

# Scale-Invariant Extinction Time Estimates for Some Singular Diffusion Equations

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## Abstract

We study three singular parabolic evolutions: the second-order total variation flow, the fourth-order total variation flow, and a fourth-order surface diffusion law. Each has the property that the solution becomes identically zero in finite time. We prove scale-invariant estimates for the extinction time, using a simple argument which combines an energy estimate with a suitable Sobolev-type inequality.

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# 1 Introduction

We shall discuss three singular parabolic PDEs: the second-order total variation flow

$$u_t = \operatorname{div} \left( \frac{\nabla u}{|\nabla u|} \right), \quad (1.1)$$

the fourth-order total variation flow

$$u_t = -\Delta \left[ \operatorname{div} \left( \frac{\nabla u}{|\nabla u|} \right) \right], \quad (1.2)$$

and the fourth-order surface diffusion law

$$u_t = -\Delta \left[ \operatorname{div} \left( \frac{\nabla u}{|\nabla u|} + \mu |\nabla u|^{q-2} \nabla u \right) \right] \quad (1.3)$$

(with  $\mu > 0$  and  $q > 1$ ). Our goal is to prove finite-time extinction, i.e. to show that the solution becomes identically zero in finite time, and to give scale-invariant estimates for the extinction time. We focus on the spatially periodic setting, but we also consider other boundary conditions. In the periodic setting the initial data  $u_0$  should have mean value 0; since  $u_t$  is a divergence, the mean value is independent of time.

Equations (1.1)–(1.3) should not be taken literally, since right hand sides are undefined when  $\nabla u = 0$ . The rigorous definition of the total variation flow is familiar: it is the  $L^2$  steepest-descent associated with the BV seminorm  $\int |\nabla u|$ . The definition of the fourth-order total variation flow is similar: it is the  $H^{-1}$  steepest-descent for the BV seminorm. The surface diffusion law is the  $H^{-1}$  steepest-descent for

$$\int |\nabla u| + \frac{\mu}{q} |\nabla u|^q. \quad (1.4)$$

These definitions will be discussed in Sections 2 and 3.

The finite-time extinction of solutions to (1.3) has been noticed in the materials science literature, where this equation (with  $q = 3$ ) has been proposed as a continuum model for the evolution of a crystal surface below the roughening temperature in the “diffusion-limited” regime, see e.g. [CRSC, HS, M, RV, SRF, SRRTCC]. The adequacy of this model is somewhat controversial, since this PDE approach to facetting has yet to be derived as the continuum limit of a more microscopic model. Numerical simulation has been the main technique for assessing the implications of (1.3). Simulation has “shown” that  $u$  becomes constant at a finite time  $T^*$ , and that  $\|u\|$  decreases linearly as  $t \uparrow T^*$ . However, to our knowledge there has been no mathematical analysis of finite-time extinction for either of the 4th-order equations (1.2) or (1.3), and it is difficult to see from simulations how the extinction time depends on the details of the initial data.

There is a substantial body of work on finite-time extinction for solutions of the second-order total variation flow (1.1); see Remark 2.7 for a brief summary. But that work relies on intrinsically second-order techniques (comparison and maximum principles). Our approach is quite different: the main tools are “energy estimates” and Sobolev inequalities. While our approach has not previously

been applied to the second-order total variation flow, similar arguments have been used for other second-order problems; see Remark 2.8 for specific references.

Our focus is mainly on “scale-invariant estimates.” To explain what this means, let us focus on the fourth-order total variation flow. It has two scale invariances,  $t \rightarrow \lambda t, u \rightarrow \lambda u, x \rightarrow x$  and  $t \rightarrow \lambda^4 t, u \rightarrow \lambda u, x \rightarrow \lambda x$ . The first invariance shows that the extinction time  $T^*(u_0)$  is positively homogeneous of degree one, i.e.  $T^*(\lambda u_0) = \lambda T^*(u_0)$  where  $u_0$  is the initial condition. This suggests an extinction time estimate of the form

$$T^* \leq C \|u_0\|_X.$$

If the constant  $C$  is unchanged when we scale the domain, then the second invariance restricts the character of the norm on the right hand side. Scale invariance is especially important for the analysis of surface diffusion, because we often use periodic boundary conditions to minimize finite-size effects. In this setting the period is arbitrary; therefore a physically meaningful estimate should not depend on it.

To capture the main idea of our analysis, let us sketch how it works for the second-order total variation flow with periodic boundary conditions in space dimension 2. The key observation is that

$$\|u\|_{L^2(\mathbf{T}^2)}(t) \leq \|u_0\|_{L^2(\mathbf{T}^2)} - Ct \text{ for } t < T^*(u_0), \quad (1.5)$$

where  $\mathbf{T}^2$  is the period cell,  $u_0$  is the initial condition (assumed to have mean value 0), and the constant  $C$  is scale-invariant. This clearly implies the extinction time estimate  $T^* \leq C^{-1} \|u_0\|_{L^2(\mathbf{T}^2)}$ . (For the rigorous version of the argument that follows, see Theorem 2.4.) Multiplying (1.1) by  $u$ , integrating over the period cell, then integrating by parts gives (in any dimension)

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbf{T}^2} |u|^2 dx = - \int_{\mathbf{T}^2} |\nabla u|.$$

The scale-invariant Sobolev inequality  $(\int_{\mathbf{T}^n} |u|^{n/(n-1)} dx)^{(n-1)/n} \leq S_n \int_{\mathbf{T}^n} |\nabla u|$  becomes, when  $n = 2$ ,

$$\left( \int_{\mathbf{T}^2} |u|^2 dx \right)^{1/2} \leq S_2 \int_{\mathbf{T}^2} |\nabla u|.$$

These inequalities combine to give (1.5) with  $C = S_2^{-1}$ .

Our analysis of the fourth-order equations (1.2) and (1.3) uses a similar technique, starting with the estimate (in any dimension  $n$ )

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbf{T}^n} u(-\Delta^{-1}u) dx \leq - \int_{\mathbf{T}^n} |\nabla u|.$$

where  $\mathbf{T}^n$  is the period cell. In space dimension 4 the argument is almost parallel to the one just given, and the scale-invariant estimate is

$$T^* \leq C \|u_0\|_{H^{-1}(\mathbf{T}^4)}$$

(see Theorem 3.3). In dimensions  $n < 4$  we must work a bit more, and the scale-invariant estimate of  $T^*$  involves interpolation between two negative norms of  $u_0$  (see Theorems 3.8 and 4.3). We do not prove finite-time extinction for the fourth-order problems in dimension  $n \geq 5$ .

Our surface diffusion law (1.3) is a special case of the more general equation

$$u_t = -\operatorname{div} \left[ M(\nabla u) \nabla \operatorname{div} \left( \frac{\nabla u}{|\nabla u|} + \mu |\nabla u|^{q-2} \nabla u \right) \right]. \quad (1.6)$$

Equations of this type (with specific formulas for the “mobility”  $M(\nabla u)$ ) have been “derived” as continuum limits of step motion laws away from the vicinity of a facet, see e.g. [CRSC, MK, OZ]. It is natural to expect that the solution of (1.6) should have finite-time extinction. However our method seems to work only when  $M$  is constant.

The paper is organized as follows. Section 2 presents our results on the second-order total variation flow. In space dimension  $n > 2$ , the analogue of (1.5) is  $\|u\|_{L^n}(t) \leq \|u_0\|_{L^n} - S_n^{-1}t$  (see Theorem 2.4). The one-dimensional case is special, because the solution can be made more or less explicit. Using this, we show in Section 2.5 that finite-time extinction can fail for the Cauchy problem if the initial data decays slowly enough at infinity.

Section 3 presents our results on the fourth-order total variation flow. The main results are Theorems 3.3 and 3.8, which give scale-invariant estimates for the extinction time in the periodic setting, for space dimensions  $n = 4$  and  $n < 4$  respectively. We also discuss some non-scale-invariant estimates (Theorem 3.11), and we briefly discuss the situation for Dirichlet or Neumann boundary conditions (Section 3.6).

Section 4 presents our results on the fourth-order surface diffusion law. As noted earlier, that equation represents the  $H^{-1}$  steepest-descent for  $\int |\nabla u| + \frac{\mu}{q} |\nabla u|^q$ , whereas the fourth-order total variation flow is  $H^{-1}$  steepest descent for  $\int |\nabla u|$ . The presence of  $|\nabla u|^q$  in the energy has a big effect on the qualitative properties of solutions, since it prevents the formation of discontinuities (which can indeed form when  $\mu = 0$  [GG10]). However the presence of  $|\nabla u|^q$  seems to have little effect on the extinction time: our analysis and the resulting estimates are only slightly different from those in Section 3.

## 2 Warming up: the second-order total variation flow

In this section we prove scale-invariant extinction time estimates for the second-order total variation flow. We begin by reviewing the sense in which this evolution is the  $L^2$  gradient flow of the BV seminorm. This interpretation is well-known, see e.g. [ACM] and [GGK]. An informal discussion of gradient flow in more or less the present setting can be found in the review article [KG].

### 2.1 Abstract framework

Let  $H$  be a real Hilbert space equipped with an inner product  $\langle \cdot, \cdot \rangle$ , and let  $\Phi$  be a convex, lower semicontinuous function on  $H$  with nonempty domain  $D(\Phi)$ . It is well-known (see e.g. [Ko], [Br], [Ba]) that the initial value problem

$$\frac{du}{dt}(t) \in -\partial\Phi(u(t)) \text{ for a.e. } t > 0, \text{ with } u|_{t=0} = u_0 \in H \quad (2.1)$$

admits a unique solution  $u \in C([0, \infty), H)$  which is absolutely continuous with values in  $H$  in any compact subset of  $(0, \infty)$ . Here  $\partial\Phi(v)$  denotes the subdifferential at  $v$ , in other words  $f \in \partial\Phi(v) \subset D(\Phi)$  if and only if

$$\Phi(v + h) - \Phi(v) \geq \langle h, f \rangle$$

holds for all  $h \in H$ .

If  $\Phi$  is homogeneous of degree  $d$ , one can show that it satisfies the ‘‘Euler equation’’  $\langle u, \partial\Phi(u) \rangle = d\Phi(u)$ ; the proof is parallel to the familiar argument for homogeneous functions on  $\mathbf{R}^n$ . For later convenience we now prove a similar property for a sum of homogeneous functionals. Note that when  $\Phi = \Phi_1 + \Phi_2$ , we only know in general that  $\partial(\Phi_1 + \Phi_2) \supset \partial\Phi_1 + \partial\Phi_2$  (see e.g. [ET]), so the following Lemma cannot be proved by adding the results for each  $\Phi_j$ .

**Lemma 2.1.** *Suppose that for  $1 \leq j \leq m$ ,  $\Phi_j$  is positively homogeneous of degree  $d_j$  in  $H$  (in other words,  $\Phi_j(\lambda v) = \lambda^{d_j}\Phi_j(v)$  for all  $\lambda > 0$  and  $v \in D(\Phi_j)$ ). Then  $\Phi = \sum_{j=1}^m \Phi_j$  satisfies*

$$\langle u, f \rangle = \sum_{j=1}^m d_j \Phi_j(u)$$

for all  $f \in \partial\Phi(u)$ .

