class } C^2,\ f_\xi < 0,\ g_\xi < 0, \text{ and} \\ \text{that there is a function } k(\xi) \in C^0 \text{ such that } g_u(u,\xi) = -k(\xi) f_\xi(u,\xi) \end{array}$$

and the minimization problem of the functional

$$\text{(1.2)} \qquad I[u] := \int_\Omega \left(\frac{\varepsilon^2 |\nabla u|^2}{2} - W(u) \right) dx$$

with constraint

$$\text{(1.3)} \qquad m = \frac{1}{|\Omega|}\int_\Omega u dx.$$

1.1. **Shadow reaction-diffusion system.** In 1975 Chafee [C75] showed that every non-constant steady state to a scalar reaction-diffusion equation with the Neumann boundary condition is unstable. Hence if a steady state is stable, then it should be constant, i.e., a homogeneous function. In 1978 Casten-Holland [CH78] and in 1979 Matano [Ma79] independently showed that the same conclusion holds for a reaction-diffusion equation on a convex domain in $\mathbb{R}^N$. Hence every model that can be described by a scalar reaction-diffusion equation does not have a stable inhomogeneous pattern when the domain is convex. In [Ma79] it was shown that there exist a scalar reaction-diffusion equation and a non-convex domain such that a stable inhomogeneous steady state exists.

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Key words and phrases. Reaction-Diffusion system; Variational problem with constraint; Stability; Hot spots; Boundary spike layer.

Spatial dimension	Scalar equation	(Shadow) system
1	constant [C75]	monotone [N94, NPY01]
N ($N \geq 2$)	constant [CH78, Ma79]	?

TABLE 1. Stable steady states on a convex domain.

Before going to the next result, we will explain the shadow system. Let us consider the Neumann problem of the reaction-diffusion system

$$\text{(FS)} \qquad u_t = D_u \Delta u + f(u,v) \ \text{ in } \ \Omega \times \mathbb{R}_+, \qquad \tau v_t = D_v \Delta v + g(u,v) \ \text{ in } \ \Omega \times \mathbb{R}_+,$$
$$\partial_\nu u = 0 \ \text{ on } \ \partial\Omega \times \mathbb{R}_+, \qquad \partial_\nu v = 0 \ \text{ on } \ \partial\Omega \times \mathbb{R}_+.$$

Let $D_v \to +\infty$. Then we can expect that $v(x,t)$ tends to a spatially homogeneous function $\xi(t)$ which depends only on t. Letting $v(x,t) = \xi(t)$ and integrating the second equation of (FS) with respect to x over Ω, we have (SS). The first equation of (SS) is a scalar homogeneous equation if ξ is fixed. Hence the techniques of analyzing a homogeneous equation can be used. (The first equation of (FS) may be an inhomogeneous equation when v is fixed.) We can expect that the two systems (SS) and (FS) are close in some sense if D_v is large. (See [Mi06a] for example.) We call (SS) *the shadow system* of (FS).

In 1994 Nishiura [N94] showed that every steady state to (SS) in a finite interval with certain conditions on f and g is unstable when u is neither constant nor monotone. Hence if an inhomogeneous steady state (u,ξ) is stable, then u should be monotone. This result was generalized by Ni-Poláčik-Yanagida [NPY01] in 2001. Table 1 shows the summary of the results.

1.2. **Activator-Inhibitor system.** We study the stable steady states to a shadow system in a high-dimensional domain. It is known that there is a stable inhomogeneous steady state to (SS) even if the domain is convex, e.g., a ball. For example, the shadow Gierer-Meinhardt system [GM72]

$$\text{(GM)} \qquad u_t = \varepsilon^2 \Delta u - u + \frac{u^p}{\xi^q} \ \text{ in } \ \Omega \times \mathbb{R}_+, \qquad \tau \xi_t = -\xi + \frac{1}{|\Omega| \xi^s} \int_\Omega u^r dx \ \text{ in } \ \mathbb{R}_+,$$
$$\partial_\nu u = 0 \ \text{ on } \ \partial\Omega \times \mathbb{R}_+,$$
$$0 < (p-1)/q < r/(s+1), \ p > 1, \ q > 0, \ r > 0, \ s \geq 0.$$

has a stable inhomogeneous steady state called a boundary spike layer even if the domain is convex. For example, see [W97] for the existence and see [Mi05] for the stability.

On the other hand, there are several classes of nonlinearities such that the system does not have a stable inhomogeneous steady state. Jimbo-Morita [JM94] showed that the reaction-diffusion system (FS) with the gradient structure does not have a stable inhomogeneous steady state provided that the domain is convex and that $\tau = 1$. Yanagida [Y02a] showed that the same conclusion holds for the reaction-diffusion system (FS) with the skew-gradient structure on a convex domain if $\tau > 0$ is not small. The skew-gradient (shadow) system includes the (shadow) Gierer-Meinhardt system when p, q, r, and s satisfy certain conditions. Yanagida's result looks to contradict the existence of a stable inhomogeneous steady state to the (shadow) Gierer-Meinhardt system. However, his result does not cover the case where τ is small, and a stable inhomogeneous steady state can exist when $\tau > 0$ is small.

Before explaining an effect of τ, we intuitively explain (FS). The activator-inhibitor system (FS) is a model describing the interaction between *the (short range) activator* u and *the (long range) inhibitor* v. The shadow system (SS) is a limit system where the diffusion coefficient of v diverges. Thus v becomes a spatially homogeneous function ξ. The activator activates the production rate of the inhibitor ($g_u > 0$), and the inhibitor suppresses the production rate of the activator ($f_v < 0$). The production rate of the inhibitor decreases as the inhibitor increases ($g_v < 0$). However, we do not assume the monotonicity of f in u. We want to consider the case where the activator reacts autocatalytically. In that case f is not monotone in u. A

typical example of f is $f(u,v) = u(1-u)(a-u) - \alpha v$ $(0 < a < 1,\ \alpha > 0)$. We call (FS) the activator-inhibitor system if f and g satisfy

(AI) $$f_v < 0, \quad g_u > 0, \quad \text{and} \quad g_v < 0.$$

Note that if (u,ξ) is a steady state to (SS) for some $\tau > 0$, then (u,ξ) is a steady state to (SS) for every $\tau > 0$. τ is the rate of reaction speeds between the activator and the inhibitor. When τ is large, the reaction speed of the inhibitor is slow. Dividing the second equation of (SS) by τ and letting $\tau \to +\infty$, we see that ξ changes slowly in time. Therefore we can expect that the behavior of the solution to (SS) is close to that of the solution to a scalar reaction-diffusion equation. We can expect that all the inhomogeneous steady states are unstable provided that the domain is convex (cf. [CH78, Ma79]). When τ is small, the inhibitor reacts quickly. This effect stabilizes an inhomogeneous steady state, and a stable inhomogeneous steady state can exist.

By the way, there is a possibility where a stable inhomogeneous steady state becomes unstable when τ is large. In this case a Hopf bifurcation occurs. [NTY01, WW03] studied in detail the pair of complex eigenvalues that pass through the imaginary axis. This change from stability to instability does not appear in a scalar equation, and appears only in a system. The range of τ for which a steady state is stable is important when one studies the stability of a steady state to a system.

1.3. **Stable patterns of (SS).** We want to find all the stable steady states. However, it is actually impossible to find all the stable steady states. Hence we will change the problem: If a steady state is stable, then what shape is it? Our strategy is to find a sufficient condition, which can be determine by the shape, for the steady state to be unstable for *all* $\tau > 0$. Then the contrapositive of the sufficient condition becomes the necessary condition for the steady state to be stable for *some* $\tau > 0$. In other words we know the shape of the stable steady states. We give an abstract sufficient condition.

Theorem 1.1 ([Mi06b])**.** *Let (u,ξ) be an inhomogeneous steady state to (SS) with (1.1). If the second eigenvalue of the eigenvalue problem*

$$\Delta\phi + f_u(u,\xi)\phi = \lambda\phi \quad \text{in} \quad \Omega, \quad \partial_\nu u = 0 \quad \text{on} \quad \partial\Omega$$

is positive, then, for each $\tau > 0$, (u,ξ) is unstable. Thus if (u,ξ) is stable for some $\tau > 0$, then the Morse index of u (with respect to the first equation of (SS)) is one.

1.4. **Example.** Let us consider the assumption (1.1). The assumptions $f_\xi < 0$ and $g_\xi < 0$ are included in (AI). Therefore those are natural in some sense. Although the last assumption seems to be artificial, (1.1) includes the following two systems:

Example 1.2. The shadow Gierer-Meinhardt system is (GM). (GM) always satisfies (AI). If $p = r - 1$, then (1.1) holds.

Example 1.3. The shadow system with the FitzHugh-Nagumo type nonlinearity is

(FHN) $$u_t = D_u\Delta u + u(1-u)(u-a) - \alpha\xi, \qquad \tau\xi_t = \frac{1}{|\Omega|}\int_\Omega \beta u dx - \gamma\xi \quad \text{in} \quad \mathbb{R}_+,$$

$$\partial_\nu u = 0 \quad \text{on} \quad \partial\Omega,$$

$$0 < a < 1, \quad \alpha > 0, \quad \beta > 0, \quad \text{and} \quad \gamma > 0.$$

(FHN) always satisfies (AI) and (1.1).

1.5. **Variational problem with constraint.** When W is a double well potential, (1.2) with (1.3) is a model arising in the van der Waals-Cahn-Hilliard theory of phase transitions. (See [Mo87] and references therein for details of the model.) The two bottoms of the well are corresponding to two stable states, and u tends to go to one of the bottoms. However, the constraint prevents u from becoming a constant stable state if m of (1.3) is in between two local minimum points of W. It is easily expected that a local minimizer exhibits a spatial pattern.

If $W(u) = (u^2-1)^2/4$, if $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and if ε is small, then the shape of energy minimizing sequences is well understood. Modica [Mo87] and Sternberg [S88] have shown that the limit of minimizers u_ε as $\varepsilon \downarrow 0$ is a function with values ± 1 almost everywhere and that the interface minimizes the area under the constraint that the ratio of $|\{u_\varepsilon \approx 1\}|$ and $|\{u_\varepsilon \approx -1\}|$ is a certain value. Luckhaus-Modica [LM89] have shown that the area of minimizing interface is a hypersurface with constant mean curvature. (If $N \leq 7$, then the interface is smooth. If $N \geq 8$, the interface may have singularities, however the Hausdorff dimension of the set of the singularities is at most $N-8$.) Sternberg-Zumbrun [SZ98] have shown that, if Ω is strictly convex, then, for some $k \geq 1$, the interface $\{a_\varepsilon + \varepsilon^k < u_\varepsilon < b_\varepsilon - \varepsilon^k\}$, the superlevel set $\{u_\varepsilon > a_\varepsilon + \varepsilon^k\}$ and the sublevel set $\{u_\varepsilon < b_\varepsilon - \varepsilon^k\}$ are connected, where a_ε and b_ε ($a_\varepsilon < b_\varepsilon$) go to two stable zeros of $-W'(u)$ as $\varepsilon \downarrow 0$. In [SZ98] the connectivity of the interface and the boundary was also shown.