*Proof.* We take  $v = u, h = (\lambda - 1)u$  in the definition of the subdifferential to get

$$\Phi(\lambda u) - \Phi(u) \geq (\lambda - 1)\langle u, f \rangle.$$

By homogeneity we have

$$\sum_{j=1}^m (\lambda^{d_j} - 1)\Phi_j(u) \geq (\lambda - 1)\langle u, f \rangle.$$

Assuming that  $\lambda > 1$ , we divide both sides by  $\lambda - 1$  and send  $\lambda$  to 1 to get

$$\sum_{j=1}^m d_j \Phi_j(u) \geq \langle u, f \rangle.$$

The opposite inequality is obtained by assuming  $\lambda < 1$  and repeating the same procedure.  $\square$

As an application we obtain a fundamental energy identity. Let  $\|u\|$  be the norm of  $u$  in  $H$ , in other words  $\|u\|^2 = \langle u, u \rangle$ .

**Lemma 2.2.** *Suppose  $\Phi$  is a convex, lower semicontinuous function with non-empty domain  $D(\Phi)$ . Assume moreover that  $\Phi = \sum_{j=1}^m \Phi_j$  where  $\Phi_j$  is positively homogeneous of degree  $d_j$  for  $j = 1, 2, \dots, m$ . Then the solution of (2.1) satisfies*

$$\frac{1}{2} \frac{d}{dt} \|u\|^2(t) = - \sum_{j=1}^m d_j \Phi_j(u), \text{ a.e. } t > 0.$$

*Proof.* Take the inner product of (2.1) with  $u$  and apply Lemma 2.1.  $\square$

It is not always easy to characterize the subdifferential of a non-smooth convex function. But when  $\Phi$  is nonnegative and positively homogeneous of degree 1, we can characterize  $\partial\Phi$  by considering an appropriate “dual problem.” We recall how this works (see e.g. Theorem 1.8 of [ACM], where the result is stated for convex functions on a Banach space). For any nonnegative function  $\Psi$  on  $H$  with nonempty domain, let  $\tilde{\Psi}$  be the nonnegative function defined by

$$\tilde{\Psi}(v) = \sup \left\{ \frac{\langle v, w \rangle}{\Psi(w)} : w \in H \right\}$$

with the conventions that  $0/0 = 0$  and  $0/\infty = 0$ . Note that  $\tilde{\Psi}$  is always convex, lower semicontinuous, and positively homogeneous of degree one in  $H$ . If  $\Psi$  is positively homogeneous of degree one,  $\tilde{\Psi}$  is nothing but the support function of the one-sublevel-set of  $\Psi$  (the set  $\{w \in H : \Psi(w) \leq 1\}$ ).

**Lemma 2.3** ([ACM], Theorem 1.8). *Suppose  $\Phi$  is convex, lower semicontinuous, nonnegative, and positively homogeneous of degree one. Then*

$$f \in \partial\Phi(v) \text{ if and only if } \tilde{\Phi}(f) \leq 1 \text{ and } \langle v, f \rangle = \Phi(v).$$

*Proof.* From Lemma 2.2 we know that  $\langle v, f \rangle = \Phi(v)$  whenever  $f \in \partial\Phi(v)$ . Under the condition  $\langle v, f \rangle = \Phi(v)$ , the assertion  $f \in \partial\Phi(v)$  is by definition equivalent to the statement that

$$\Phi(v+h) \geq \langle v+h, f \rangle$$

for all  $h \in H$ . In other words  $\Phi(w) \geq \langle w, f \rangle$  for all  $w \in H$ . This is equivalent to saying that  $\tilde{\Phi}(f) \leq 1$ .  $\square$

## 2.2 The total variation flow

As noted in the Introduction, the second-order total variation flow can be formally written as

$$u_t = \operatorname{div} \left( \frac{\nabla u}{|\nabla u|} \right). \tag{2.2}$$

We now discuss its rigorous definition for various boundary conditions. This requires defining a convex, lower semicontinuous functional  $\Phi$  such that  $\partial\Phi$  is formally of the form  $-\operatorname{div}(\nabla u/|\nabla u|)$ .

**Periodic Boundary Condition.** This is the easiest case. We consider the problem in  $L^2(\mathbf{T}^n)$ , where

$$\mathbf{T}^n = \prod_{i=1}^n (\mathbf{R}/\omega_i \mathbf{Z})$$

with  $\omega_i > 0$ . In other words  $v \in L^2(\mathbf{T}^n)$  if  $v$  is a locally square integrable function defined for  $x \in \mathbf{R}^n$  which is periodic in each  $x_i$  with period  $\omega_i$ . This is of course a Hilbert space with inner product

$$\langle v, w \rangle = \int_0^{\omega_1} \cdots \int_0^{\omega_n} v(x) w(x) dx = \int_{\mathbf{T}^n} v w dx.$$

Let  $BV(\mathbf{T}^n) \subset L^2(\mathbf{T}^n)$  be the space of periodic functions with bounded variation, and consider the function  $\Phi_\pi$  on  $H$  defined by

$$\Phi_\pi(v) = \begin{cases} \int_{\mathbf{T}^n} |\nabla v| & \text{if } v \in BV(\mathbf{T}^n) \\ \infty & \text{otherwise.} \end{cases}$$

Here  $\int_{\mathbf{T}^n} |\nabla v|$  denotes the total variation of the vector-valued measure  $\nabla v$  in  $\mathbf{T}^n$ , i.e.,

$$\int_{\mathbf{T}^n} |\nabla v| = \sup \left\{ \int_{\mathbf{T}^n} v \operatorname{div} \varphi \, dx : \sup_x |\varphi(x)| \leq 1, \varphi \in C^1(\mathbf{T}^n, \mathbf{R}^n) \right\}.$$

It is easy to see that  $\Phi_\pi$  is convex and lower semicontinuous, see e.g. [Gi]. It is convenient to consider the closed subspace  $L^2_{\text{av}}(\mathbf{T}^n)$  of  $L^2(\mathbf{T}^n)$  consisting of functions with mean value zero:

$$L^2_{\text{av}}(\mathbf{T}^n) = \left\{ v \in L^2(\mathbf{T}^n) : \int_{\mathbf{T}^n} v \, dx = 0 \right\}.$$

The rigorous interpretation of the second-order total variation flow (1.1) with a periodic boundary condition is equation (2.1) with  $\Phi = \Phi_\pi$  and  $H = L^2_{\text{av}}(\mathbf{T}^n)$ . It has a unique solution, for any initial data  $u_0 \in H$ .

**Neumann Boundary Condition.** Let  $\Omega$  be a domain in  $\mathbf{R}^n$ . We set  $H = L^2(\Omega)$  and define a function on  $H$  of the form

$$\Phi_N(v) = \begin{cases} \int_{\Omega} |\nabla v| & \text{if } v \in BV(\Omega) \cap L^2(\Omega) \\ \infty & \text{otherwise.} \end{cases}$$

Here  $BV(\Omega)$  denotes the space of functions with bounded variation in  $\Omega$ . It is easy to see that  $\Phi_N$  is convex and lower semicontinuous. The rigorous interpretation of the second-order total variation flow (1.1) with a homogeneous Neumann boundary condition  $\partial u / \partial \nu = 0$  is equation (2.1) with this choice of  $H$  and  $\Phi = \Phi_N$ . If  $\Omega = \mathbf{R}^n$ , the solution just defined solves the Cauchy problem. If  $\Omega$  has bounded Lebesgue measure, we may alternatively take  $H = L^2_{\text{av}}(\Omega)$ , the space of  $L^2$  functions with mean value zero.

**Dirichlet Boundary Condition.** To impose a Dirichlet condition at  $\partial\Omega$ , it is tempting to suggest that  $\Phi(v) = \infty$  unless  $v \in BV(\Omega)$  vanishes at  $\partial\Omega$ . But this does not work – the resulting  $\Phi$  would not be lower semicontinuous. We must therefore proceed a little differently. Let  $\Omega$  be a bounded domain in  $\mathbf{R}^n$ . For any  $u \in L^2(\Omega)$ , let  $\tilde{u}$  denote its extension by 0 to all  $\mathbf{R}^n$ . Then we define

$$\Phi_D(u) = \begin{cases} \int_{\mathbf{R}^n} |\nabla \tilde{u}| & \text{if } \tilde{u} \in BV(\mathbf{R}^n) \\ \infty & \text{otherwise.} \end{cases}$$

This  $\Phi_D$  is convex and lower semicontinuous function as a function on  $H = L^2(\Omega)$ . The rigorous interpretation of the second-order total variation flow with a boundary condition  $u = 0$  at  $\partial\Omega$  is (2.1) with this choice of  $H$  and  $\Phi = \Phi_D$ . (See [GGK] for a more general discussion, with a nonconstant weight in the definition of  $\Phi$  and inhomogeneous Dirichlet data.)

### 2.3 A scale-invariant Sobolev inequality

Our bound for the extinction time will make use of the scale-invariant Sobolev (isoperimetric) inequality

$$\left( \int_{\Omega} |u|^{\frac{n}{n-1}} dx \right)^{\frac{n-1}{n}} \leq S_n \int_{\Omega} |\nabla u|. \quad (2.3)$$

Such an estimate holds

- (a) when  $\Omega = \mathbf{R}^n$ , for any  $u \in BV(\Omega)$ ;
- (b) when  $\Omega \subset \mathbf{R}^n$  has finite Lebesgue measure, for any  $u$  with mean value 0; and
- (c) when  $\Omega = \mathbf{T}^n$ , for any periodic  $u$  with mean value 0.

It also holds with  $u$  replaced by  $\tilde{u}$  (the extension by 0 off  $\Omega$ ), when  $\Omega$  is a bounded domain with Lipschitz boundary and  $u \in BV(\Omega)$ . These familiar facts can be found, for example, in the appendix of [ACM].

We call (2.3) a *scale-invariant* estimate, because in cases (b) and (c) it holds for  $\Omega$  if and only if it holds for the dilated domain  $\lambda\Omega$ , with a constant independent of  $\lambda$ . The estimate still depends, however, on the *shape* of the domain. To explain, let us focus on the periodic setting (c) in space dimension 2. Let  $C_L$  be the best constant for (2.3) when  $u$  is periodic with mean value 0 and the period cell is  $[0, L] \times [0, 1/L]$ . Restricting attention to  $u(x) = f(x_1)$ , we have

$$\left( \int_0^L |f(x_1)|^2 dx_1 \right)^{1/2} L^{-1/2} \leq C_L L^{-1} \int_0^L |f'(x_1)|$$

for any  $f$  with mean value 0. Changing variables by  $x_1 = Lz$  gives

$$\left( \int_0^1 |g(z)|^2 dz \right)^{1/2} \leq C_L L^{-1} \int_0^1 |g'(z)|$$

for any  $g$  with mean value 0. It follows that  $C_L/L$  stays bounded away from 0 as  $L \rightarrow \infty$ . Thus, in the periodic setting the best constant in (2.3) tends to infinity when the period cell becomes highly eccentric.

The situation when  $u$  has a Dirichlet boundary condition is different. The constant  $S_n$  can be taken independent of  $\Omega$  in that setting, since (2.3) holds with  $u$  replaced by  $\tilde{u}$  and  $S_n = S_n(\mathbf{R}^n)$ . (In fact, one can show that for any bounded domain  $\Omega$ , the *best* constant for (2.3) with a Dirichlet boundary condition is  $S_n(\Omega) = S_n(\mathbf{R}^n)$ .)

### 2.4 An upper bound for the extinction time

We are interested in the extinction time  $T^*(u_0)$  of the second-order total variation flow with one of the boundary conditions discussed in Section 2.2. It is defined by

$$T^*(u_0) = \inf\{t \in (0, \infty) : u(x, \tau) = 0 \text{ for } \tau \geq t\},$$

where  $u$  is the solution with initial data  $u_0$ . Our main result is



**Theorem 2.4.** *For the periodic problem in dimension  $n \geq 2$  with initial data  $u_0 \in L^2_{\text{av}}(\mathbf{T}^n)$ , the extinction time satisfies*

$$T^*(u_0) \leq S_n \|u_0\|_{L^n}.$$

*For the Neumann problem in a bounded domain (defined using  $\Phi = \Phi_N$  and  $H = L^2_{\text{av}}(\Omega)$ ) the same estimate holds with no further hypothesis when  $n = 2$ , and provided  $\partial\Omega$  is smooth if  $n \geq 3$ . For the Cauchy problem (defined using  $\Phi = \Phi_N$  and  $H = L^2(\mathbf{R}^n)$ ) the same estimate holds for  $n \geq 2$ . Finally, for the Dirichlet problem in a bounded domain (defined using  $\Phi = \Phi_D$  and  $H = L^2(\Omega)$ ) the same conclusion holds for  $n = 2$ . (We do not assert the estimate for the Dirichlet problem when  $n \geq 3$ .)*

*Proof.* What is clear is the case  $n = 2$ . In this case by Lemma 2.2 we have

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |u|^2 dx = - \int_{\Omega} |\nabla u| \tag{2.4}$$

since  $\Phi_{\pi}$ ,  $\Phi_D$  and  $\Phi_N$  are positively homogeneous of degree one (in the periodic setting  $\Omega = \mathbf{T}^n$ ). By the scale-invariant Sobolev inequality (2.3) we have

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |u|^2 dx \leq -\frac{1}{S_2} \left( \int_{\Omega} |u|^2 dx \right)^{1/2},$$

from which it follows that

$$\frac{d}{dt} \|u\|_{L^2}(t) \leq -S_2^{-1} \text{ provided } \|u\|_{L^2}(t) \neq 0.$$

This argument applies also to the Dirichlet boundary problem, provided we replace  $u$  by  $\tilde{u}$ . Thus in all the various settings we have

$$\|u\|_{L^2}(t) \leq \|u_0\|_{L^2} - S_2^{-1} t \text{ for } t < T^*(u_0).$$

Since the left hand side is nonnegative, we conclude that  $T^*(u_0) \leq S_2 \|u_0\|_{L^2}$ .

For  $n \geq 3$  the proof is more involved. We begin with a formal calculation. If  $n \geq 2$  is even, we multiply the PDE (2.2) by  $u^{n-1}$  to get

$$\frac{1}{n} \frac{d}{dt} \int_{\Omega} u^n dx = \int_{\Omega} u^{n-1} u_t dx = \int_{\Omega} u^{n-1} \operatorname{div}(\nabla u / |\nabla u|) dx.$$

Integrating by parts, we find that the right hand side equals

$$\begin{aligned} - (n-1) \int_{\Omega} u^{n-2} \nabla u \cdot \nabla u / |\nabla u| dx + \int_{\partial\Omega} u^{n-1} (\partial u / \partial \nu) / |\nabla u| dS = \\ - (n-1) \int_{\Omega} u^{n-2} |\nabla u| dx = - \int_{\Omega} |\nabla u^{n-1}| dx. \end{aligned}$$

Here we used the Dirichlet or Neumann boundary condition to conclude that the boundary integral was zero. (If  $\Omega = \mathbf{R}^n$  or  $\mathbf{T}^n$ , there is no boundary integral.) Applying the scale-invariant Sobolev estimate (2.3) we get

$$\frac{1}{n} \frac{d}{dt} \int_{\Omega} u^n dx \leq -S_n^{-1} \left( \int_{\Omega} u^n dx \right)^{\frac{n-1}{n}}.$$

We thus obtain (formally) that

$$\frac{1}{n} \frac{d}{dt} \|u\|_{L^n}^n \leq -S_n^{-1} \|u\|_{L^n}^{n-1}. \quad (2.5)$$

If  $n \geq 3$  is odd, we obtain the same inequality (formally) by multiplying the PDE by  $|u|^{n-2}u$  instead of  $u^{n-1}$ . It is easy to see that (2.5) implies our extinction time estimate.

There are at least two ways to make this argument rigorous for the Cauchy problem, the periodic problem, and the Neumann boundary condition. One way is to use a characterization of the subdifferential we'll discuss later (see Lemmas 3.5 and 3.12) combined with integration by parts. This approach uses the truncation of  $u$  defined by

$$u_k = \begin{cases} u & \text{if } -k \leq u \leq k \\ k & \text{if } u > k \\ -k & \text{if } u < -k. \end{cases}$$

One argues essentially as in the formal argument, but using  $u_k$  instead of  $u$ ; for example, when the boundary condition is periodic one gets

$$\int_{\mathbf{T}^n} |u_k|^{n-2} u_k u_t = - \int_{\mathbf{T}^n} |\nabla |u_k|^{n-1}|$$

However, this argument is not very straightforward: it requires a characterization of the subdifferential that's more specific than Lemma 3.5, and the treatment of the term  $|\nabla |u_k|^{n-1}|$  is difficult.

Our second approach, which uses an approximation argument, is easier. We shall focus on the case of  $\Phi_N$  with a bounded domain  $\Omega$  (the argument is similar but easier in the periodic setting and for the Cauchy problem). We approximate the original energy by a smooth one, whose gradient flow is uniformly parabolic:

$$\Phi_N^\varepsilon(u) = \int_{\Omega} \sqrt{|\nabla u|^2 + \varepsilon^2} dx + \frac{\varepsilon^2}{2} \int_{\Omega} |\nabla u|^2.$$

Its gradient flow is formally equivalent to

$$\frac{\partial u}{\partial t} = \operatorname{div} \left( \frac{\nabla u}{\sqrt{|\nabla u|^2 + \varepsilon^2}} \right) + \varepsilon^2 \Delta u \quad (2.6)$$

with  $\partial u / \partial \nu = 0$  at  $\partial \Omega$ . There is a unique, smooth, global-in-time solution for any  $L^2$  initial data  $u_0^\varepsilon$ . Since  $\Phi_N^\varepsilon \rightarrow \Phi_N$  in the Mosco sense, we know that the solutions converge ( $u^\varepsilon \rightarrow u$  in

$C([0, T], L^2(\Omega))$ , where  $u$  is the solution of (2.1) with  $\Phi = \Phi_N$  and  $H = L_{\text{av}}^2(\Omega)$  provided  $u_0^\varepsilon \rightarrow u_0$  in  $L^2(\Omega)$ . This convergence follows from a general theorem due to J. Watanabe [W] and H. Brezis and A. Pazy [BP]. (Note that without the  $\Delta u$  term our perturbed problem is essentially the one studied by [LT].)

We first assume that  $u_0$  is Lipschitz and bounded. Multiplying (2.6) by  $|u|^{n-2}u$  for odd  $n \geq 3$  and integrating by parts gives

$$\frac{1}{n} \frac{d}{dt} \int_{\Omega} |u|^n dx = -(n-1) \int_{\Omega} |u|^{n-2} \frac{|\nabla u|^2}{\sqrt{|\nabla u|^2 + \varepsilon^2}} dx - (n-1)\varepsilon^2 \int_{\Omega} |u|^{n-2} |\nabla u|^2 dx. \quad (2.7)$$

Since  $X^2/(X^2 + \varepsilon^2)^{1/2} \geq X - \delta$  for all  $X \geq \delta$  provided that  $\delta \geq \varepsilon/2$ , this inequality yields

$$\begin{aligned} \frac{1}{n} \frac{d}{dt} \int_{\Omega} |u|^n dx &\leq -(n-1) \int_{\Omega} |u|^{n-2} (|\nabla u| - \delta)_+ dx \\ &= - \int_{\Omega} |\nabla u|^{n-1} dx + \delta(n-1) \int_{\Omega \cap \{|\nabla u| > \delta\}} |u|^{n-2} dx \\ &\quad + (n-1) \int_{\Omega \cap \{|\nabla u| \leq \delta\}} |u|^{n-2} |\nabla u| dx \end{aligned} \quad (2.8)$$

for  $\delta \geq \varepsilon/2$ .

The function  $u$  depends on  $\varepsilon$ , and we are interested in the limit  $\varepsilon \rightarrow 0$ , so we henceforth write  $u^\varepsilon$  rather than  $u$ . Since we have assumed that  $u_0$  is Lipschitz and bounded we may take  $u_0^\varepsilon$  such that  $\|\nabla u_0^\varepsilon\|_{L^\infty} \leq L$  uniformly in  $0 < \varepsilon < 1$  and  $u_0^\varepsilon \rightarrow u_0$  in  $L^p$  ( $1 \leq p \leq \infty$ ). Then by the weak maximum principle for  $|\nabla u^\varepsilon|^2$  there is an  $\alpha > 0$  independent of  $\varepsilon$  such that

$$\|\nabla u^\varepsilon\|_\infty(t) \leq e^{\alpha t} L$$

for all  $t \geq 0$  independent of  $\varepsilon$ . The constant  $\alpha > 0$  depends on the curvature of  $\partial\Omega$  (see [GOS] and [AG]). If  $\Omega$  is convex, then one can take  $\alpha = 0$ ; in general, however, one cannot take  $\alpha = 0$  (see [GOS], which examines the Neumann problem for the level set formulation of mean curvature flow).

Now choose  $T$  such that  $T < T^*(u_0)$ . Since  $u^\varepsilon \rightarrow u$  in  $C([0, T], L^2)$ , by interpolation with our uniform estimate of the gradient we conclude that  $u^\varepsilon \rightarrow u$  in  $C([0, T], L^p)$ . Applying the Sobolev inequality, we obtain from (2.8) that

$$\|u^\varepsilon\|_{L^n}^{n-1} \frac{d}{dt} \|u^\varepsilon\|_{L^n}(t) \leq -S_n^{-1} \|u^\varepsilon\|_{L^n}^{n-1}(t) + \delta(n-1) \|u^\varepsilon\|_{L^{n-2}}^{n-2}(t). \quad (2.9)$$

Here we invoke the assumption that  $u_0 \in L_{\text{av}}^2(\Omega)$  so that  $u^\varepsilon \in L_{\text{av}}^2(\Omega)$ . Since  $\|u^\varepsilon\|_{L^n}(t)$  is decreasing in  $t$  by (2.7) and since  $T < T^*(u_0)$ ,  $\|u^\varepsilon\|_{L^n}$  is bounded from below by a positive constant  $\eta$ . Thus (2.9) yields

$$\frac{d}{dt} \|u^\varepsilon\|_{L^n}(t) \leq -S_n^{-1} + \delta(n-1) \|u^\varepsilon\|_{L^{n-2}}^{n-2}(t) \eta^{-(n-1)}$$

or

$$\|u^\varepsilon\|_{L^n}(t) \leq \|u_0^\varepsilon\|_{L^n} - S_n^{-1} t + \delta(n-1) \eta^{-(n-1)} \int_0^t \|u^\varepsilon\|_{L^{n-2}}^{n-2}(\tau) d\tau. \quad (2.10)$$

Since  $\Omega$  is bounded, the integral on the RHS is bounded for  $\varepsilon \in (0, 1)$ . Sending  $\varepsilon$  to zero and then  $\delta$  to zero we obtain the desired estimate

$$\|u\|_{L^n}(t) \leq \|u_0\|_{L^n} - S_n^{-1}t \text{ for } t \in [0, T^*(u_0)) \quad (2.11)$$

when  $\Omega$  is bounded.

The preceding discussion addresses the Neumann problem in dimension  $n$ , when  $n \geq 3$  is odd. The argument for the Neumann problem with  $n \geq 2$  even is similar. So is the argument for the periodic problem. For the Cauchy problem, i.e. when  $\Omega = \mathbf{R}^n$ , the boundedness of  $\|u^\varepsilon\|_{L^{n-2}}$  does not follow from the boundedness of  $\|u^\varepsilon\|_{L^2}$  and  $\|u^\varepsilon\|_{L^n}$  when  $n = 3$ . However, multiplying an approximation of  $\text{sgn } u$  with (2.6) and integrating by parts yields  $d\|u\|_{L^1}(t)/dt \leq 0$  (we shall do something similar in the next subsection). So (2.10) yields the desired estimate (2.11).