In the same research direction as ours Carr-Gurtin-Slemrod [CGS84] have shown that every non-constant local minimizer is monotone when the domain is a finite interval. Gurtin-Matano [GM88] studied the shape of the local (and global) minimizers when Ω is a disk, annulus or cylinder. In [GM88] they have shown that when Ω is a disk, each global minimizer is monotone in some direction. However, they used the rearrangement technique, and their method is not applicable to the local minimizers. On the other hand, we use the following sufficient condition for u not to be a local minimizer (unstable):

Theorem 1.4. *Let u be a critical point of (1.2) with (1.3). If the second eigenvalue of the eigenvalue problem*

$$\varepsilon^2 \Delta \phi + W''(u)\phi = \mu \phi \ \text{ in } \ \Omega, \qquad \partial_\nu \phi = 0 \ \text{ on } \ \partial\Omega$$

is positive, then u is not a local minimizer. Thus if u is a local minimizer, then the Morse index of u is one.

2. Solution with Morse index one

Our problem can be reduced the following problem: If the Morse index of u is one, then what shape is u? However, there are not so many results of this problem. It is because there is a potential problem. How do we describe the function defined in a high-dimensional domain? As far as the scalar equation is concerned, every stable steady state is constant. Hence we need not answer the problem. In the case of the shadow system in an interval, every inhomogeneous stable steady state can be described as monotone. Our answer (or suggestion) here is the following: Using the number and the locations of the critical points, we describe the function defined in a high-dimensional domain.

From now on, we consider the case where the domain is a disk $D := \{x \in \mathbb{R}^2;\ |x| < 1\}$.

We are in a position to state the main result.

Theorem 2.1 ([Mi06b, Mi07a, Mi10])**.** *Suppose that $\Omega = D$. Let u be a non-constant solution of*

$$\Delta u + h(u) = 0 \ \text{ in } \ D, \quad \partial_\nu u = 0 \ \text{ on } \ \partial D. \tag{2.1}$$

If the Morse index of u is one, then u satisfies the following (a) and (b):
(a) u has exactly two critical points in $\overline{D}$ and those are on ∂D. In particular, u attains its maximum and minimum at those two points and there is no critical point in D.
(b) For every $c \in (\min_{x\in\overline{D}} u(x), \max_{x\in\overline{D}} u(x))$, the c-level set of u is a unique C^1-curve whose edges hit ∂D at two different points and it divides D into exactly two simply connected subdomains.

Fig. 1 shows the shape of u when (u, ξ) is stable. We do not assume smallness or largeness of the diffusion coefficient D_u in (SS). The proof does not rely on the singular perturbation technique. When D_u is small, there are many results about the shape of inhomogeneous steady states. This theorem says that only the steady state whose shape is like a boundary spike layer can be stable even if D_u is not small. If D_u is larger than a certain value, then we can show that u is symmetric with respect to a line containing the center of the disk [Mi07b]. It is well-known

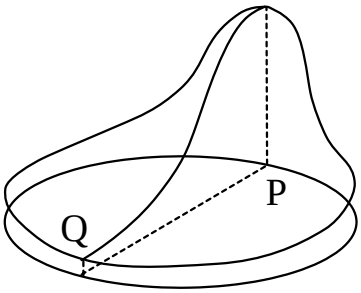

FIGURE 1. The shape of a solution with Morse index one.

that every positive steady state is radially symmetric if the Dirichlet boundary condition is imposed. However, there seems to be few results about the symmetry of the steady state to a Neumann problem when the steady state is not the least-energy solution.

If u has an interior peak (e.g., spike or spot), then the top of the peak is a critical point, hence the steady state is unstable. The stable pattern does not have an interior peak.

3. Proofs and related results

3.1. **Proofs.** The proof of Theorem 2.1 consists of several lemmas including the following two:

Lemma 3.1 ([Mi06b, Lemma 3.4])**.** *Suppose that $\Omega = D$. Let u be a non-constant solution of (2.1). By $U(\theta)$ we define $U(\theta) := u(\cos\theta, \sin\theta)$. If $\mathcal{Z}[U_\theta(\,\cdot\,)](:= \sharp\{U_\theta(\theta) = 0;\ \theta \in \mathbb{R}/2\pi\mathbb{Z}\}) \geq 3$, then the second eigenvalue of the eigenvalue problem*

$$\Delta\phi + h'(u)\phi = \lambda\phi \quad in \quad \Omega, \quad \partial_\nu\phi = 0 \quad on \quad \partial\Omega \tag{3.1}$$

is positive.

Lemma 3.2 ([Mi07a, Lemma C])**.** *Suppose that $\Omega = D$. Let u be a non-constant solution to (2.1). If u has a critical point inside D, then the second eigenvalue of (3.1) is positive. Here we say that (x_0, y_0) is a critical point of u if $u_x(x_0, y_0) = u_y(x_0, y_0) = 0$.*

In Lemmas 3.1 and 3.2 we do not impose an assumption on h except $h \in C^2$.

In the proofs of Lemmas 3.1 and 3.2 the detailed analysis of the zero level set of $-(x-x_0)u_y + (y-y_0)u_x$ is done. The zero-level set (or the nodal curve) gives a relation between the shape of the solution and the Morse index. The zero-level set is corresponding to the zero-number in a one-dimensional case.

3.2. **Extension of Lemma 3.2.** We consider Lemma 3.2 when Ω is a convex domain. It is expected that the following holds:

Conjecture 3.3 ([Y06, Yanagida])**.** *Let $\Omega \subset \mathbb{R}^N$ be a convex domain. Let u be a non-constant solution to (2.1). If u has a critical point inside Ω, then the second eigenvalue of (3.1) is positive.*

E. Yanagida pointed out that this conjecture is a nonlinear version of the "hot spots" conjecture of J. Rauch.

Conjecture 3.4 ([R74, Rauch])**.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. The maximum and the minimum of each non-zero eigenfunction corresponding to the second eigenvalue of the Neumann Laplacian are attained on the boundary.*

The "hot spots" conjecture immediately follows from Conjecture 3.3. Lemma 3.2 is the positive answer of Conjecture 3.3 when the domain is a disk. Table 2 shows the relation among known results and conjectures. In particular, Conjectures 3.3 and 3.4 can be seen as nonlinear and high-dimensional versions of Sturm-Liouville theory.

Morse index	Linear	Nonlinear
0	The first eigenfunction of Δ_N is constant	[CH78, Ma79]
1	Conjecture 3.4	Conjecture 3.3
n	What shape is an eigenfunction?	What shape is u?

TABLE 2. The relation among [CH78, Ma79] and Conjectures 3.4 and 3.3.

	Equation	Domain	Solution
Conjecture 3.3	$\Delta u + h(u) = 0$	Convex domain	Any solution
Ni-Takagi	$\varepsilon^2 \Delta u - u + u^p = 0$	Any domain	Least-energy sol.

TABLE 3. The relation between Conjecture 3.3 and a Ni-Takagi problem.

Table 3 shows the relation between Conjecture 3.3 and a problem of Ni and Takagi [NT91, NT93]. If we ignore the restriction on the domain, then Conjecture 3.3 can be seen as a generalization of a problem of Ni-Takagi.

We consider Conjecture 3.4. It is known that there are several counterexamples of Conjecture 3.4. In 1999 Burdzy-Werner [BW99] gave a counterexample. Their domain is a planar domain with three holes. Burdzy [B05] later gave another counterexample which is a planar domain with one hole. There are classes of planar domains for which the conjecture holds. Banuelos-Burdzy [BB99] proved the conjecture for planar convex domains with two axes of symmetry. However, another technical assumption is imposed in [BB99]. Jerison-Nadirashivili [JN00] removed the technical assumption. The method of [JN00] is very different of that of [BB99]. When the symmetry is not assumed, Atar-Burdzy [AB04] proved the conjecture for a long domain called *the lip domain*. This class of domains includes a non-convex domain. It is widely believed that the conjecture holds for a convex domain. In general it is difficult to prove the conjecture when the domain does not have symmetries. The author obtained a partial positive answer.

Theorem 3.5 ([Mi09]). *Let Ω be a planar convex domain. Let $d := \sup_{p,q\in\Omega} |p-q|$. If*
(i) $d^2/|\Omega| < 1.378$, or
(ii) Ω is in a strip with width l and $dl/|\Omega| < 1.219$,
then Conjecture 3.4 holds.

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Ill-posedness of the 3D-Navier-Stokes equations near BMO^{-1}

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We consider the nonstationary incompressible Navier-Stokes equations in $\mathbb{R}^3$:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + (u \cdot \nabla)u + \nabla p = 0, \quad \text{div } u = 0 \quad \text{in} \quad x \in \mathbb{R}^3, t \in (0,T), \\ u|_{t=0} = u_0, \end{cases} \tag{0.1}$$

where $u = u(t) = (u^1(x,t), u^2(x,t), u^3(x,t))$ and $p = p(t) = p(x,t)$ denote the velocity vector field and the pressure of fluid at the point $(x,t) \in \mathbb{R}^3 \times (0,T)$, respectively, while $u_0 = (u_0^1(x), u_0^2(x), u_0^3(x))$ is a given initial velocity vector field.

We are concerned with the ill-posedness of the Cauchy problem for (0.1). More precisely for a given function space $X = X(\mathbb{R}^3)$ we say that the Cauchy problem is well-posed in X if there exists a space $Y \subset C([0,T), X)$ such that for all $u_0 \in X$ there exists a unique solution $u \in Y$ for (0.1) and the flow map $u_0 \to u = \Phi(u_0)$ is continuous from X to $C([0,T), X)$. Also we say that the Cauchy problem is ill-posed in X if it is not. The classical results on the existence theorem of the mild solution were shown by Kato [6] and Giga-Miyakawa [3]. Making use of the iteration procedure, they constructed a global solution in the class $C([0,\infty); L^n(\mathbb{R}^n)) \cap C((0,\infty); L^p(\mathbb{R}^n))$ for $n < p \le \infty$, when an initial data u_0 is small enough in $L^n(\mathbb{R}^n)$. To construct a solution in more general classes of initial data is very important problem. Giga-Miyakawa [4], Kato [7] and Taylor [12] proved the well-posedness in certain Morrey spaces. Cannone [2] and Kozono-Yamazaki [9] investigated this problem in Besov spaces. In particular, Koch and Tataru [8] obtained the global solvability for (0.1), when the initial data u_0 is small enough in BMO^{-1}. BMO^{-1} includes above function spaces and it has been considered as the largest space of initial data (see Lemarié-Rieusset [10]). On the other hand, Montgomery-Smith [11] introduce an equation similar to Navier-Stokes equation and proved ill-posedness in the Besov space $B^{-1}_{\infty,\infty}$, which is larger than BMO^{-1}. In 2008, Bourgain-Pavlović [1] showed that (0.1) is ill-posed in $B^{-1}_{\infty,\infty}$ by showing norm inflation phenomena of the solution for some initial data. More precisely, they proved that for any $\delta > 0$ there exist initial data u_0 with $\|u_0\|_{B^{-1}_{\infty,\infty}} < \delta$ such that the corresponding solution u satisfies $\|u(t)\|_{B^{-1}_{\infty,\infty}} > 1/\delta$ for some $t < \delta$. This shows that the flow map Φ is not continuous. On the other hand, Germain [5] proved that the flow map is not C^2 in the Besov spaces $B^{-1}_{\infty,q}$ for $q > 2$. However he did not treat ill-posed problem in such spaces. The purpose of my talk is to show ill-posedness

of 3D-Navier-Stokes equations in Besov spaces $B^{-1}_{\infty,q}$ $(q > 2)$ (see [13]). Thus our result is an extension of both Bourgain-Pavlović's and Germain's results.