We have now shown (2.11) in all the cases covered by the theorem, for initial data satisfying  $u_0 \in L^1 \cap L^\infty$  with  $\nabla u_0 \in L^\infty$ . But the estimate then follows for all  $u_0 \in L^2 \cap L^n$ , by an approximation argument.  $\square$

Theorem 2.4 addresses the Dirichlet problem only in dimension  $n = 2$  because the justification of our formal estimate (2.5) is more difficult for the Dirichlet problem. For example, for the Dirichlet problem we do not have a uniform-in- $\varepsilon$  Lipschitz bound for general domains (though we have it in some cases, for example if  $\Omega$  is mean-convex). We prefer not to pursue this issue, since it would take us too far afield.

## 2.5 The 1D problem and some explicit solutions

This section considers the one dimensional problem

$$u_t = \partial_x(\text{sgn } u_x), \quad u|_{t=0} = u_0. \quad (2.12)$$

We focus first on the periodic setting, turning afterward to the Cauchy problem. (There is no need to consider the Dirichlet or Neumann settings separately, since in 1D the evolution with a homogeneous Dirichlet or Neumann boundary condition can be reduced to the periodic problem, see e.g. [GG98].)

By the 1D periodic setting, we mean steepest descent for  $\int |u_x|$  using  $H = L^2_{\text{av}}(\mathbf{T}^1)$ . As in Section 2.4 we approximate  $u$  by solving

$$u_t^\varepsilon = \partial_x \left( \frac{u_x^\varepsilon}{\sqrt{(u_x^\varepsilon)^2 + \varepsilon^2}} \right) + \varepsilon^2 u_{xx}^\varepsilon, \quad u^\varepsilon|_{t=0} = u_0^\varepsilon \quad (2.13)$$

in  $\mathbf{T}^1 \times (0, \infty)$  where  $\mathbf{T}^1 = \mathbf{R}/\omega\mathbf{Z}$ . The convergence  $u^\varepsilon \rightarrow u$  is uniform in  $\mathbf{T}^1 \times [0, T]$  as proved in [GG98]. We multiply (2.13) by  $f'(u)$ , where  $f(u)$  is a convex approximation of  $|u|$  such as  $f(u) = \sqrt{u^2 + \sigma^2}$ . Integrating by parts gives

$$\begin{aligned} \frac{d}{dt} \int_0^\omega f(u^\varepsilon) dx &\leq - \int_0^\omega \frac{f''(u^\varepsilon) (u_x^\varepsilon)^2}{\sqrt{(u_x^\varepsilon)^2 + \varepsilon^2}} dx \\ &\leq - \int_0^\omega f''(u^\varepsilon) (|u_x^\varepsilon| - \delta)_+ dx \end{aligned}$$

if  $\delta \geq \varepsilon/2$ . (In the last inequality we used that  $X^2/(X^2 + \varepsilon^2)^{1/2} \geq X - \delta$  for  $X \geq \delta$  with  $\delta \geq \varepsilon/2$ .) Integrating over  $(0, t)$ , we get

$$\begin{aligned} & \int_0^\omega f(u^\varepsilon(x, t)) dx \\ & \leq \int_0^\omega f(u_0^\varepsilon(x)) dx - \int_0^t \int_0^\omega |(f'(u^\varepsilon))_x| dx ds + \delta \int_0^t \int_0^\omega f''(u^\varepsilon(x, s)) dx ds \\ & \leq \int_0^\omega f(u_0^\varepsilon) dx - \int_0^t \|f'(u^\varepsilon)\|_{L^\infty}(s) ds + \delta \int_0^t \int_0^\omega f''(u^\varepsilon(x, s)) dx ds. \end{aligned}$$

Sending  $\varepsilon \rightarrow 0$  then  $\delta \rightarrow 0$ , we conclude that

$$\int_0^\omega f(u(x, t)) dx + \int_0^t \|f'(u)\|_{L^\infty}(s) ds \leq \int_0^\omega f(u_0) dx$$

since  $f''(u^\varepsilon(x, s))$  is bounded. Now sending  $\sigma$  (the parameter in the definition of  $f$ ) to zero we get

$$\|u\|_{L^1}(t) \leq \|u_0\|_{L^1} - t \text{ for } t \in [0, T^*(u_0)).$$

This argument also works for the Cauchy problem, since we may approximate an  $L^2$  initial condition by one satisfying  $u_0 \in L^1 \cap L^\infty$  and  $u_{0x} \in L^\infty$ , so that  $\|u_x^\varepsilon\|_{L^\infty}(t) \leq \|u_{0x}^\varepsilon\|_{L^\infty} \leq \|u_{0x}\|_{L^\infty}$  and  $\|u^\varepsilon\|_{L^p}(t) \leq \|u_0\|_{L^p}$  for any  $1 \leq p \leq \infty$ ,  $t \geq 0$ . Altogether, we have proved:

**Theorem 2.5.** *In the 1D periodic setting, let  $u$  be the solution of the second order total variation flow (2.12) with initial condition  $u_0 \in L^2_{\text{av}}(\mathbf{T}^1)$ . Then  $\|u\|_{L^1}(t) \leq \|u_0\|_{L^1} - t$  for  $t \in (0, T^*(u_0))$ . In particular,  $T^*(u_0) \leq \|u_0\|_{L^1}$ . The same estimate also holds for the Dirichlet problem and for the Neumann problem. In addition, the estimate holds for the Cauchy problem for any  $u_0 \in L^2(\mathbf{R})$ .*

The preceding theorem suggests that for the Cauchy problem,  $T^*$  could be infinite if  $u_0 \notin L^1(\mathbf{R})$ . We shall prove that this is the case; in fact, for broad class of initial data (to be specified in a moment), we'll show that finite-time extinction occurs if and only if  $u_0 \in L^1$ .

Suppose  $u_0 \in C^1(\mathbf{R})$  is a positive, even function of  $x$ , with  $u_{0x} \leq 0$  for  $x > 0$ . Suppose further that  $u_0(x) \equiv A > 0$  for  $x \in (-r_0, r_0)$  and  $u_0 < A$ ,  $|u_{0x}| > 0$  outside  $(-r_0, r_0)$ , and let  $U(x, t)$  be the solution of (2.12) with this initial condition. This solution is explicitly computable (see e.g. [GG98], or Section 4.1 of [ACM], which addresses the radial setting in any space dimension):

$$U(x, t) = \begin{cases} A - \int_0^t \lambda(s) ds & \text{for } |x| \leq r(t) \\ u_0(x) & \text{for } |x| \geq r(t), \end{cases}$$

where  $\lambda = 1/r(t)$  is the ‘‘crystalline curvature’’ of the facet  $(-r(t), r(t))$  (the flat part of the graph of  $U(x, t)$ ) and  $r(t)$  is determined by the condition that  $U$  be continuous:

$$u_0(r(t)) = A - \int_0^t \lambda(s) ds. \tag{2.14}$$

The function  $U$  satisfies (2.1), because

$$\sigma(x) = \begin{cases} -\lambda x & \text{for } |x| \leq r(t) \\ u_x/|u_x| & \text{for } |x| > r(t) \end{cases}$$

is such that  $U_t = \sigma_x \in -\partial\Phi$  when  $\Phi = \int_{\mathbf{R}} |u_x|$  and  $H = L^2(\mathbf{R})$ .

Let us explore the properties of this solution. Differentiation of (2.14) gives an ODE for  $r(t)$ :

$$u'_0(r) \frac{dr}{dt} = -r^{-1},$$

from which it follows that

$$\int_{r_0}^{r(t)} -su'_0(s) ds = t.$$

Integration by parts yields

$$-[su_0(s)]_{r_0}^{r(t)} + \int_{r_0}^{r(t)} u_0(s) ds = t.$$

The left hand side equals

$$\int_{r_0}^r [u_0(s) - u_0(r)] ds + r_0[u_0(r_0) - u_0(r)] \text{ with } r = r(t).$$

Using the inverse function  $u_0^{-1}$ , we observe that

$$\int_{r_0}^r [u_0(s) - u_0(r)] ds = \int_{u_0(r)}^{u_0(r_0)} u_0^{-1}(t) dt.$$

Sending  $r$  to infinity yields

$$\lim_{r \rightarrow \infty} \int_{r_0}^r [u_0(s) - u_0(r)] ds = \int_0^{u_0(r_0)} u_0^{-1}(t) dt = \|u_0\|_{L^1(r_0, \infty)}.$$

In particular,  $r(t)$  reaches infinity at a finite time  $t = T^*$  if and only if  $u_0 \in L^1(\mathbf{R})$ . We have proved

**Theorem 2.6.** *Suppose  $u_0 \in C^1(\mathbf{R})$  is a positive, even function of  $x$ , with  $u_{0x} \leq 0$  for  $x > 0$ . Suppose further that  $u_0(x) \equiv A > 0$  for  $x \in (-r_0, r_0)$  and  $u_0 < A$ ,  $|u_{0x}| > 0$  outside  $(-r_0, r_0)$ . Then the solution of (2.12) becomes identically zero in finite time if and only if  $u_0 \in L^1(\mathbf{R})$ .*

We conclude this section with some remarks on related work.

**Remark 2.7.** *To the best of our knowledge, the first proof of finite-time extinction for the second-order total variation flow (2.1) was given by Andreu, Caselles, Diaz, and Mazón in [ACDM]. That paper proves*

$$T^*(u_0) \leq \frac{d(\Omega)}{n} \|u_0\|_{\infty}$$

in space dimension  $n$ , for the Dirichlet and the Neumann problems in a bounded domain  $\Omega$  with Lipschitz boundary. Here  $d(\Omega)$  denotes the radius of the smallest ball containing  $\Omega$ , in other words half the diameter of  $\Omega$ . Their argument uses a comparison principle, and the fact that the PDE has explicit solutions of the form  $\lambda(t)\chi_B$  when  $B$  is a ball. Our estimates of  $T^*$  in Theorems 2.4 and 2.5 are quite different in character, since they are scale-invariant (i.e. the constants are invariant as one changes  $\Omega$  by dilation). Our proofs are also very different, since (at least at the formal level) they rely on energy and Sobolev-type inequalities rather than comparison.