We give a sketch of the proof briefly. First, we introduce initial data which is composed by a sum of r cosine functions. The idea of setting of the initial data is proposed by [1] and [5]. We take a lacunary frequency set, and the norm of initial data in $B^{-1}_{\infty,q}$ $(q > 2)$ is controlled by r. Second, we extract an inflation term from second approximation. Third, we estimate the remainder term y. The remainder term satisfies certain integral equation composed by first and second approximations including an inflation term. We also control the remainder term by r. Since we set refined initial data from Bourgain-Pavlović's setting, we can get better estimate of second approximation than their estimate. According to their setting of initial data, using BMO^{-1} norm to estimate remainder term y is important. Since we got better estimate of second approximation, we can use the bilinear estimate of a class of bounded uniformly continuous functions (equipped with the L^∞ norm).

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Analysis of the motion of a membrane touching a solid plane (abstract)

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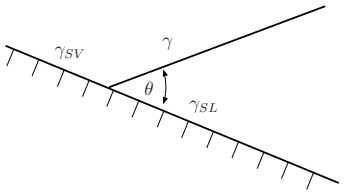
We are interested in the motion of a membrane that is in contact with a rigid plane.



In many cases, the membrane is described by some partial differential equation (such as heat equation) and on the free boundary (points where the membrane touches the plane) a contact angle condition is prescribed which originates in the physical properties of the materials in contact (i.e., surface tensions $\gamma, \gamma_{SV}, \gamma_{SL}$).

A pioneering beautiful paper on the mathematical aspect of the problem by Alt and Caffarelli (1981) deals with the stationary case

$$\Delta u = 0 \quad \text{in } \Omega \cap \{u > 0\}, \qquad |\nabla u| = Q,\ u = 0 \quad \text{on } \Omega \cap \partial\{u > 0\}.$$

They study the functional

$$\int_\Omega \left(|\nabla u|^2 + Q^2 \chi_{u>0}\right) dx$$

and show that it possesses minima which are Lipschitz continuous and have linear growth away from the free boundary. For such harmonic functions they find a representation formula and show that the minima are weak solutions, while the free boundary is a smooth surface except of a set of zero $(n-1)$-dimensional Hausdorff measure.

On the other hand, Caffarelli and Vázquez (1995) studied the evolutionary problem

$$u_t - \Delta u = 0 \quad \text{in } \{u > 0\}, \qquad |\nabla u| = 1,\ u = 0 \quad \text{on } \partial\{u > 0\}$$

by a different technique. They regularize the problem by adding an absorption term in the following way

$$u^\varepsilon_t - \Delta u^\varepsilon = -\tfrac{1}{2}\chi'_\varepsilon(u^\varepsilon), \qquad u^\varepsilon \geq 0.$$

Here, χ_ε is a smoothing of the characteristic function in the interval $(0, \varepsilon)$. The authors show uniform estimates for the solution of the regularized equation (Lipschitz in space and Hölder in time) and use them to construct a weak solution of the original problem. They also study the regularity of free boundary in case of shrinking support.

We are interested in the study of the evolutionary problem with volume constraint

$$\int_\Omega u(t,x)\, dx = V \qquad \forall t,$$

which appears, for example, in the free boundary problem modelling the motion of bubbles or droplets on a surface. The problem becomes

$$u_t - \Delta u = \lambda \quad \text{in } \{u > 0\}, \qquad |\nabla u|^2 = 2\gamma \quad \text{on } \partial\{u > 0\}$$

and its regularized version is

$$u_t = \Delta u - \gamma\chi_\varepsilon'(u) + \chi_{u>0}\lambda_\varepsilon \qquad \text{in } (0,T)\times\Omega, \qquad \text{where} \quad \lambda_\varepsilon = \int_\Omega \left[u_t u + |\nabla u|^2 + \gamma\chi_\varepsilon'(u)u\right] dx.$$

Here λ (or λ_ε) are nonlocal terms coming from the volume constraint.

In the regularized problem the volume constraint gives rise to an obstacle-type problem with a nonlocal obstacle function. Accordingly, the sharp contact angle limit $\varepsilon \to 0$ is expected to have two factors influencing the behaviour on the free boundary: the stronger linear growth due to contact angle condition and the weaker quadratic growth (curvature) originating in the volume constraint.

With the view of numerical approximation and because of the presence of the global constraint we analyse the regularized obstacle problem by a minimization method introduced by K. Rektorys and developed by N. Kikuchi. In this method time variable is discretized and the functional

$$J_n(u) = \int_\Omega \Big(\frac{|u-u_{n-1}|^2}{2h} + \frac{1}{2}|\nabla u|^2 + \gamma\chi_\varepsilon(u) \Big)\, dx$$

is minimized. Here we define a special constrained space

$$\mathcal{K}^\delta = \big\{ u \in H_0^1(\Omega)\,; \quad \int_\Omega \chi_\delta(u)u\, dx = V \big\}$$

as the admissible space for minimization.

The (regularization of) characteristic function in the admissible space is essential in order to satisfy the obstacle condition. Indeed, the minimizers are shown to exist and be nonnegative. The weak solution is then constructed by deriving uniform estimates in h and δ and taking $h, \delta \to 0$. (See, Svadlenka & Omata, 2009 for details.)

The analysis for the sharp limit $\varepsilon \to 0$ is yet to be done. Yamaura constructed L^2 - generalized minimizing movement corresponding to the considered energy without taking into account the volume constraint. It is expected that a similar technique will basically work for the constrained problem.

Our future plan is to consider the application of phase-field approximation to the contact angle problem. The phase-field method is superior to the above scalar approach in the sense that it addresses surfaces and can therefore express contact angles larger than right angle. It is assumed that one can derive a boundary parabolic monotonicity formula (see [5]) and thus rigorously construct a hypersurface evolving according to its mean curvature with a prescribed contact angle on the rigid boundary.

Another challenging task is the contact problem arising, e.g., in the modelling of collision of elastic curves with an obstacle. For this phenomenon, there are numerical results but a suitable mathematical approach to this hyperbolic free boundary problem is still unknown.

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ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO NONLINEAR SCHRÖDINGER EQUATIONS

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1. Cubic nonlinear Schrödinger equation

We consider the nonlinear Schrödinger equation

$$i\partial_t u + \frac{1}{2m}\Delta u = \lambda |u|^2 u \tag{1.1}$$

in $\mathbf{R}$, where $\Delta = \partial_x^2, m$ is mass of particle, $\lambda \in \mathbf{C}$. Equation (1.1) is non relativistic version of nonlinear Klein-Gordon equation

$$\frac{1}{2c^2 m}\partial_t^2 v - \frac{1}{2m}\Delta v + \frac{mc^2}{2} v = -\lambda |v|^2 v, \tag{1.2}$$

where c is the speed of light. Indeed we change $v = e^{-itmc^2}u$ to get

$$\frac{1}{2c^2 m}\partial_t^2 u - i\partial_t u - \frac{1}{2m}\Delta u = -\lambda |u|^2 u.$$

If we let $c \to \infty$, then we can obtain (1.1). We survay results on asymptotic behavior of small solutions to the initial value problem and the final value problem for (1.1). We use the following factorization formula for the free Schrödinger evolution group $\mathcal{U}\left(\frac{t}{m}\right) = \exp\left(\frac{i}{2m}\Delta\right)$such that

$$\mathcal{U}\left(\frac{t}{m}\right) = M^m(t)\mathcal{D}\left(\frac{t}{m}\right)\mathcal{V}\left(\frac{t}{m}\right)\mathcal{F}.$$

This formula is useful to study asymptotic behavior of solutions and used in paper [8]. We have from the above

$$\mathcal{F}\mathcal{U}\left(-\frac{t}{m}\right) = \mathcal{V}\left(-\frac{t}{m}\right) i\overline{E}^{\frac{1}{m}}(t)\mathcal{D}\left(\frac{m}{t}\right),$$

where we denote

$$M(t) = e^{\frac{i}{2t}|x|^2}, E(t) = e^{\frac{it}{2}|\xi|^2},$$

a dilation operator

$$(\mathcal{D}(t)\phi)(x) = \frac{1}{it}\phi(\frac{x}{t})$$

and

$$\mathcal{V}\left(\frac{t}{m}\right) = \mathcal{F}M^m(t)\mathcal{F}^{-1}.$$

Note that

$$\mathcal{D}\left(\frac{m}{t}\right)M^m(t) = E^{\frac{1}{m}}(t)\mathcal{D}\left(\frac{m}{t}\right).$$

Key words and phrases. Modified wave operator, Nonlinear Schrödinger equations.