**Remark 2.8.** While our energy-based approach has not been applied to the total variation flow, similar arguments have been used to study finite-time extinction for other nonlinear parabolic equations. The example closest to what we have presented here is the analysis of finite-time extinction for diffusion equation associated with the  $p$ -Laplacian

$$u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \quad \text{when } 1 < p < 2.$$

This topic is discussed at length by DiBenedetto in [D] for the Cauchy problem in  $\mathbf{R}^n$  and the Dirichlet problem in a bounded domain. He proves

$$T^*(u_0) \leq C \|u_0\|_{L^s}^{2-p} \quad \text{with } s = \frac{n(2-p)}{p}, \text{ when } n \geq 2 \text{ and } 1 < p < \frac{2n}{n+2} \quad (2.15)$$

for both the Cauchy and Dirichlet problems, with  $C$  depending only on  $p$  and  $n$  (see Propositions 2.1 and 3.1 in Chapter VII of [D]). For the Dirichlet problem, he also proves

$$T^*(u_0) \leq C \|u_0\|_{L^2}^{2-p} |\Omega|^{(n(p-2)+2p)/2n} \quad \text{when } p > 1 \text{ and } \frac{2n}{n+2} < p < 2.$$

While DiBenedetto did not consider  $p = 1$ , his estimate (2.15) reduces to our result  $T^*(u_0) \leq C \|u_0\|_{L^n}$  when  $p = 1$ ; moreover his proof is a lot like ours. The same type of argument was used earlier by Benilan and Crandall to study the finite-time extinction of solutions to  $u_t = \Delta(|u|^m \operatorname{sgn} u)$  with  $0 < m < (n-2)/n$  in space dimension  $n \geq 3$  (see Proposition 10 of [BC]). They proved that if  $u_0 \in L^{\beta+1}(\mathbf{R}^n) \cap L^1(\mathbf{R}^n)$  with  $\beta = (2 - 2^*m)/(2^* - 2)$  (here  $1/2^* = 1/2 - 1/n$ , and one verifies that  $\beta > 0 \Leftrightarrow m < (n-2)/n$ ) then

$$T^*(u_0) \leq C \left( \int_{\mathbf{R}^n} u_0^{\beta+1} dx \right)^{2/n}.$$

The heuristic version of their proof is similar to our heuristic argument for Theorem 2.4: it starts by multiplying the PDE by  $u^\beta$  then integrating by parts. Their approach to making the argument rigorous is somewhat different from ours (they use implicit-in-time discretization rather than regularization). The paper [BC] also proves that finite time extinction never occurs for  $m \geq (n-2)/n$  and (nonzero) initial data  $u_0 \in L^1(\mathbf{R}^n)$ .

**Remark 2.9.** Another steepest-descent law with finite-time extinction is the mean curvature flow. Evans and Spruck showed in [ES3] that

$$T^*(\Sigma) \leq C (\mathcal{H}^{n-1}(\Sigma))^{2/(n-1)}$$

for the mean curvature flow of a closed hypersurface  $\Sigma$  in  $\mathbf{R}^n$ . A lower bound for the extinction time was given by Giga and Yama-uchi in [GY], namely

$$T^* \geq 2|D|^2 (\mathcal{H}^{n-1}(\Sigma))^{-2},$$

where  $D$  is the bounded open set surrounded by  $\Sigma$  and  $|D|$  is its volume. In dimension  $n = 2$  the extinction time is known exactly:  $T^*(\Sigma) = \frac{1}{2\pi}|D|$ , since for the curve-shortening flow in the plane

$$\frac{d}{dt}|D(t)| = - \int_{\Sigma(t)} \text{inward normal velocity } ds = - \int_{\Sigma(t)} \text{curvature } ds = -2\pi.$$

### 3 The fourth-order total variation flow

In this section we prove scale-invariant extinction time estimates for the fourth-order total variation flow in space dimension  $1 \leq n \leq 4$ . Recall that the formal PDE is

$$u_t = -\Delta \left[ \operatorname{div} \left( \frac{\nabla u}{|\nabla u|} \right) \right]. \quad (3.1)$$

We will discuss the rigorous definition of this evolution (as an  $H^{-1}$  gradient flow) in Section 3.2.

#### 3.1 Some Hilbert spaces

As usual, the Sobolev space  $H^1(\Omega)$  is equipped with the inner product

$$(f, g)_{H^1} = ((f, g))_1 + \int_{\Omega} fg \, dx, \quad ((f, g))_1 = \int_{\Omega} \nabla f \cdot \nabla g \, dx. \quad (3.2)$$

In the periodic setting,  $\Omega$  is the period cell  $\mathbf{T}^n$ .

We want to work in a subspace on which  $((f, g))_1$  is an equivalent inner product. For the Dirichlet problem on a bounded domain the convenient choice is

$$H_0^1(\Omega) = \{f \in H^1(\Omega) : f = 0 \text{ at } \partial\Omega\},$$

since by Poincaré's inequality we have  $\|f\|_{L^2(\Omega)} \leq C\|\nabla f\|_{L^2(\Omega)}$  when  $f = 0$  at  $\partial\Omega$ . In the periodic setting or for the Neumann problem on a bounded domain the convenient choice is the space of mean-value-zero functions:

$$H_{\text{av}}^1(\Omega) = \left\{ f \in H^1(\Omega) : \int_{\Omega} f \, dx = 0 \right\},$$

since when  $f$  has mean value zero we have  $\|f\|_{L^2(\Omega)} \leq C\|\nabla f\|_{L^2(\Omega)}$ .

When  $\Omega$  is a bounded domain, we take  $H^{-1}(\Omega)$  to be the dual of  $H_0^1(\Omega)$ . It can be viewed as a subspace of the space  $\mathcal{D}'(\Omega)$  of distributions.



In the periodic setting, we take  $H_{\text{av}}^{-1}(\mathbf{T}^n)$  to be the dual of  $H_{\text{av}}^1(\mathbf{T}^n)$ . It is the closure of the space of smooth mean-value-zero periodic functions under the  $H_{\text{av}}^{-1}(\mathbf{T}^n)$  norm. Its inner product can be expressed as

$$((f, g))_{-1} = \int_{\mathbf{T}^n} (-\Delta)^{-1} f \cdot g \, dx, \quad (3.3)$$

where  $-\Delta$  denotes the isometry from  $H_{\text{av}}^1(\mathbf{T}^n)$  to  $H_{\text{av}}^{-1}(\mathbf{T}^n)$  defined by  $f \mapsto ((f, \cdot))_1$ .

In connection with the Neumann problem, we will use  $H_{\text{av}}^{-1}(\Omega)$ , defined to be the dual  $H_{\text{av}}^1(\Omega)$ . It is not a subspace of the space  $\mathcal{D}'(\Omega)$  of distributions on  $\Omega$ . As in the periodic setting, its inner product can be expressed by (3.3), where  $-\Delta$  is the isometry from  $H_{\text{av}}^1(\Omega)$  to  $H_{\text{av}}^{-1}(\Omega)$  defined by  $f \mapsto ((f, \cdot))_1$ .

### 3.2 $H^{-1}$ steepest descent

We now discuss the rigorous definition of the fourth-order total variation flow with various boundary conditions.

**Periodic Boundary Condition.** The rigorous meaning of (3.1) with a periodic boundary condition is steepest descent for

$$\Phi_\pi(v) = \begin{cases} \int_{\mathbf{T}^n} |\nabla v| & \text{if } v \in BV(\mathbf{T}^n) \cap H_{\text{av}}^{-1}(\mathbf{T}^n) \\ \infty & \text{otherwise.} \end{cases}$$

in the Hilbert space  $H = H_{\text{av}}^{-1}(\mathbf{T}^n)$ . (We use the same notation  $\Phi_\pi$  as in Section 2.2 because this is essentially the same function – only the choice of  $H$  has changed. Note, however, that in the present context  $\int_{\mathbf{T}^n} v \, dx = 0$  whenever  $\Phi_\pi(v) < \infty$ .) The evolution is thus characterized by

$$\frac{du}{dt}(t) \in -\partial_{H^{-1}} \Phi_\pi(u(t)) \text{ for a.e. } t > 0, \text{ with } u|_{t=0} = u_0 \in H_{\text{av}}^{-1}(\mathbf{T}^n), \quad (3.4)$$

where  $\partial_{H^{-1}}$  denotes the subdifferential in  $H_{\text{av}}^{-1}(\mathbf{T}^n)$ .

The subdifferential  $\partial_{H^{-1}} \Phi_\pi$  is formally equal to  $-\Delta \partial \Phi_\pi$ , where  $\partial$  denotes the subdifferential in  $L^2$  sense. We now state and prove a rigorous version of this assertion. As in Lemma 2.3 we consider an associated function  $\tilde{\Phi}$ ; in fact we consider two such functions, one using the Hilbert space  $H_{\text{av}}^{-1}$  and the other using  $L_{\text{av}}^2$ : let

$$\begin{aligned} \tilde{\Phi}_{\pi H^{-1}}(v) &= \sup \left\{ ((v, w))_{-1} / \Phi_\pi(w) : w \in H_{\text{av}}^{-1}(\mathbf{T}^n) \right\} \\ \tilde{\Phi}_{\pi L^2}(v) &= \sup \left\{ \int_{\mathbf{T}^n} v w \, dx / \Phi_\pi(w) : w \in L_{\text{av}}^2(\mathbf{T}^n) \right\}. \end{aligned}$$

**Lemma 3.1.** *The condition  $f \in \partial_{H^{-1}} \Phi_\pi(v)$  is equivalent to conditions that  $\tilde{\Phi}_{\pi L^2}((-\Delta)^{-1} f) \leq 1$  and  $\int_{\mathbf{T}^n} (-\Delta)^{-1} f \cdot v \, dx = \Phi_\pi(v)$ . In particular,  $f \in \partial_{H^{-1}} \Phi_\pi(v)$  if and only if  $(-\Delta)^{-1} f \in \partial \Phi_\pi(v)$  provided that  $v \in L_{\text{av}}^2(\mathbf{T}^n)$ .*

*Proof.* It is easy to see that

$$\tilde{\Phi}_{\pi H^{-1}}(v) = \tilde{\Phi}_{\pi L^2}((-\Delta)^{-1}v) \text{ for } v \in H_{\text{av}}^{-1}(\mathbf{T}^n)$$

since  $((v, w))_{-1} = \int_{\mathbf{T}^n} (-\Delta)^{-1}vw \, dx$  and  $L_{\text{av}}^2$  is dense in  $H_{\text{av}}^{-1}(\mathbf{T}^n)$ . By Lemma 2.3 the condition  $f \in \partial_{H^{-1}}\Phi(v)$ ,  $v \in H_{\text{av}}^{-1}(\mathbf{T}^n)$  is equivalent to the assertion that

$$\tilde{\Phi}_{\pi L^2}((-\Delta)^{-1}f) \leq 1 \text{ and } ((v, f))_{-1} = \Phi_{\pi}(v),$$

or in other words that

$$\tilde{\Phi}_{\pi L^2}((-\Delta)^{-1}f) \leq 1 \text{ and } \int_{\mathbf{T}^n} (-\Delta)^{-1}f \cdot v \, dx = \Phi_{\pi}(v).$$

By Lemma 2.3 this is equivalent to the statement that  $(-\Delta)^{-1}f \in \partial\Phi_{\pi}(v)$ , if  $v \in L^2(\mathbf{T}^n)$ .  $\square$

**Comment.** The requirement that  $v$  be in  $L^2(\mathbf{T}^n)$  arises because otherwise the  $L^2$  subdifferential  $\partial\Phi_{\pi}(v)$  is not well-defined. But when  $v \in H_{\text{av}}^{-1}(\mathbf{T}^n)$  is not in  $L^2$ , it is natural to define the  $L^2$ -subdifferential  $\partial\Phi_{\pi}(v)$  as the set of all  $g \in H_{\text{av}}^1(\mathbf{T}^n)$  such that

$$\Phi_{\pi}(v+h) - \Phi_{\pi}(v) \geq \int_{\mathbf{T}^n} hg \, dx$$

for all  $h \in H_{\text{av}}^{-1}(\mathbf{T}^n)$ . With this definition we can extend Lemma 2.3 to assert that  $g \in \partial\Phi_{\pi}(v)$  if and only if  $g \in H_{\text{av}}^1(\mathbf{T}^n)$  satisfies  $\tilde{\Phi}_{L^2}(g) \leq 1$  and  $\int_{\mathbf{T}^n} gv \, dx = \Phi(v)$ . Since  $(-\Delta)^{-1}f$  is always in  $H_{\text{av}}^1(\mathbf{T}^n)$ , we may delete the restriction “ $v \in L_{\text{av}}^2(\mathbf{T}^n)$ ” in Lemma 3.1 by using this extended interpretation of the subdifferential.