Multiplying both sides of (1.1) by $\mathcal{FU}\left(-\frac{t}{m}\right)$ and putting $w = \mathcal{FU}\left(-\frac{t}{m}\right)u$, we obtain

$$i\partial_t w = \lambda \mathcal{FU}\left(-\frac{t}{m}\right)|u|^2 u.$$

Asymptotic behavior of small solutions to the Cauchy problem for (1.1) is obtained by showing the right hand side of the above is decomposed into two terms:

$$\lambda \mathcal{FU}\left(-\frac{t}{m}\right)|u|^2 u = \lambda \frac{m}{t}|w|^2 w + R,$$

where R is considered as a remainder term. Therefore asymptotic behavior of solutions for (1.1) is determined by the ordinary differential equation

$$i\partial_t \widehat{w} = \lambda \frac{m}{t}|\widehat{w}|^2 \widehat{w}.$$

Indeed, for the final value problem, we can find a solution in the neighborhood of solutions of the ordinary differential equations. We let $\widehat{w} = re^{i\psi}$, then we have

$$i\partial_t r - r\partial_t \psi = \lambda \frac{m}{t}|r|^2 r$$

from which it follows that if $\lambda = \lambda_1 + i\lambda_2, \lambda_1, \lambda_2 \in \mathbf{R}$

$$\partial_t r = \lambda_2 \frac{m}{t} r^3, -\partial_t \psi = \lambda_1 \frac{m}{t}|r|^2.$$

By a given function ϕ, we have for $\lambda_2 \leq 0$

$$r(t) = \frac{\left|\widehat{\phi}\right|}{\left(1 - 2m\lambda_2 \left|\widehat{\phi}\right|^2 \log t\right)^{\frac{1}{2}}},$$

and for $\lambda_2 < 0$

$$\begin{aligned} \psi(t) &= \frac{\lambda_1}{2\lambda_2} \int^t \frac{d\left(-2\lambda_2 m \left|\widehat{\phi}\right|^2 \log t\right)}{\left(1 - 2m\lambda_2 \left|\widehat{\phi}\right|^2 \log t\right)} \\ &= \frac{\lambda_1}{2\lambda_2} \log\left(1 - 2m\lambda_2 \left|\widehat{\phi}\right|^2 \log t\right) \end{aligned}$$

for $\lambda_2 = 0$

$$\psi(t) = -\lambda_1 m \left|\widehat{\phi}\right|^2 \log t.$$

Thus we have the solution of ordinary differential equation such that

$$\begin{aligned} \widehat{w} &= re^{i\psi} \\ &= \frac{\left|\widehat{\phi}\right| \exp\left(i\frac{\lambda_1}{2\lambda_2} \log\left(1 - 2m\lambda_2 \left|\widehat{\phi}\right|^2 \log t\right)\right)}{\left(1 - 2m\lambda_2 \left|\widehat{\phi}\right|^2 \log t\right)^{\frac{1}{2}}}, \end{aligned}$$

for $\lambda_2 < 0$ and

$$\begin{aligned} \widehat{w} &= re^{i\psi} \\ &= \left|\widehat{\phi}\right| \exp\left(-i\lambda_1 m \left|\widehat{\phi}\right|^2 \log t\right), \end{aligned}$$

for $\lambda_2 = 0$. We make a changing of variable $\widehat{\varphi}(t) = \left|\widehat{\phi}\right| \exp\left(i\frac{\lambda_1}{2\lambda_2}\log\left(1 - 2m\lambda_2\left|\widehat{\phi}\right|^2 \log t\right)\right)$ or $\widehat{\varphi}(t) = \left|\widehat{\phi}\right| \exp\left(-i\lambda_1 m\left|\widehat{\phi}\right|^2 \log t\right)$. Then

$$\widehat{w} = \frac{\widehat{\varphi}(t)}{\left(1 - 2m\lambda_2\left|\widehat{\varphi}(t)\right|^2 \log t\right)^{\frac{1}{2}}}, \lambda_2 < 0$$

$$\widehat{w} = \widehat{\varphi}(t) \exp\left(-i\lambda_1 m\left|\widehat{\varphi}(t)\right|^2 \log t\right), \lambda_2 = 0.$$

2. System of NLS in 2D

In this section we report the recent results obtained in [5]. We consider a system of nonlinear Schrödinger equations

$$\begin{cases} i\partial_t v_1 + \frac{1}{2m_1}\Delta v_1 = \lambda\overline{v_1}v_2 \\ i\partial_t v_2 + \frac{1}{2m_2}\Delta v_2 = \mu v_1^2 \end{cases} \tag{2.1}$$

in $\mathbf{R}^2$, where $\Delta = \sum_{j=1}^2 \partial_j^2, \partial_j = \partial/\partial x_j, m_1, m_2$ are masses of particles and $\lambda, \mu \in \mathbf{C}$. We make the scaling $v_1 = \frac{1}{\sqrt{|\lambda\mu|}}u_1$ and $v_2 = \frac{\mu}{|\lambda\mu|}u_2$, to exclude the constants λ and μ from system (2.1) to get

$$\begin{cases} i\partial_t u_1 + \frac{1}{2m_1}\Delta u_1 = \gamma\overline{u_1}u_2, \\ i\partial_t u_2 + \frac{1}{2m_2}\Delta u_2 = u_1^2, \end{cases} \tag{2.2}$$

where $\gamma = \frac{\lambda\mu}{|\lambda\mu|} \in \mathbf{C}$, $|\gamma| = 1$. We assume the mass condition

$$2m_1 = m_2 \tag{2.3}$$

which is called the resonance condition. We also consider the case

$$2m_1 \neq m_2, m_1 \neq m_2 \tag{2.4}$$

which is call the non resonance condition.

The system (2.2) is non relativistic version of a system of nonlinear Klein-Gordon equations

$$\begin{cases} \frac{1}{2c^2 m_1}\partial_t^2 v_1 - \frac{1}{2m_1}\Delta v_1 + \frac{m_1 c^2}{2}v_1 = -\gamma\overline{v_1}v_2, \\ \frac{1}{2c^2 m_2}\partial_t^2 v_2 - \frac{1}{2m_2}\Delta v_2 + \frac{m_2 c^2}{2}v_2 = -v_1^2, \end{cases} \tag{2.5}$$

where c is the speed of light.

We introduce the weighted Sobolev space

$$\mathbf{H}^{m,s} = \left\{ f = (f_1, f_2) \in \mathbf{L}^2; ||f||_{\mathbf{H}^{m,s}} = \sum_{j=1}^2 ||f_j||_{\mathbf{H}^{m,s}} < \infty \right\},$$

where

$$||f||_{\mathbf{H}^{m,s}} = \left\| (1-\Delta)^{\frac{m}{2}} \left(1 + |x|^2\right)^{\frac{s}{2}} f \right\|_{\mathbf{L}^2}.$$

We write $\mathbf{H}^m = \mathbf{H}^{m,0}$ for simplicity.

Under the resonance condition (2.3) we prove

Theorem 1. *Let $2m_1 = m_2$, $\gamma < 0, t \geq 1, \widehat{w_1} \in \mathbf{H}^{2,0}$ and $|\widehat{w_1}(\xi)| \geq \delta > 0$. Then there exists an $\varepsilon > 0$ such that (2.2) has a unique global solution*

$$(u_1(t), u_2(t)) \in \mathbf{C}\left([1,\infty); \mathbf{L}^2 \times \mathbf{L}^2\right)$$

satisfying the asymptotics

$$\left\| u_1(t) - \frac{1}{m_1}\mathcal{U}\left(\frac{t}{m_1}\right)\mathcal{F}^{-1}\psi_{1+}\left(t, \frac{\cdot}{m_1}\right)\right\|_{\mathbf{L}^2} + \left\| u_2(t) - \frac{1}{m_2}\mathcal{U}\left(\frac{t}{m_2}\right)\mathcal{F}^{-1}\psi_{2+}\left(t, \frac{\cdot}{m_2}\right)\right\|_{\mathbf{L}^2} \leq Ct^{-b}$$

for $t \geq 1$ and any $\widehat{w_1}$ such that $\|\widehat{w_1}\|_{\mathbf{H}^{s,0}} \leq \varepsilon$, where $\frac{1}{2} < b < 1$,

$$\psi_{1+}(t,\xi) = \frac{\widehat{w_1}(\xi)}{1 + \sqrt{|\gamma|}\,|\widehat{w_1}(\xi)|\log t}$$

and

$$\begin{aligned}\psi_{2+}(t,\xi) &= \frac{1}{\sqrt{|\gamma|}}\frac{1}{|\widehat{w_1}(\xi)|}\frac{\widehat{w_1}(\xi)^2}{1 + \sqrt{|\gamma|}\,|\widehat{w_1}(\xi)|\log t} \\ &= \frac{1}{\sqrt{|\gamma|}}\frac{\psi_{1+}(t,\xi)^2}{|\psi_{1+}(t,\xi)|}.\end{aligned}$$

Theorem 2. *Let $2m_1 = m_2$, $\gamma > 0, t \geq 1$, $\widehat{w_1} \in \mathbf{H}^{2,0}$ and $|\widehat{w_1}(\xi)| > \delta$. Then the same result as in Theorem 1 holds for*

$$\psi_{1+}(t) = i\widehat{w_1}(\xi)\, e^{i\sqrt{\frac{\gamma}{2}}|\widehat{w_1}(\xi)|\log t}$$

and

$$\psi_{2+}(t) = -i\frac{1}{\sqrt{2\gamma}}\frac{\widehat{w_1}(\xi)^2}{|\widehat{w_1}(\xi)|}e^{2i\sqrt{\frac{\gamma}{2}}|\widehat{w_1}(\xi)|\log t} = i\frac{1}{\sqrt{2\gamma}}\frac{\psi_{1+}(t,\xi)^2}{|\psi_{1+}(t,\xi)|}.$$

for $t \geq 1$, where $\frac{1}{2} < b < 1$.

It is known that by the above theorems, the identity $\mathcal{U}\left(\frac{t}{m}\right) = M^m(t)\,\mathcal{D}\left(\frac{t}{m}\right)\mathcal{F}M^m(t)$ we see that

$$\begin{aligned}&\left\| u_1(t) - \frac{1}{it}M^{m_1}(t)\,\psi_{1+}\left(t, \frac{\cdot}{t}\right)\right\|_{\mathbf{L}^2} \\ &+ \left\| u_2(t) - \frac{1}{it}M^{m_2}(t)\,\psi_{2+}\left(t, \frac{\cdot}{t}\right)\right\|_{\mathbf{L}^2} \\ \leq\ & Ct^{-b}.\end{aligned}$$

Under the non resonance condition we prove

Theorem 3. *Let $2m_1 \neq m_2$, $m_1 \neq m_2$, $(\phi_{1+}, \phi_{2+}) \in \mathbf{H}^{0,2} \cap \dot{\mathbf{H}}^{0,-2b}$. Then there exists an $\varepsilon > 0$ such that (2.2) has a unique global solution*

$$(u_1(t), u_2(t)) \in \mathbf{C}\left([1,\infty); \mathbf{L}^2 \times \mathbf{L}^2\right)$$

for any (ϕ_{1+}, ϕ_{2+}) satisfying $\sum_{j=1}^{2}\|\phi_{j+}\|_{\mathbf{H}^{s,0}} \leq \varepsilon$. Furthermore we have the asymptotics

$$\left\| u_1(t) - \mathcal{U}\left(\frac{t}{m_1}\right)\phi_{1+}\right\|_{\mathbf{L}^2} + \left\| u_2(t) - \mathcal{U}\left(\frac{t}{m_2}\right)\phi_{2+}\right\|_{\mathbf{L}^2} \leq Ct^{-b}$$

for $t \geq 1$, where $\frac{1}{2} < b < 1$.