**Dirichlet or Neumann boundary conditions.** The treatment is parallel to the periodic setting, so we shall be brief. For the Dirichlet problem, the solution is steepest descent for  $\Phi_D$  using the Hilbert space  $H^{-1}(\Omega)$ , in other words

$$\frac{du}{dt}(t) \in -\partial_{H^{-1}}\Phi_D(u(t)) \text{ for a.e. } t > 0, \text{ with } u|_{t=0} = u_0 \in H^{-1}(\Omega), \quad (3.5)$$

where  $\partial_{H^{-1}}$  denotes the subdifferential in  $H^{-1}(\Omega)$ . This evolution is formally equivalent to (3.1) with the boundary condition

$$\operatorname{div} \left( \frac{\nabla u}{|\nabla u|} \right) = 0 \quad \text{and} \quad u = 0 \text{ at } \partial\Omega.$$

By the Dirichlet analog of Lemma 3.1, it is natural to write (3.5) formally as

$$\partial_t u = -\Delta_D \operatorname{div}(\nabla u/|\nabla u|) \text{ in } \Omega \times (0, \infty)$$

with  $u = 0$  on  $\partial\Omega$ , where  $\Delta_D$  denotes the Laplacian with the Dirichlet boundary condition.

Similarly, for the Neumann problem, the solution is steepest descent for  $\Phi_N$  using the Hilbert space  $H_{\text{av}}^{-1}(\Omega)$ . It is described by the analogue of (3.5) with  $\Phi_D$  replaced by  $\Phi_N$ . The associated boundary condition is formally

$$\frac{\partial}{\partial\nu} \operatorname{div} \left( \frac{\nabla u}{|\nabla u|} \right) = 0 \quad \text{and} \quad \frac{\partial u}{\partial\nu} = 0 \text{ at } \partial\Omega.$$

### 3.3 Upper bounds on the extinction time in the periodic context

Here and in Sections 3.4 and 3.5 we shall focus for simplicity on the spatially periodic setting. Some of our results extend, however, to the case of a homogeneous Dirichlet or Neumann boundary condition. This is discussed in Section 3.6.

Our basic tool is the abstract energy identity Lemma 2.2. Specialized to the present context, it becomes:

**Lemma 3.2.** *Let  $u$  be the solution of (3.4) with initial data  $u_0 \in H_{\text{av}}^{-1}(\mathbf{T}^n)$ . Then*

$$\frac{1}{2} \frac{d}{dt} \|u\|_{H^{-1}}^2(t) = - \int_{\mathbf{T}^n} |\nabla u(\cdot, t)| \quad \text{for a.e. } t > 0 \quad (3.6)$$

where  $\|u\|_{H^{-1}}$  denotes the norm of  $u$  in  $H_{\text{av}}^{-1}(\mathbf{T}^n)$ , i.e.,  $\|u\|_{H^{-1}}^2 = ((u, u))_1$ . (The left hand side makes sense, since the solution of (3.4) is absolutely continuous as a function of  $t$  taking values in  $H_{\text{av}}^{-1}(\mathbf{T}^n)$ .)

Note that (3.6) is the estimate obtained formally by multiplying the equation by  $(-\Delta)^{-1}u$ :

$$\frac{1}{2} \frac{d}{dt} \|u\|_{H^{-1}}^2(t) = \int_{\mathbf{T}^n} (-\Delta)^{-1}u \, u_t \, dx = \int_{\mathbf{T}^n} u \, \operatorname{div} \left( \frac{\nabla u}{|\nabla u|} \right) \, dx = - \int_{\mathbf{T}^n} |\nabla u|.$$

To use this estimate, we need a scale-invariant inequality relating the  $H^{-1}$  and BV norms. Combining a standard Sobolev estimate with the Calderón-Zygmund inequality, we have

$$\|u\|_{H^{-1}} = \|(-\Delta)^{-1/2}u\|_{L^2} \leq A' \|\nabla(-\Delta)^{-1/2}u\|_{L^p} \leq A_p \|u\|_{L^p} \quad (3.7)$$

when  $1/2 = 1/p - 1/n$ . Here  $A'$  and  $A_p$  are scale-invariant estimates (i.e. they depend on the shape of the period cell, but not on its size). When  $n = 4$  we may take  $p = 4/3$  to get

$$\|u\|_{H^{-1}} \leq A_{4/3} \|u\|_{L^{4/3}} \leq A_{4/3} S_4 \int_{\mathbf{T}^n} |\nabla u| \quad \text{for any } u \in BV(\mathbf{T}^n). \quad (3.8)$$

Thus (3.6) implies

$$\frac{1}{2} \frac{d}{dt} \|u\|_{H^{-1}}^2(t) \leq -(A_{4/3} S_4)^{-1} \|u\|_{H^{-1}}(t) \quad \text{for a.e. } t > 0$$

whence

$$\frac{d}{dt} \|u\|_{H^{-1}}(t) \leq -(A_{4/3} S_4)^{-1} \text{ for } t < T^*(u_0).$$

We have proved the following extinction time estimate for  $n = 4$ :

**Theorem 3.3.** *Let  $u$  be the solution of (3.4) with initial data  $u_0 \in H_{\text{av}}^{-1}(\mathbf{T}^n)$ . If  $n = 4$ , then  $\|u\|_{H^{-1}}(t) \leq \|u_0\|_{H^{-1}} - (A_{4/3} S_4)^{-1} t$  for all  $t < T^*(u_0)$ . In particular,*

$$T^*(u_0) \leq A_{4/3} S_4 \|u_0\|_{H^{-1}}.$$

The fourth-order total variation flow in dimension 4 is like the second-order total variation flow in dimension 2: for these special dimensions, a scale-invariant extinction time estimate emerges directly from the basic “energy identity” through the use of scale-invariant Sobolev-type estimates. In the second-order setting our result for  $n > 2$  was obtained by multiplying the equation by  $u^{n-1}$  and integrating by parts. A similar argument seems unavailable in the fourth-order setting; as a result, we have no results on finite-time extinction in dimensions  $n > 4$ .

We can, however, estimate the extinction time of the fourth-order total variation flow in dimensions  $n = 1, 2, 3$ . The basic idea is to replace (3.8) by a scale-invariant interpolation estimate of the form

$$\|u\|_{H^{-1}} \leq C \|u\|_X^{1-\theta} \left( \int_{\mathbf{T}^n} |\nabla u| \right)^\theta$$

where  $X$  is a suitable (negative) norm. We will choose  $X$  such that  $\frac{d}{dt} \|u\|_X$  can be estimated using the PDE – so that while  $\|u\|_X$  is not constant it is controlled by the initial data. Then an argument parallel to that of Theorem 3.3 will give an upper bound on the extinction time.

The rest of this section is devoted to carrying out this program. We begin with the interpolation inequality. For any  $1 \leq p \leq \infty$ , define the  $\dot{W}^{-1,p}$  norm of a periodic function  $w$  by

$$\|w\|_{\dot{W}^{-1,p}} = \sup \left\{ \int_{\mathbf{T}^n} \varphi w \, dx : \varphi \in C_0^1(\mathbf{T}^n), \|\nabla \varphi\|_{L^{p'}} \leq 1 \right\} \quad \text{where } \frac{1}{p} + \frac{1}{p'} = 1.$$

**Lemma 3.4.** *Suppose  $1 \leq n \leq 4$ ,  $1 \leq p \leq \infty$ , and  $1/2 \leq \theta \leq 1$  are related by  $1 + \frac{n}{2} = \theta(n-1) + (1-\theta)(3 + \frac{n}{p})$ . Then there is a constant  $C_*$  such that*

$$\|u\|_{H^{-1}} \leq C_* \|(-\Delta)^{-1}u\|_{\dot{W}^{-1,p}}^{1-\theta} \left( \int_{\mathbf{T}^n} |\nabla u| \right)^\theta$$

for all  $u \in H_{\text{av}}^{-1}(\mathbf{T}^n) \cap BV(\mathbf{T}^n)$ . The constant is scale-invariant, i.e.  $C_*$  depends on the eccentricity of the periodic cell but not on its size. In addition,  $C_*$  is independent of  $p$ .

The proof of Lemma 3.4 will be given in Section 3.5. We turn now to question of how the PDE can be used to bound  $\frac{d}{dt} \|(-\Delta)^{-1}u\|_{\dot{W}^{-1,p}}$ . Formally, the idea is clear:  $(-\Delta)^{-1}u_t = \text{div}(\nabla u/|\nabla u|)$  is the divergence of an  $L^\infty$  vector field with magnitude 1, so we expect the  $\dot{W}^{-1,p}$  norm of  $(-\Delta)^{-1}u$  to grow at most linearly with rate  $\|\text{div}(\nabla u/|\nabla u|)\|_{\dot{W}^{-1,p}} \sim \|\nabla u/|\nabla u|\|_{L^p} \sim |\mathbf{T}^n|^{1/p}$ . To give a rigorous version of this calculation, we need some information about the  $L^2$  and  $H^{-1}$  subdifferentials of  $\Phi_\pi$ . Starting with the former:

**Lemma 3.5** (Characterization of  $\partial\Phi_\pi$ ). *Suppose  $v \in D(\Phi_\pi) \subset L_{\text{av}}^2(\mathbf{T}^n)$ . If  $f \in \partial\Phi_\pi(v)$ , then there exists a periodic  $z \in L^\infty(\mathbf{T}^n, \mathbf{R}^n)$  such that  $f = -\text{div} z$ ,  $\|z\|_{L^\infty} \leq 1$  and*

$$\int_{\mathbf{T}^n} f v \, dx = \Phi_\pi(v).$$

*The converse is also true.*

*Proof.* The analogous statement for  $\Phi_N$  is Lemma 1.10 of [ACM], and the proof given there applies equally to  $\Phi_\pi$ . The heart of the proof is the demonstration that

$$\tilde{\Phi}_{\pi L^2}(v) = \inf \{ \|z\|_{L^\infty} : z \in L^\infty(\mathbf{T}^n, \mathbf{R}^n), \operatorname{div} z \in L^2(\mathbf{T}^n), v = -\operatorname{div} z \}$$

from which the result follows easily, using our Lemma 2.3.  $\square$

Turning now to  $\partial_{H^{-1}}\Phi_\pi(v)$ , the subdifferential of  $\Phi_\pi$  defined using the  $H_{\text{av}}^{-1}(\mathbf{T}^n)$  inner product:

**Lemma 3.6** (Characterization of  $\partial_{H^{-1}}\Phi_\pi(v)$ ). *Suppose  $v \in D(\Phi_\pi) \subset H_{\text{av}}^{-1}(\mathbf{T}^n)$ . If  $f \in H_{\text{av}}^{-1}(\mathbf{T}^n)$  belongs to  $\partial_{H^{-1}}\Phi_\pi(v)$  then there exists a periodic  $z \in L^\infty(\mathbf{T}^n, \mathbf{R}^n)$  such that  $(-\Delta)^{-1}f = \operatorname{div} z$ ,  $\|z\|_{L^\infty} \leq 1$ , and*

$$\int_{\mathbf{T}^n} (-\Delta)^{-1}f \cdot v \, dx = \Phi_\pi(v).$$

*The converse is also true.*

*Proof.* This follows immediately from Lemma 3.5, using Lemma 3.1.  $\square$

We now use Lemma 3.6 to control the growth of  $\|(-\Delta)^{-1}u\|_{\dot{W}^{-1,p}}$ , for the solution of the fourth-order total variation flow.