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On dispersive estimates for Schrödinger equations

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Abstract

Let $H = -\Delta + V(x)$ be three dimensional Schrödinger operator with the real potential $V(x)$ which decays at infinity. Let P_c be the projection onto the continuous spectral subspace of $L^2(\mathbb{R}^3)$ for H. Suppose that 0 is not an eigenvalue nor a resonance of H. Then, we show under suitable decay and smoothness conditions on V that the propagator e^{-itH} for the Schrödinger equation $i\partial_t u = Hu$ admits the expansion as $t \to \infty$ of the form

$$\left\| \langle x \rangle^{-k-\varepsilon} \left(e^{-itH} P_c u - \sum_{j=0}^{k} t^{-\frac{3}{2}-j} A_j u \right) \right\|_{L^\infty} \leq C t^{-\frac{3}{2}-k-\sigma} \| \langle x \rangle^{k+\varepsilon} u \|_{L^1} \qquad (1)$$

where $0 < \sigma < \varepsilon$ and A_j, $j = 0, 1, \ldots, k$ are finite rank operators, We discuss the extension of the expansion formula (1) for the case when 0 is an eigenvalue or/and a resonace of H. The work is in progress and the precise result will be presented in the talk.

UNIQUE CONTINUATION AND NONLINEAR DISPERSIVE EQUATIONS

GUSTAVO PONCE

The aim of this talk is to present recent results obtained in collaboration with L. Escauriaza, C. E. Kenig, and L. Vega concerning unique continuation properties of solutions of Schrödinger equations.

First, we shall consider Schrödinger equations of the form

$$i\partial_t u + \Delta u = V(x,t)u, \ \text{ in } \ \mathbb{R}^n \times [0,1]. \tag{1}$$

Our first goal is to obtain sufficient conditions on a solution u, the potential V and the behavior of the solution at two different times, $t_0 = 0$ and $t_1 = 1$, which guarantee that $u \equiv 0$ in $\mathbb{R}^n \times [0,1]$.

In the case when the potential $V \equiv 0$ one has, defining the Fourier transform of a function f as

$$\widehat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}n} e^{-i\,\xi\cdot x}\, f(x)\, dx,$$

the identity

$$e^{it\Delta}u_0(x) = u(x,t)$$

$$= (4\pi it)^{-\frac{n}{2}} \int_{\mathbb{R}n} e^{\frac{i|x-y|^2}{4t}} u_0(y)\, dy = (2\pi it)^{-\frac{n}{2}}\, e^{\frac{i|x|^2}{4t}} \widehat{e^{\frac{i|\cdot|^2}{4t}} u_0}\left(\frac{x}{2t}\right), \tag{2}$$

This shows that this kind of problem (the decay of the Schrödinger equation at two different times) for the free solution of the Schrödinger equation with data u_0

$$i\partial_t u + \triangle u = 0, \qquad u(x,0) = u_0(x), \quad (x,t) \in \mathbb{R}^n \times \mathbb{R},$$

is intrinsically related to "uncertainty principles" concerning the decay of a function f and its Fourier transform, $\widehat{f}$.

Among these uncertainty principles one has the following one due to G. H. Hardy ([4]) :

If $f(x) = O(e^{|x|^2/\beta^2})$, $\widehat{f}(\xi) = O(e^{-4|\xi|^2/\alpha^2})$ and $\alpha\beta < 4$, then $f \equiv 0$.

Also, if $\alpha\beta = 4$, f is a constant multiple of $e^{-|x|^2/\beta^2}$.

Using (2), Hardy uncertainty principle can be rewritten in terms of the free solution of the Schrödinger equation :

$$\text{If } \ u_0(x) = O(e^{-\frac{|x|^2}{\beta^2}}), \ \ e^{it\Delta}u_0(x) = O(e^{-\frac{|x|^2}{\alpha^2}}), \ \text{ and } \ \alpha\,\beta < 4t, \ \text{then } u_0 \equiv 0.$$

In the context of the Schrödinger equation we shall present an extension of this results to solution of the equation (1).

As an application we shall consider the semi-linear Schrödinger equation

$$i\partial_t u + \Delta u \pm |u|^a u = 0, \ \text{ in } \ \mathbb{R}^n \times [0,1], \ \ a > 0. \tag{3}$$

and give some answers to the following question : given $u_1,\ u_2$ solutions of (3), what do we have to know about their difference $(u_1 - u_2)(x,t)$ at two times $t = 0$ and $t = 1$ to guarantee that they are equal?

We shall also study the relation of the space decay properties of the global in time solution of (1) and the following stationary result of Meshkov [7]:

Let $w \in H^2_{loc}(\mathbb{R}^n)$ be a solution of

$$\Delta w + \widetilde{V}(x) w = 0, \quad x \in \mathbb{R}^n, \quad \text{with} \ \ \widetilde{V} \in L^\infty(\mathbb{R}^n). \tag{4}$$

$$\text{If} \quad \int e^{2a|x|^{4/3}} |w|^2 dx < \infty, \quad \forall\, a > 0, \ \text{ then } \quad w \equiv 0. \tag{5}$$

Moreover, the exponent $4/3$ in (5) is optimal for complex valued potentials $\widetilde{V}(x)$.

As an application we shall obtain results concerning the possible concentration profiles of blow up solutions and the possible profiles of the traveling waves solutions of semi-linear Schrödinger equations .

In addition, we shall describe a recent result obtained in collaboration with G. Fonseca concerning the Benjamin-Ono equation. More precisely, for the initial value problem associated to the Benjamin-Ono equation

$$\begin{cases} \partial_t u + \mathcal{H}\partial_x^2 u + u\partial_x u = 0, & t, x \in \mathbb{R}, \\ u(x,0) = u_0(x), \end{cases} \tag{6}$$

we establish sharp persistence properties of the solution flow in the weighted Sobolev spaces $H^s(\mathbb{R}) \cap L^2(|x|^{2r}dx)$, $s \in \mathbb{R}$, $s \geq 1$ and $s \geq r$. These generalize previous works of R. Iorio [5] and [6].

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MODELING HYDRODYNAMICS IN 1D

MARCUS WUNSCH

In this talk, we will present several model equations in one space dimension for the Euler equations in three and two space dimensions, respectively.

We will first discuss results on the generalized Constantin-Lax-Majda (gCLM) equation,

$$(1)\quad \begin{cases} \omega_t(t,x) + a\, v\omega_x = v_x\, \omega \qquad t>0, \quad a\in\mathbb{R}, \\ v_x(t,x) = H\omega(t,x) = (P.V.)\ \int_{-\pi}^{\pi}\ \omega(t,y)\ \cot\left(\frac{x-y}{2}\right)\ dy \\ \omega(0,x) = \omega_0(x), \qquad\qquad x\in\mathbb{S}^1\simeq\mathbb{R}/2\pi\mathbb{Z}. \end{cases}$$

This equation was introduced and analyzed by Okamoto, Sakajo & Wunsch (*Nonlinearity*, 2008).

If $a = 0$, (1) reduces to the well-known vorticity model equation $\omega_t = \omega H\omega$ of P. Constantin, Lax & Majda (1985), which has an abundance of solutions blowing up in finite time. In the presence of a convective derivative ($a = 1$), one obtains the vorticity model of De Gregorio (1990).

Finally, if $a = -1$, the gCLM equation (1) becomes the model equation of A. Córdoba, D. Córdoba & Fontelos (2005) for the 2D quasi-geostrophic equations and the Birkhoff-Rott equations describing the evolution of vortex sheets with surface tension.

A general, heuristic motivation for the study of the gCLM equation is the paradigm of Ohkitani & Okamoto (2005) that the *interplay* of convection $v\omega_x$ and stretching $v_x\omega$ leads to creation or depletion of finite-time singularities: the size of the parameter a in (1) thus reflects the impact of the convection.

As an illustration of the adequacy of the gCLM equation (1) for testing this paradigm, it can be shown that if $a = \infty$, corresponding to an "absolutely dominating" convection, solutions persist for all times. Moreover, we will demonstrate that there is a continuation criterion for (1) closely resembling the breakdown criterion of Beale, Kato & Majda (1984) for the incompressible Euler equations. Finally, it will be mentioned that the gCLM equation (1) with parameter $a = -1/2$ has an interesting geometric interpretation: It describes the geodesic

flow of a fractional Sobolev metric on the Lie group of orientation-preserving circle diffeomorphisms modulo rotations (cf. Wunsch, *J. Nonl. Math. Phys.* (2010)).

In the second part of the talk, we will discuss the generalized Proudman-Johnson (gPJ) equation on the real line, which can be regarded as a nonlocal perturbation of the Burgers equation:

$$\begin{cases} u_t(t,x) + \left(\frac{u^2}{2}\right)_x = \frac{\mathfrak{a}+3}{4} \left\{ \int_{-\infty}^{x} - \int_{x}^{\infty} \right\} u_x(t,\zeta)^2 \, d\zeta, \quad t > 0 \\ u(0,x) = u_0(x), \ x \in \mathbb{R}. \end{cases} \tag{2}$$

The equation for the axisymmetric Euler flow in 2D, corresponding to the case $\mathfrak{a} = 1$, was first derived by Proudman & Johnson (1962); it is obtained by separating the space variables in the stream function of the velocity vector $\mathbf{u}$ solving the incompressible Euler equations in 2D. Setting $\mathfrak{a} = -\frac{m-3}{m-1}$ in (2), one obtains the axisymmetric Euler flow in $\mathbb{R}^m$. Moreover, (2) reduces to the Hunter-Saxton equation (Hunter & Saxton (1991)) modeling orientation waves in nematic liquid crystals if $\mathfrak{a} = -2$, and to the Burgers equation from gas dynamics if $\mathfrak{a} = -3$.

Reviewing the papers of Cho & Wunsch (J. Differential Equations, 2010), Wunsch (*J. Math. Fluid Mech.*, 2009), and A. Constantin & Wunsch (*Proc. Japan Acad. Ser. A Math. Sci.*, 2009), we will present several new results on the initial value problem (2) and the periodic boundary problem for the gPJ equation,

$$\begin{cases} u_{txx} + u u_{xxx} = \mathfrak{a} u_x u_{xx} \\ u(0,x) = u_0(x), \quad x \in \mathbb{S}^1. \end{cases} \tag{3}$$

We will state a novel blowup criterion for (3) and show that certain geometric properties of the initial data u_0 are preserved for all times of existence. Moreover, we will see that a modification of the method of characteristics yields global weak solutions for (2) for certain parameter values of $\mathfrak{a}$.

In the final part of this presentation, we will see that both the gCLM (1) equation with $a = 0$ and the gPJ equation (2) are embedded in a wider family of a two-component systems: the generalized Hunter-Saxton system (Wunsch, *SIAM J. Math. Anal.* (2010))

$$\begin{cases} v_t(t,x) + v v_x = \left\{ \int_{-\infty}^{x} - \int_{x}^{\infty} \right\} \left[\frac{\alpha+2}{4} \, v_x(t,\zeta)^2 - \frac{\kappa}{4} \, w(t,\zeta)^2 \right] \, d\zeta, \\ \quad v(0,x) = v_0(x) \\ w_t(t,x) + v w_x = \alpha \, v_x(t,x) w(t,x), \quad t > 0, \\ \quad w(0,x) = w_0(x), \quad x \in \mathbb{R}, \end{cases} \tag{4}$$

where α and κ denote numerical constants. This system comprises the model equations of Hou & Li (2008) for the 3D axisymmetric Euler flow with swirl if $(\alpha, \kappa) = (1, 1)$, the Hunter-Saxton system modeling the nonlinear dynamics of non-dissipative dark matter if $(\alpha, \kappa) = (-1, \pm 1)$ (cf. Wunsch, *DCDS B* (2009)), the gPJ equation if $w = \sqrt{-1}\, v_x$ and $\mathfrak{a} = 2\alpha - 1$, and the CLM equation if $\alpha = \kappa = \infty$. We will give evidence that the periodic boundary problem corresponding to (4) not only has blowup solutions but also solutions existing for all times, and that on the real line there are global weak solutions as in the case for the gPJ equation (2).

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Global existence for supercritical wave equations with random initial data

ZHONG Sijia

In this talk, we will consider about the following nonlinear wave equations

$$\begin{cases} \partial_t^2 v - \triangle v + |x|^2 v + |v|^\alpha v = 0, \\ v(0) = f_1,\ \partial_t v(0) = f_2, \end{cases} \tag{0.1}$$

here $v : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$.

Our main result is

Theorem 0.1. *Suppose that $\alpha < \frac{4d}{(d+1)(d-2)}$ is positive. Let us fix a real number p such that $\max\{\frac{2(2d+3)\alpha}{12-(d-2)\alpha}, \frac{2(d+1)}{d-1}\} < p < \frac{2d}{d-2}$. Let $(h_n(w), l_n(w))_{n=0}^\infty$ be a sequence of independent random variables on a probability space $(\Omega, \mathcal{A}, p)$, in which h_n and l_n are standard Gaussian random variables. Consider (0.1) with radial initial data*

$$f_1^w = \sum_{n=1}^\infty \frac{h_n(w)}{\lambda_n} e_n,\ f_2^w = \sum_{n=1}^\infty l_n(w) e_n, \tag{0.2}$$

where (λ_n^2) is the eigenvalues of the harmonic oscillator $H = -\triangle + |x|^2$, $\lambda_n = \sqrt{2n+d}$, and $(e_n)_{n=0}^\infty$ is the orthonormal basis associated to λ_n^2. Then for every $s < 0$, almost surely in $w \in \Omega$, the problem (0.1) has a unique global solution

$$v^w \in C(\mathbb{R}_t, \mathcal{H}^s(\mathbb{R}^d)) \bigcap L^p(< t >^{-1} dt, \mathcal{W}^{\theta(p)-,p}(\mathbb{R}^d)),$$

with $\theta(p) = \frac{1}{3} - \frac{d}{3}(\frac{1}{2} - \frac{1}{p})$. $\mathcal{H}^s$ and $\mathcal{W}^{\theta(p)-,p}$ will be defined later.

Furthermore, the solution is a perturbation of the linear solution

$$v^w(t) = \cos(t\sqrt{H}) f_1^w + \frac{\sin(t\sqrt{H})}{\sqrt{H}} f_2^w + \tilde{v}^w(t),$$

where $\tilde{v}^w \in C(\mathbb{R}_t, \mathcal{H}^\sigma(\mathbb{R}^d))$ for some $0 < \sigma = \frac{1}{3} + \frac{d}{3} - \frac{2d+3}{3p}-$. Moreover

$$\|v^w\|_{\mathcal{H}^s(\mathbb{R}^d)} \le C(w, s) \ln(2 + |t|)^{\frac{1}{2}}. \tag{0.3}$$

Remark 0.2. *By the result of this Theorem, we can see that, for $s < 0$, the critical α is smaller than $\frac{4}{d}$ which is strictly smaller than $\frac{4d}{(d+1)(d-2)}$. So for $\frac{4}{d} < \alpha < \frac{4d}{(d+1)(d-2)}$, it is supercritical, which means when we choose some special kind of the initial data, the result would be better. In particular, for $d = 2$, the theorem holds for any $\alpha > 0$.*

Remark 0.3. *By the same idea of [5] Lemma 3.2, (please also refer to Lemma of our paper), we can see that almost surely,*

$$(f_1^w, f_2^w) \in \bigcap_{s<0} (\mathcal{H}^s(\mathbb{R}^d) \times \mathcal{H}^{s-1}(\mathbb{R}^d)),$$

but the probability of the event $\{(f_1^w, f_2^w) \in \mathcal{H}^0(\mathbb{R}^d) \times \mathcal{H}^{-1}(\mathbb{R}^d)\}$ is zero. Thus the randomization process has no smoothing property in the scale of $\mathcal{H}^s$ regularity, and in the above statement we obtain global solutions for data which are not in $\mathcal{H}^0(\mathbb{R}^d) \times \mathcal{H}^{-1}(\mathbb{R}^d)$. On the other hand, our result is not a "small data result".

Remark 0.4. *By the result of Koch and Tatăru [9], this theorem might hold for any $V(x)$ that is radial and behaves like $|x|^2$ for $|x| \to \infty$, for example $< x >^2$. For the sake of conciseness, we just state the special case of $V(x) = |x|^2$.*

By the previous work [6], Burq and Tzvetkov have developed a general theory for constructing local strong solutions to nonlinear wave equations, posed on compact Riemannian manifolds with supercritical random initial data. Then in [7], they showed that in a particular case, which is the nonlinear wave equation with Dirichlet boundary condition posed on the unit ball of $\mathbb{R}^3$, there would be global solutions by combining the local theory with some invariant measure arguments in [1], [2], [10], [12] and [5]. Thomann in [11] got some local well posedness for the Schrödinger equation with a confining potential on the whole space, and then extended it to the one without the potential. Then recently, Burq, Thomann and Tzvetkov in [4] proved the global existence of solutions of Schrödinger equations with random initial data in $\mathbb{R}$. The purpose of our paper is considering global strong solution of the wave equation with the harmonic potential on the whole space. So we will use some idea from [6], [7], [11], [4] and so on. But first of all, we need to prove the Strichartz estimate for (0.1).

Let us consider about the linear wave equation without the potential term first, i.e.

$$\begin{cases} \partial_t^2 v - \triangle v = 0 \\ v(0) = v_0,\ \partial_t v(0) = v_1, \end{cases}$$

then, there is some Strichartz estimate:

$$\|v\|_{L^p((0,T),L^q(\mathbb{R}^d))} \leq C(\|v_0\|_{H^s(\mathbb{R}^d)} + \|v_1\|_{H^{s-1}(\mathbb{R}^d)}), \tag{0.4}$$

where $H^s(\mathbb{R}^d)$ is the usual Sobolev space, and admissible pair (p,q) satisfies $2 \le p \le \infty$, $2 \le q < \infty$ and

$$\frac{1}{p} + \frac{d}{q} = \frac{d}{2} - s, \quad \frac{2}{p} + \frac{d-1}{q} \le \frac{d-1}{2}. \tag{0.5}$$

There are lots of results about Strichartz estimates of the above type on the whole space $\mathbb{R}^d$, compact manifolds with or without boundary, noncompact manifolds and spaces with other geometric conditions.

It is well known that there are some similar properties between the problem on the compact manifolds with the one associated to the harmonic oscillator, so what about our case?

Theorem 0.5. *For $x \in \mathbb{R}^d$, (p_1, q_1), (p_2, q_2) satisfying (0.5), and*

$$\frac{1}{p_1} + \frac{d}{q_1} = \frac{d}{2} - s = \frac{1}{p_2'} + \frac{d}{q_2'} - 2,$$

we have the following estimates for solutions v to (0.1)

$$\|v\|_{L^{p_1}((0,1),L^{q_1}(\mathbb{R}^d))} \le C(\|f_1\|_{\mathcal{H}^s(\mathbb{R}^d)} + \|f_2\|_{\mathcal{H}^{s-1}(\mathbb{R}^d)} + \|F\|_{L^{p_2'}((0,1),L^{q_2'}(\mathbb{R}^d))}), \tag{0.6}$$

here F is the nonlinear term of the equation.

Remark 0.6. *Our result is uniformly with respect to time.*

Remark 0.7. *This result is not only right for $|x|^2$, but also for any $V(x) = \sum_{j=1}^d a_j x_j^2$, with $a_j > 0$ and even some $V(x)$ behaving roughly like $|x|^2$, for example $< x >^2$.*

To prove this Theorem, we will use the idea from [8] and so on. First, we do the dyadic decomposition by the idea of [3], and reduce the problem to a fixed frequency. Then, we try to write out the approximation expression of the operator $e^{-it\sqrt{H}}$ ($H = -\triangle + |x|^2$). By calculating the dispersion of the operator, the result of Theorem is gained by applying the idea of Keel and Tao [9].

The difference between the proof of (0.4) with (0.6) is that there are cases that the growth of $|x|$ might be much larger than $|\xi|$. Fortunately, for this cases, by estimating the Hessian Matrix, the dispersive effect would even be better.

By the above Theorem, we will prove Theorem 0.1 by the idea of [7]. However, there are some points we should pay attention to. First, without the periodic condition, we show that there is some decaying of time t, i.e. $v^w \in L^p(< t >^{-1} dt, \mathcal{W}^{\theta(p)-,p}(\mathbb{R}^d))$. This would be enough to get the global result and could be applied to more general cases. Secondly, because we are dealing with the whole space case, there are some differences in the interpolation theory.

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Family of two-dimensional ideal fluid dynamics related to surface quasi-geostrophic equation

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Abstract

We study 2D surface quasi-geostrophic (SQG) equation numerically and theoretically. After reviewing recent results, we consider a generalised class of equations of ideal fluid, where the active scalar is a fractional power α of Laplacian applied to the stream function. This includes 2D SQG and 2D Euler equations as special cases. We present some numerical results of the generalised system and compare them for some different values of α. In an attempt to unify the whole family systematically, a successive approximation is introduced to treat the SQG equation.

I. INTRODUCTION

Mathematical study on the SQG equation was initiated in [1, 2]. Since then many papers have been published regarding the analyses of this equation, which are too numerous to cite here. Numerical studies have been done, e.g. in [1–6]. Mathematically, the following is the best result known for its regularity. We consider the SQG equation with hypo-viscous dissipativity either in $\mathbb{R}^2$ or in $\mathbb{T}^2$

$$\frac{\partial \theta}{\partial t} + (\boldsymbol{u} \cdot \nabla)\theta = -\nu(-\triangle)^{\gamma}\theta \quad (0 \leq \gamma \leq 1),$$

with an initial datum $\theta(\boldsymbol{x}, 0) = \theta_0(\boldsymbol{x})$. The velocity $\boldsymbol{u} = -\nabla^{\perp}(-\triangle)^{-1/2}\theta$ is a skewed Riesz transform of θ, where $\nabla^{\perp} = (\partial_y, -\partial_x)$. It has been proved that when $\gamma \geq \frac{1}{2}$ we have no blow-up [7, 8]. The hypo-viscous equation has been studied numerically in [9]. See also [10] for more related works.