**Lemma 3.7.** *Let  $u$  be the solution of (3.4) with initial data  $u_0 \in H_{\text{av}}^{-1}(\mathbf{T}^n)$ . Then for any  $1 \leq p \leq \infty$  we have*

$$\|(-\Delta)^{-1}u\|_{\dot{W}^{-1,p}}(t) \leq |\mathbf{T}^n|^{1/p}t + \|(-\Delta)^{-1}u_0\|_{\dot{W}^{-1,p}}$$

*for all  $t > 0$ . Here  $|\mathbf{T}^n|$  denotes the volume of the period cell, i.e.  $|\mathbf{T}^n| = \omega_1 \cdots \omega_n$ .*

*Proof.* Since  $\|\cdot\|_{\dot{W}^{-1,p}}$  is a norm, we have

$$\frac{d}{dt} \|(-\Delta)^{-1}u\|_{\dot{W}^{-1,p}}(t) \leq \|(-\Delta)^{-1}u_t\|_{\dot{W}^{-1,p}}(t).$$

By the characterization of  $\partial_{H^{-1}}\Phi_\pi$  in Lemma 3.6, we know that

$$\|(-\Delta)^{-1}u_t\|_{\dot{W}^{-1,p}}(t) = \|\operatorname{div} z\|_{\dot{W}^{-1,p}}(t)$$

for some  $z \in L^\infty(\mathbf{T}^n, \mathbf{R}^n)$  such that  $\|z\|_{L^\infty} \leq 1$ . The right hand side is, by definition,

$$\begin{aligned} \|\operatorname{div} z\|_{\dot{W}^{-1,p}} &= \sup \left\{ \int_{\mathbf{T}^n} (-\nabla\varphi) \cdot z \, dx : \varphi \in C_0^\infty(\mathbf{T}^n), \|\nabla\varphi\|_{L^{p'}} \leq 1 \right\} \\ &\leq \|z\|_{L^p} \leq |\mathbf{T}^n|^{1/p}. \end{aligned}$$

We thus conclude that

$$\frac{d}{dt} \|(-\Delta)^{-1}u\|_{\dot{W}^{-1,p}}(t) \leq |\mathbf{T}^n|^{1/p},$$

which yields the desired result.  $\square$

We are ready to state and prove our main extinction time estimate. The following result includes Theorem 3.3 (it is the special case  $n = 4$ ,  $\theta = 1$ ).

**Theorem 3.8.** *Suppose  $1 \leq n \leq 4$ ,  $1 \leq p \leq \infty$ , and  $1/2 < \theta \leq 1$  are related by  $1 + \frac{n}{2} = \theta(n-1) + (1-\theta)(3 + \frac{n}{p})$ . Let  $u$  be the solution of (3.4) with initial data  $u_0 \in H_{\text{av}}^{-1}(\mathbf{T}^n)$ . Then*

$$\|u\|_{H^{-1}}(t)^{2-(1/\theta)} \leq \|u_0\|_{H^{-1}}^{2-(1/\theta)} - \left(2 - \frac{1}{\theta}\right) C_*^{-1/\theta} \int_0^t A(s)^{1-(1/\theta)} ds$$

for all  $t < T^*(u_0)$ , with  $A(t) = |\mathbf{T}^n|^{1/p}t + \|(-\Delta)^{-1}u_0\|_{\dot{W}^{-1,p}}$ . As a consequence we have

$$T^*(u_0) \leq \frac{1}{2 - (1/\theta)} C_*^{1/\theta} \|(-\Delta)^{-1}u_0\|_{\dot{W}^{-1,p}}^{(1/\theta)-1} \|u_0\|_{H^{-1}}^{2-(1/\theta)}.$$

The constant  $C_*$  comes from the interpolation inequality of Lemma 3.4; in particular it is scale-invariant.

*Proof.* Let  $y(t) = \|u\|_{H^{-1}}(t)$ . Combining the energy inequality (3.6) and the interpolation inequality (Lemma 3.4) we have

$$y \frac{dy}{dt} \leq - \int_{\mathbf{T}^n} |\nabla u|^2 \leq -My^{1/\theta} \|(-\Delta)^{-1}u\|_{\dot{W}^{-1,p}}^{(\theta-1)/\theta}, \quad M = C_*^{-1/\theta}.$$

By Lemma 3.7 we have

$$\|(-\Delta)^{-1}u\|_{\dot{W}^{-1,p}}(t) \leq A(t), \quad A(t) = |\mathbf{T}^n|^{1/p}t + \|(-\Delta)^{-1}u_0\|_{\dot{W}^{-1,p}}.$$

Thus for  $t < T^*(u_0)$  we have

$$y^{1-(1/\theta)} \frac{dy}{dt} \leq -MA(t)^{(\theta-1)/\theta}.$$

Integrating over  $(0, t)$  gives

$$\begin{aligned} \frac{1}{2 - (1/\theta)} \left( y(t)^{2-1/\theta} - y_0^{2-1/\theta} \right) &\leq -M \int_0^t A(s)^{1-1/\theta} ds \\ &= -\frac{M}{a} \frac{1}{2 - (1/\theta)} \left\{ (at + A_0)^{2-1/\theta} - A_0^{2-1/\theta} \right\} \end{aligned}$$

with  $A_0 = A(0)$ ,  $a = |\mathbf{T}^n|^{1/p}$ ,  $y_0 = y(0)$ . We have thus proved the desired estimate for  $\|u\|_{H^{-1}}(t)$ . This implies the extinction time estimate

$$\begin{aligned} y_0^{2-(1/\theta)} &\geq M \left[ (aT^* + A_0)^{2-(1/\theta)} - A_0^{2-(1/\theta)} \right] / a \\ &= MA_0^{2-(1/\theta)} \left[ (1 + aT^*A_0^{-1})^{2-(1/\theta)} - 1 \right] / a \\ &\geq MA_0^{2-(1/\theta)} cT^*A_0^{-1} = cC_*^{-1/\theta} A_0^{1-(1/\theta)} T^* \end{aligned}$$

with  $c = 2 - (1/\theta)$ . The proof is now complete.  $\square$

The restriction  $1 + \frac{n}{2} = \theta(n-1) + (1-\theta)(3 + \frac{n}{p})$  permits  $\theta = 1$  only when  $n = 4$ . Thus while Theorem 3.8 permits some freedom in the selection of  $\theta$  and  $p$ , for  $n < 4$  its estimate of the extinction time always involves interpolation. It is natural to ask whether  $T^*(u_0) \leq C\|u_0\|_X$  where  $X$  is a suitably-defined Besov space. While we are unable to prove this, we can get something similar as follows.

**Proposition 3.9.** *Suppose  $n, p$ , and  $\theta$  are as in Theorem 3.8, with  $n < 4$ ,  $\theta < 1$ , and  $p < \infty$ . Let  $S : H_{\text{av}}^{-1}(\mathbf{T}^n) \rightarrow \mathbf{R}$  be the convexification of the extinction time  $T^*$ . Then*

$$S(u_0) \leq C\|u_0\|_X$$

with a scale-invariant constant  $C$ , when  $X$  is the real interpolation space  $(W_{\text{av}}^{-3,p}, H_{\text{av}}^{-1})_{\mu,1}$  with  $\mu = 2 - \theta^{-1}$ .

**Comment.** Since  $n < 4$ , the space  $W_{\text{av}}^{-3,p}$  is bigger than  $H_{\text{av}}^{-1}$  by the Sobolev embedding. The real interpolation space  $X$  defined above is a kind of Besov space; for example, if  $p = 2$  then  $X$  is the Besov space  $B_{2,1}^{2\mu-3}$  (see e.g. Theorem 6.2.4 of [BL]). By the way the reason Proposition 3.9 requires  $p < \infty$  is that the norm  $\|(-\Delta)^{-1}u\|_{\dot{W}^{-1,\infty}}$  is not equivalent to the  $W^{-3,\infty}$  norm.

*Proof of Proposition 3.9.* Since the fourth-order total variation flow is invariant under the scaling  $u \rightarrow \lambda u$ ,  $t \rightarrow \lambda t$ ,  $x \rightarrow x$ , the extinction time  $T^*$  is homogeneous of degree one; moreover  $T^*$  is clearly symmetric ( $T^*(-u_0) = -T^*(u_0)$ ), and  $T^*(u_0) = 0$  only when  $u_0 = 0$ . But we do not know that  $T^*$  is a norm, because we do not know it is convex. Therefore we consider its convexification  $S$ . One verifies easily that  $S$  is a seminorm, and clearly  $S \leq T^*$ . Therefore the Proposition is a consequence of our extinction time estimate (Theorem 3.8) combined with the general result stated below as Lemma 3.10.  $\square$

**Lemma 3.10.** *Suppose a pair of Banach spaces  $E$  and  $F$  are continuously embedded in a linear Hausdorff space so that  $E + F$  and  $E \cap F$  are well-defined. Let  $S(a)$  be a seminorm defined on  $E \cap F$ . If*

$$S(a) \leq C\|a\|_E^{1-\mu}\|a\|_F^\mu$$

for all  $a \in E \cap F$ , then

$$S(a) \leq C\|a\|_X \text{ with } X = (E, F)_{\mu,1}.$$

*Proof.* This well-known result can be found for example as Theorem 3.9.1 of [BL]. The proof is easy. First, observe that the interpolation inequality implies

$$S(a) \leq C2^{-j\mu}J(2^j, a, E, F)$$

for all  $j \in \mathbf{Z}$ , where

$$J(t, a, E, F) = \max\{\|a\|_E, t\|a\|_F\}.$$

If  $a = \sum u_j$ , we obtain

$$S(a) \leq \sum S(u_j) \leq C \sum 2^{-j\mu}J(2^j, u_j, E, F).$$

The infimum of the right hand sum over all decompositions  $a = \sum u_j$  is, by definition, the norm  $\|a\|_X$ .  $\square$

### 3.4 Proof of the interpolation estimate

This section presents the proof of Lemma 3.4. The case  $p = \infty$  is quite easy, so we address it first. Since

$$\|u\|_{H^{-1}}^2 = ((u, u))_{-1} = \int_{\mathbf{T}^n} (-\Delta)^{-1} u \cdot u \, dx,$$

we use the definition of the  $\dot{W}^{-1, \infty}$ -norm to get

$$\|u\|_{H^{-1}}^2 \leq \left( \int_{\mathbf{T}^n} |\nabla u| \right) \|(-\Delta)^{-1} u\|_{\dot{W}^{-1, \infty}};$$

this is the estimate for  $\theta = 1/2, p = \infty$  with  $C_* = 1$ . When  $n < 3$  and  $p = \infty$  the relation  $1 + \frac{n}{2} = \theta(n-1) + (1-\theta)(3 + \frac{n}{p})$  forces  $\theta = 1/2$ , so those cases are complete. When  $n = 4$  and  $p = \infty$  the relation permits  $1/2 \leq \theta \leq 1$ . But the estimate for  $\theta = 1/2$  has just been proved, and the estimate for  $\theta = 1$  is equation (3.8); the estimate follows for  $1/2 \leq \theta \leq 1$  by interpolation.

There is nothing further to prove when  $n = 4$ , since the relation linking  $n, p$ , and  $\theta$  requires  $p = \infty$  or  $\theta = 1$  when  $n = 4$ , and the inequality has already been proved in those cases.

For the remaining cases, when  $n \leq 3$  and  $1 \leq p < \infty$ , we proceed differently. Our overall strategy is to write  $u = u_r + u_s$  as the sum of a regular part  $u_r$  and a singular part  $u_s$ . This well-known approach to interpolation inequalities is especially convenient for negative norms. A recent application in [KoOt] defined  $u_r$  via convolution with a mollifier. Here we shall take  $u_r = e^{t\Delta} u$ , where  $e^{t\Delta} u$  is the solution of the heat equation with initial data  $u$ . This choice has been used by many authors; for example, the Nash-type inequality  $\|u\|_{L^2}^2 \leq C \|u\|_{L^1}^{4/(n+2)} \|\nabla u\|_{L^2}^{2-4/(n+2)}$  is proved using such a method in [VCC] (see Remark II, 3.3(a)). Another example is the proof of the Gagliardo-Nirenberg inequality in Chapter 6 of [GGS].