II. GENERALISED SQG EQUATION FOR INVISCID FLUIDS

We consider a generalised version of SQG equation [3, 11] for inviscid fluids

$$\frac{\partial \theta}{\partial t} + (\boldsymbol{u} \cdot \nabla)\theta = 0, \tag{1}$$

with $\theta(\boldsymbol{x}, 0) = \theta_0(\boldsymbol{x})$. Here the velocity $\boldsymbol{u}$ is given by

$$\boldsymbol{u} = \nabla^{\perp}\psi, \quad \Lambda^{\alpha}\psi = \theta \quad (0 \leq \alpha \leq 2).$$

Here $\Lambda \equiv (-\triangle)^{1/2}$ is Zygmund operator defined by Fourier transform $\tilde{\Lambda} = |\boldsymbol{k}|$. The system reduces to the 2D Euler equations if $\alpha = 2$, to the 2D SQG equation if $\alpha = 1$, and to a trivially steady state if $\alpha = 0$.

III. PERTURBATION THEORY: ODE ANALOGY

We recall a perturbation theory à la Poincaré of an ordinary differential equation (ODE) which depends upon a parameter μ, see e.g. [12, 13]. (We note that notations used in this section are independent from those in the rest of the extended abstract.)

Consider an ODE

$$\frac{dy}{dx} = f(x, y, \mu), \text{ with an initial datum } y(x_0, \mu) = y_0,$$

which is assumed to be solvable for $\mu = \mu_0$. If we consider a variation

$$z(x, \mu) = \frac{\partial y(x, \mu)}{\partial \mu}, \text{ with an initial datum } z(x_0, \mu) = 0,$$

it satisfies

$$\frac{dz}{dx} = \left.\frac{\partial f(x, Y, \mu)}{\partial Y}\right|_{Y=y(x,\mu)} z + \left.\frac{\partial f(x, Y, \mu)}{\partial \mu}\right|_{Y=y(x,\mu)},$$

which is called an equation of variation.

An approximation for $y(x, \mu)$ for small $|\mu - \mu_0|$ may be written

$$y(x, \mu) - y(x, \mu_0) = \sum_{n=1}^{\infty} (\mu - \mu_0)^n C_n(x),$$

where $C_n(x)$ are suitable coefficients, e.g.

$$z(x, \mu_0) = \lim_{\mu \to \mu_0} \frac{y(x, \mu) - y(x, \mu_0)}{\mu - \mu_0} = C_1(x).$$

IV. SUCCESSIVE APPROXIMATIONS

We apply the above idea to the generalised SQG equation. We illustrate how this is done for the first variation. If we take the variation of (1) with respect to α, we find

$$\frac{D}{Dt}\frac{\partial \theta}{\partial \alpha} \equiv \frac{\partial}{\partial t}\frac{\partial \theta}{\partial \alpha} + \boldsymbol{u} \cdot \nabla \frac{\partial \theta}{\partial \alpha} = -\frac{\partial \boldsymbol{u}}{\partial \alpha} \cdot \nabla \theta.$$

In $\mathbb{R}^2$, we find more explicitly after straightforward manipulations [14]

$$\frac{D}{Dt}\frac{\partial\theta}{\partial\alpha} = \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{(\boldsymbol{x}-\boldsymbol{y})^\perp}{|\boldsymbol{x}-\boldsymbol{y}|^2}\frac{\partial\theta(\boldsymbol{y})}{\partial\alpha}d\boldsymbol{y}\cdot\nabla\theta(\boldsymbol{x}) + \frac{1}{4\pi}\int_{\mathbb{R}^2}\left(\log|\boldsymbol{x}-\boldsymbol{y}|\right)^2\nabla^\perp\theta(\boldsymbol{y})d\boldsymbol{y}\cdot\nabla\theta(\boldsymbol{x}).$$

In principle, the equations for higher-order variations may be obtained by successive differentiations. Given these, we may write, for example, near the 2D Euler limit $\alpha = 2$

$$\theta(\boldsymbol{x},t,\alpha) = \theta(\boldsymbol{x},t,2) + \sum_{n=1}^{\infty}(\alpha-2)^n\theta_n(\boldsymbol{x},t),$$

where $\theta_n(\boldsymbol{x},t) \equiv \frac{\partial^n\theta}{\partial\alpha^n}(\boldsymbol{x},t)$.

V. CONCLUSION

In fact, under periodic boundary conditions we can carry out the analyses more systematically. A formal analysis in this case indicates that all the members in the family behave similarly with respect to a 'new time variable' $\xi = \alpha t$. We discuss the implications of this scaling, in connection with numerical simulations. These are to be reported in detail in [14].

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Finite volume method for degenerate diffusion problems

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1 Introduction

The finite volume method (FVM) is a discretization method based on local conservation properties of equations so that it is well suited for PDEs of conservation laws. Although the range of application seems to be smaller than that of the finite element method (FEM), FVM has its own advantages. For example, FVM naturally satisfies the discrete maximum principle, if it is applied to a linear diffusion problem. We recall that the discrete maximum principle in FEM holds only when some shape conditions on the triangulation are satisfied, and such a restriction often causes some difficulties. In this paper, we shall reveal another advantage of FVM through the degenerate diffusion problems and the nonlinear semigroup theory.

The purpose of this paper is to report some operator theoretical properties of FVM applied to a degenerate elliptic equation of the form $u-\lambda\Delta f(u)=g$ for $\lambda>0$ and $g\in L^1(\Omega)$ under the homogeneous Dirichlet boundary condition. The function f is assumed to be continuous and non-decreasing with $f(0)=0$. As is well-known, L^1 theory of Brezis and Strauss ([2]) is of great use to deal with this problem. Below, we shall see that FVM is a suitable discretization method for this problem in the sense that the discrete version of [2] can be applied. Consequently, we immediately deduce the generation of the nonlinear semigroup, namely, the unique existence of a time global solution to a semidiscrete (in space) FVM for a degenerate parabolic equation of the form $u_t-\Delta f(u)=0$. Then, we readily obtain stability results in L^1 and L^∞, and order-preserving property for finite volume solutions by the nonlinear semigroup theory. This is totally new approach to study FVM for degenerate elliptic and parabolic problems.

As an application, we shall consider a degenerate Keller-Segel system of chemotaxis. We shall propose a FVM that preserves the conservation of positivity and total mass. The time discretization makes use of the forward Euler method, and some numerical examples will be presented.

Remarks. (1) There are several classes of FVM. We shall concentrate our attention to a cell-centered classical finite volume method described in [4].
(2) Though we shall restrict our consideration to the two dimensional polygon in what follows, it is not difficult to extend those results to smooth domains and the three dimensional cases.

2 Degenerate parabolic equation and FVM

We consider the finite volume approximation applied to the initial-boundary value problem for a degenerate parabolic equation,

$$\begin{cases} u_t - \Delta f(u) = 0 \quad \text{in} \quad \Omega \times (0,T), \\ u = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \qquad u|_{t=0} = u_0(x) \quad \text{on} \quad \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ denotes a polygonal domain, T an arbitrary positive constant, and f a non-decreasing continuous function defined on $\mathbb{R}$ satisfying $f(0) = 0$. As is well-known, Problem (1) describes, for instance, the flow of homogeneous fluid in porous media, the fast (singular) diffusion problem, and the two phase Stefan problem in enthalpy formulation.

Supposing that $\overline{\Lambda}$ is an index set (set of finite number of positive integers), we let $\mathscr{D} = \{D_i\}_{i\in\overline{\Lambda}}$ be a set of open convex polygonal subsets in Ω satisfying the following conditions (see, for example, Fig. 1 and 2):

(A1) $\overline{\Omega} = \cup\{\overline{D_i} \mid i \in \overline{\Lambda}\}$.

(A2) Any $\overline{D_i}$ and $\overline{D_j}$ with $i \neq j$ meet only in entire common sides or in vertices.

(A3) There exists a set of points $\{P_i\}_{i\in\overline{\Lambda}}$ such that $P_i \in \overline{D_i}$ and $P_i \notin D_j$ with $j \neq i$. Further, the line segment connecting P_i with P_j is orthogonal to the line including σ_{ij}, if $\overline{D_i}$ and $\overline{D_j}$ share a common side σ_{ij}.

(A4) If there is a side σ of D_i such that $\sigma \subset \partial\Omega$, the accompanying point P_i is in $\partial\Omega$.

Following [4], we consider a family $\{\mathscr{D} = \mathscr{D}_h\}_h$ of $\mathscr{D}$'s above and call it the *admissible meshes* of Ω, where $h = h_{\mathscr{D}} = \max\{\text{diam}\,(D_i) \mid i \in \overline{\Lambda}\}$ is the granularity parameter. Moreover we call D_i the *control volume*. We let $\overline{\Lambda} = \Lambda \cup \partial\Lambda$, where $\partial\Lambda = \{i \in \overline{\Lambda} \mid \text{the length of } (\partial D_i \cap \partial\Omega) > 0\}$ and $\Lambda = \overline{\Lambda}\backslash\partial\Lambda$. Further we set $\Lambda_i = \{j \in \overline{\Lambda} \mid \overline{D_i} \text{ and } \overline{D_j} \text{ share a common side } \sigma_{ij}\}$. Let ψ_i be the characteristic function of D_i for any $i \in \overline{\Lambda}$. Then, we introduce sets of piecewise constant functions

$$X_h = \text{span}\,\{\psi_i\}_{i\in\overline{\Lambda}}, \qquad V_h = \{v_h \in X_h \mid v_h(P_i) = 0 \ (i \in \partial\Lambda)\}.$$

In what follows, we write v_i to express $v_h(P_i)$ for $v_h \in X_h$ and $i \in \overline{\Lambda}$.

Now, we can state a semidiscrete (in space) finite volume approximation for (1): find $u_h \in C^1([0,T];V_h)$ such that

$$\begin{cases} m_i \dfrac{d}{dt} u_i(t) = \displaystyle\sum_{j\in\Lambda_i} \gamma_{ij} \left[f(u_j(t)) - f(u_i(t)) \right] \quad (i \in \Lambda, t \in (0,T)), \\ u_i(0) = u_{0,i} \equiv \dfrac{1}{m_i} \displaystyle\int_{D_i} u_0(x)\, dx \quad (i \in \partial\Lambda), \end{cases} \tag{2}$$

where $m_i =$ the area of D_i, and $\gamma_{ij} =$ the transmissibility $= m_{ij}/d_{ij}$. ($m_{ij} =$ the length of σ_{ij}, and $d_{ij} =$ the distance from P_i to P_j.)

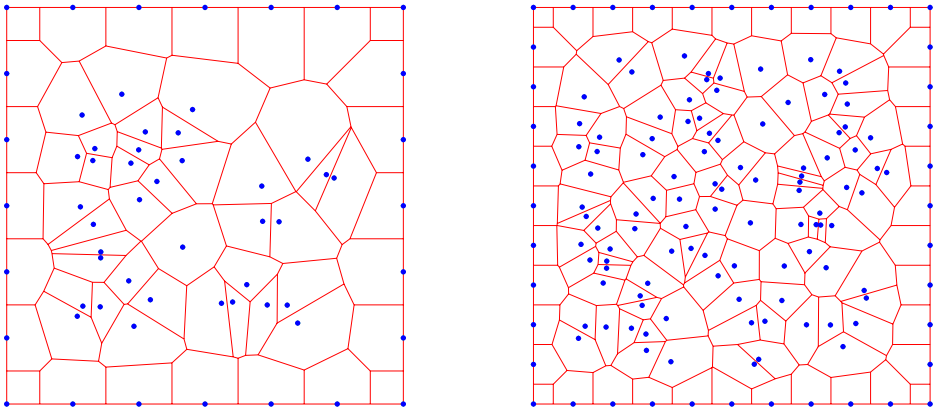

Figure 1: Admissible meshes by Voronoi diagram.

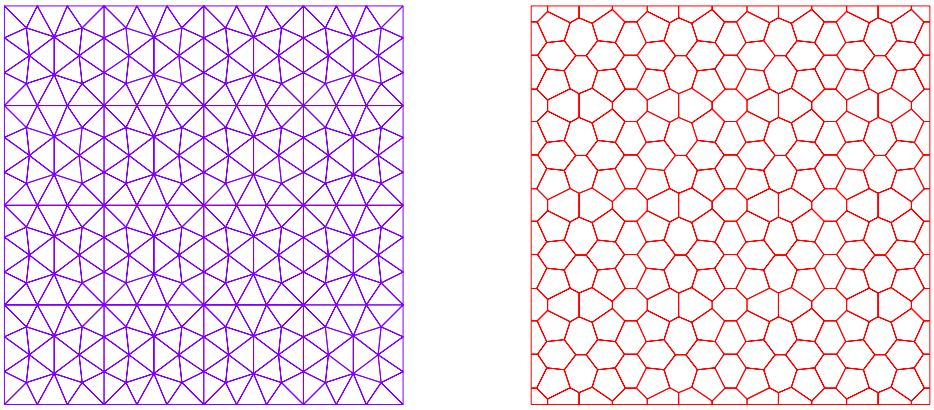

Figure 2: (left) Acute triangulation; (right) Admissible mesh based on circumcentric dual mesh of the acute triangulation.

We introduce an operator $A_h : V_h \to V_h$ defined as

$$(A_h v_h)(P_i) = -\frac{1}{m_i} \sum_{j \in \Lambda_i} \gamma_{ij} \left[f(v_j) - f(v_i) \right] \quad (i \in \Lambda)$$

for $v_h \in V_h$. Then, Problem (2) is equivalent to

$$\frac{d}{dt} u_h(t) + A_h u_h(t) = 0 \quad (0 < t < T), \qquad u_h(0) = u_{0,h}. \tag{3}$$

At this stage, we recall the L^1 theory to (1) that was developed in early 1970's in use of nonlinear semigroup. To summarize it, we set $V = L^1(\Omega)$ and introduce operators L and A in V by $Lv = -\Delta v$ for $v \in D(L) = \{v \in W_0^{1,1}(\Omega) \mid Lv \in V\}$ and $Av = L f(v)$ for $v \in D(A) = \{v \in V \mid f(v) \in D(L)\}$, respectively. Then, Problem (1) is reduced to the nonlinear evolution equation in V:

$$\frac{d}{dt} u(t) + Au(t) = 0 \quad (0 < t < T), \qquad u(0) = u_0. \tag{4}$$

It is proved in Brezis and Strauss [2] that the operator $-A$ is *m-dissipative* in V. This means that $R(I + \lambda A) = \overline{D(A)} = V$ and also

$$\|v - \hat{v}\|_1 \le \|v - \hat{v} + \lambda A v - \lambda A \hat{v}\|_1 \quad (v, \hat{v} \in D(A);\ \lambda > 0),$$

where $\|\cdot\|_p = \|\cdot\|_{L^p(\Omega)}$ for $1 \le p \le \infty$. Then, we can apply theory of Crandall and Liggett [3] to obtain the generation of semigroup $\{S(t)\}_{t\ge 0}$ on V by

$$S(t) = \underset{m\to\infty}{s\text{-}\lim} \left(I + \frac{t}{m}A\right)^{-m},$$

and $u(t) = S(t)u_0$ is regarded as the solution of (4). Another important property of A is the order-preserving, that is,

$$(I+\lambda A)^{-1} g \ge (I+\lambda A)^{-1}\hat{g} \qquad (g, \hat{g} \in V \text{ s.t. } g \ge \hat{g};\ \lambda > 0).$$

It is also proved in [2] that the L^∞ stability of the resolvent

$$\left\|(I+\lambda A)^{-1} g\right\|_\infty \le \|g\|_\infty \quad (g \in L^\infty(\Omega);\ \lambda > 0)$$

holds. This implies L^∞ stability of the semigroup

$$\|S(t)u_0\|_\infty \le \|u_0\|_\infty \quad (u_0 \in L^\infty(\Omega)).$$

Our first purpose is to prove the nonlinear finite volume operator A_h has analogous properties with the nonlinear operator A, which allows us to apply nonlinear semigroup theory in analysis of the finite volume method. Actually, we have the following.

Theorem 1. *For any $\lambda > 0$, the operator A_h has the following properties:*

(i) $R(I+\lambda A_h) = V_h$.

(ii) $\|v_h - \hat{v}_h\|_1 \le \|v_h - \hat{v}_h + \lambda A_h v_h - \lambda A_h \hat{v}_h\|$ *for any* $v_h, \hat{v}_h \in V_h$.

(iii) $(I+\lambda A_h)^{-1} g_h \ge (I+\lambda A_h)^{-1}\hat{g}_h$ *for* $g_h, \hat{g}_h \in V_h$ *such that* $g_h \ge \hat{g}_h$.

(iv) $\|(I+\lambda A_h)^{-1} g_h\|_\infty \le \|g_h\|_\infty$ *for* $g_h \in V_h$.

Then, we immediately deduce the following corollary.

Corollary 1. (i) *The operator $-A_h$ is m-dissipative in V_h with respect to the L^1 norm. Therefore, Problem (2) is uniquely solvable globally in time and the solution is given as $u_h(t) = S_h(t)u_{0,h}$, where*

$$S_h(t) = \lim_{m\to\infty} \left(I + \frac{t}{m}A_h\right)^{-m}.$$

(ii) $\|S_h(t)u_{0,h} - S_h(t)\hat{u}_{0,h}\|_1 \le \|u_{0,h} - \hat{u}_{0,h}\|_1$ *for* $u_{0,h}, \hat{u}_{0,h} \in V_h$ *and* $t \in [0,T]$.

(iii) $S_h(t)u_{0,h} \ge S_h(t)\hat{u}_{0,h}$ *for* $u_{0,h}, \hat{u}_{0,h} \in V_h$ *such that* $u_{0,h} \ge \hat{u}_{0,h}$ *and* $t \in [0,T]$.

(iv) $\|S_h(t)u_{0,h}\|_\infty \le \|u_{0,h}\|_\infty$ *for* $u_{0,h} \in V_h$ *and* $t \in [0,T]$.

In a previous paper, Mizutani et al. [6], we proposed a semidiscrete (in space) finite element approximation provided with order-preserving and L^1 contraction properties, making use of piecewise linear trial functions and the lumping mass technique. The crucial step of analysis was to prove that the finite element approximation of A has analogous properties of (i)–(iv) above. However, we could not follow the method of [2], since we confronted some issues. For example, $f(v)$ is not a piecewise linear function, even if v is a piecewise linear function. So, we had to take totally different approach from [2]. For example, we used a discrete Kato's inequality to prove the L^1 contraction property, and we used the nonlinear Chernoff formula and a special time-discretization of [1] to prove the L^∞ stability of (discrete) semigroup. Consequently, the proof was long and intricate.

The proof of Theorem 1, however, can be done in the essentially similar way as [2]. Thus, in this sense, the finite volume approximation is a suitable discretization method for the operator $\Delta f(u)$.

The second purpose of this paper is to make error analysis. The goal of this end is to derive

$$\lim_{h\downarrow 0} \sup_{t\in[0,T]} \|u_h(t) - u(t)\|_1 = 0. \tag{5}$$

In fact, we have the following result in the similar manner as [6].

Theorem 2. *If Ω is a convex polygon, u_0 is continuous on $\overline{\Omega}$ with the boundary value zero on $\partial\Omega$, f is strictly increasing continuous function with $f(0) = 0$, and the admissible mesh is regular in the sense of* [4]*, then we have the convergence of the semigroup* (5).

3 Degenerate Keller-Segel system

As an application of the previous consideration, we consider the finite volume approximation for a degenerate Keller-Segel system,

$$\begin{cases} u_t = \nabla\cdot(\nabla f(u) - u\nabla\varphi(v)) & \text{in} \quad \Omega\times(0,T), \\ kv_t = D_v\Delta v - k_1 v + k_2 u & \text{in} \quad \Omega\times(0,T), \\ \dfrac{\partial}{\partial\nu} f(u) - u\dfrac{\partial}{\partial\nu}\varphi(u) = 0, \quad \dfrac{\partial v}{\partial\nu} = 0 & \text{on} \quad \partial\Omega\times(0,T), \\ u|_{t=0} = u_0, \quad v|_{t=0} = v_0 & \text{on} \quad \Omega, \end{cases} \tag{6}$$

where Ω denotes a polygonal domain in $\mathbb{R}^2$, f and φ are non-decreasing continuous functions defined on $\mathbb{R}$ with $f(0) = 0$, ν is the outer unit normal vector to $\partial\Omega$, and D_v, k, k_1, k_2, T are positive constants. In the non-degenerate case $f(u) = D_u u$ with a positive constant D_u, Problem (6) describes the aggregation of slime molds resulting from their chemotactic features. Here, u is defined to be the density of the cellular slime molds, v the concentration of the chemical substance secreted by molds themselves, k the relaxation time, $\varphi(v)$ the sensitive function, and $k_1 v - k_2 u$ the ratio of

generation/extinction. We have developed conservative finite element methods for the non-degenerate case, cf. [7], [8]. Our schemes made use of Baba-Tabata's upwind technique combined with the mass-lumping based on the barycentric domain and a semi-implicit time discretization with a time-increment control. That is, at every discrete time step $t_n = \Delta t_1 + \cdots + \Delta t_n$, we adjust the time-increment Δt_n in order to obtain a positive solution. Consequently, our finite element approximations have positivity and mass conservation properties which are important features of the original system. Furthermore, we succeeded in establishing optimal/quasi-optimal error estimates in $L^p \times W^{1,\infty}$ with a suitable $p > 2$.

We shall propose a finite volume scheme for the degenerate case (6) that preserves the conservation of positivity and total mass. The time discretization makes use of the forward Euler method. (Our scheme may be regarded as the fully explicit version of Filbet's one [5].) Some numerical results will be also presented.

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