Writing  $u$  as

$$u = u_r + u_s = e^{t\Delta} u - \int_0^t \frac{d}{ds} e^{s\Delta} u \, ds = e^{t\Delta} u - \int_0^t \Delta e^{s\Delta} u \, ds$$

we have

$$\begin{aligned} \|u\|_{H^{-1}}^2 &= \int (-\Delta)^{-1} u \cdot u \, dx \\ &\leq \left| \int (-\Delta)^{-1} u \cdot u_r \, dx \right| + \left| \int (-\Delta)^{-1} u \cdot u_s \, dx \right| = I_1 + I_2. \end{aligned}$$

Here and for the rest of the proof we suppress the domain of integration  $\mathbf{T}^n$  to simplify the notation. By the definition of the  $\dot{W}^{-1, p}$  norm the term involving  $u_r$  can be estimated by

$$I_1 \leq \|\nabla e^{t\Delta} u\|_{L^{p'}} \|(-\Delta)^{-1} u\|_{\dot{W}^{-1, p}}.$$

By a well-known  $L^p - L^q$  estimate for the heat semigroup (see e.g. [GGS] when  $\Omega = \mathbf{R}^n$ ) we have

$$\|\nabla e^{t\Delta} u\|_{L^{p'}} = \|e^{t\Delta} \nabla u\|_{L^{p'}} \leq M_1 t^{-n/2p} \int |\nabla u| \tag{3.9}$$



with a scale-invariant constant  $M_1$  (depending on  $n$  and the eccentricity of the periodic cell) that's independent of  $p$ . We thus obtain

$$I_1 \leq M_1 t^{-n/2p} \left( \int |\nabla u| \right) \|(-\Delta)^{-1} u\|_{\dot{W}^{-1,p}}. \quad (3.10)$$

Now we estimate the term associated with the singular part  $u_s$ . We have

$$I_2 = \left| \int_0^t \left( \int e^{s\Delta} u \cdot u \, dx \right) ds \right| \leq \|u\|_{H^{-1}} \int_0^t \|\nabla e^{s\Delta} u\|_{L^2} \, ds.$$

By (3.9) with  $p = 2$ ,

$$\|\nabla e^{s\Delta} u\|_{L^2} \leq M_2 s^{-n/4} \int |\nabla u|.$$

Since  $1 \leq n \leq 3$ ,  $s^{-n/4}$  is integrable in  $(0, t)$ , and we get

$$I_2 \leq M_3 t^{1-n/4} \|u\|_{H^{-1}} \int |\nabla u|. \quad (3.11)$$

Estimates (3.10) and (3.11) combine to give

$$\|u\|_{H^{-1}}^2 \leq \left( M_1 t^{-n/2p} \|(-\Delta)^{-1} u\|_{\dot{W}^{-1,p}} + M_3 t^{1-n/4} \|u\|_{H^{-1}} \right) \int |\nabla u|$$

for all  $t > 0$ . Taking  $t$  so that

$$M_1 t^{-n/2p} \|(-\Delta)^{-1} u\|_{\dot{W}^{-1,p}} = M_3 t^{1-n/4} \|u\|_{H^{-1}}$$

we conclude that

$$\|u\|_{H^{-1}} \leq 2M_3 \left( M_1 \|(-\Delta)^{-1} u\|_{\dot{W}^{-1,p}} / M_3 \|u\|_{H^{-1}} \right)^{(1-n/4)/\beta} \int |\nabla u|$$

with  $\beta = 1 - n/4 + n/2p$ . This yields the desired estimate.

### 3.5 Non-scale-invariant estimates

It was our desire for a scale-invariant estimate that made us work so hard in Section 3.3 when  $n \leq 3$ . Non-scale-invariant estimates are easier. The crucial tool is the inequality

$$\|u\|_{H^{-1}} \leq K_n |\mathbf{T}^n|^\alpha \int_{\mathbf{T}^n} |\nabla u| \quad \text{with } \alpha = \frac{2}{n} - \frac{1}{2}, \text{ for } n = 1, 2, 3, \quad (3.12)$$

for any spatially periodic  $u$  with mean value 0. Here  $|\mathbf{T}^n|$  is the volume of the period cell, and constant  $K_n$  is scale-invariant but (for  $n > 1$ ) it depends on the eccentricity of the unit cell. To justify (3.12) it suffices, by scaling, to consider the case when  $|\mathbf{T}^n| = 1$ . For  $n = 3$  we use (3.7), Hölder's inequality, and (2.3) to get

$$\|u\|_{H^{-1}} \leq A_{6/5} \|u\|_{L^{6/5}} \leq A_{6/5} \|u\|_{3/2} \leq S_3 A_{6/5} \int |\nabla u|$$

when  $|\mathbf{T}^3| = 1$ . For  $n = 2$  we combine the Poincaré-type inequality  $\|u\|_{H^{-1}} \leq B|\mathbf{T}^n|^{1/n}\|u\|_{L^2}$  with (2.3) to get

$$\|u\|_{H^{-1}} \leq B\|u\|_{L^2} \leq BS_2 \int |\nabla u|$$

when  $|\mathbf{T}^2| = 1$ . For  $n = 1$  we argue similarly, combining the Poincaré-type inequality with Hölder's inequality and the fact that  $\|u\|_{L^\infty} \leq \int |u_x|$  to get

$$\|u\|_{H^{-1}} \leq B\|u\|_{L^2} \leq B\|u\|_{L^\infty} \leq B \int |u_x|$$

when  $|\mathbf{T}^1| = 1$ .

**Theorem 3.11.** *Suppose  $1 \leq n \leq 3$ , and let  $u$  be the solution of (3.4) with initial data  $u_0 \in H_{\text{av}}^{-1}(\mathbf{T}^n)$ . Then*

$$\|u\|_{H^{-1}}(t) \leq \|u_0\|_{H^{-1}} - (K_n|\mathbf{T}^n|^\alpha)^{-1}t \text{ for } t < T^*(u_0),$$

where  $K_n$  is the constant in (3.12) and  $\alpha = \frac{2}{n} - \frac{1}{2}$ . In particular,

$$T^*(u_0) \leq K_n|\mathbf{T}^n|^\alpha\|u_0\|_{H^{-1}}.$$

*Proof.* This is an immediate consequence of Lemma 3.2 and equation (3.12).  $\square$

### 3.6 Dirichlet or Neumann boundary condition

Recall that the fourth-order total variation flow with a Dirichlet boundary condition is defined by (3.5), and the evolution with a Neumann boundary condition is defined by the analogue of (3.5) with  $\Phi_D$  replaced by  $\Phi_N$  and  $H^{-1}(\Omega)$  replaced by  $H_{\text{av}}^{-1}(\Omega)$ .

Our 4-dimensional result Theorem 3.3 extends straightforwardly to the Dirichlet and Neumann settings. In fact, the argument used for the proof applies equally well to  $\Phi_D$  and  $\Phi_N$ .

It is natural to ask whether our more general Theorem 3.8 also extends to the Dirichlet and Neumann settings. We suppose the answer should be yes, but we have not worked out a complete proof. The main obstacle is the analogue of our interpolation inequality (Lemma 3.4). In the Dirichlet setting, for example, the argument we used for Theorem 3.8 would require the estimate

$$\|u\|_{H^{-1}(\Omega)} \leq C\|(-\Delta_D)^{-1}u\|_{\dot{W}^{-1,p}(\Omega)}^{1-\theta} \left( \int_{\Omega} |\nabla u| + \int_{\partial\Omega} |u| \right)^\theta$$

for all  $u \in H^{-1}(\Omega) \cap BV(\Omega)$ . A proof directly parallel to that of Lemma 3.4 (using the heat semigroup with a Dirichlet or Neumann boundary condition) is not straightforward, because the argument involves BV norms, and because  $\nabla e^{t\Delta}u \neq e^{t\Delta}\nabla u$  in the Dirichlet and Neumann settings (such a commutation relation was used in (3.9)).

The interpolation estimate is the only obstacle. Indeed, the other main ingredient in our proof of Theorem 3.8 was a bound on the growth of  $\|(-\Delta)^{-1}u\|_{\dot{W}^{-1,p}}$ . In the periodic setting we relied on the characterization of  $\partial\Phi_\pi$  given by Lemma 3.5. There are in fact analogous characterizations of  $\Phi_N$  and  $\Phi_D$ :

**Lemma 3.12** (Characterization of  $\partial\Phi_N$ ). *Let  $\Omega$  be a bounded domain with Lipschitz boundary, or else  $\Omega = \mathbf{R}^n$ . Suppose  $v \in D(\Phi_N) \subset L^2(\Omega)$ . Then  $f \in L^2(\Omega)$  belongs to  $\partial\Phi_N(v)$  if and only if there exists  $z \in L^\infty(\Omega, \mathbf{R}^n)$  such that  $f = -\operatorname{div} z$  in  $\Omega$ ,  $\|z\|_{L^\infty} \leq 1$ ,  $z \cdot \nu = 0$  on  $\partial\Omega$  and  $\int_\Omega f v \, dx = \Phi_N(v)$ .*

**Lemma 3.13** (Characterization of  $\partial\Phi_D$ ). *Let  $\Omega$  be a bounded domain with Lipschitz boundary, and suppose  $v \in D(\Phi_D) \subset L^2(\Omega)$ . Then  $f \in L^2(\Omega)$  belongs to  $\partial\Phi_D(v)$  if and only if there exists  $z \in L^\infty(\Omega, \mathbf{R}^n)$  such that  $f = -\operatorname{div} z$  in  $\Omega$ ,  $\|z\|_{L^\infty} \leq 1$ , and  $\int_\Omega f v \, dx = \Phi_D(v)$ .*

Lemma 3.12 is essentially Proposition 1.10 of [ACM], and Lemma 3.13 follows from Lemma 5.13 of [ACM]. Using these results in place of Lemma 3.5, one can obtain the analogue of Lemma 3.6 with  $\Phi_\pi$  replaced by  $\Phi_N$  or  $\Phi_D$ , and one can estimate the growth rate of  $\|(-\Delta_N)^{-1}u\|_{\dot{W}^{-1,p}}$  or  $\|(-\Delta_D)^{-1}u\|_{\dot{W}^{-1,p}}$  by arguing as for Lemma 3.7.

## 4 The fourth-order surface diffusion law

We turn now to the fourth-order surface diffusion law

$$u_t = -\Delta \left[ \operatorname{div} \left( \frac{\nabla u}{|\nabla u|} + \mu |\nabla u|^{q-2} \nabla u \right) \right] \quad (4.1)$$

with  $q > 1$  and  $\mu > 0$ , focusing for simplicity on the periodic setting. As noted in the Introduction, it represents  $H^{-1}$  steepest descent for

$$\int_{\mathbf{T}^n} |\nabla u| + \frac{\mu}{q} \int_{\mathbf{T}^n} |\nabla u|^q \, dx.$$

To define the solution rigorously, we observe that

$$\Phi_\pi^q(u) = \begin{cases} \Phi_\pi(u) + \frac{\mu}{q} \int_{\mathbf{T}^n} |\nabla u|^q \, dx, & \nabla u \in L^q(\mathbf{T}^n) \cap H_{\text{av}}^{-1}(\mathbf{T}^n) \\ \infty, & \text{otherwise.} \end{cases} \quad (4.2)$$

is a convex, lower semicontinuous function on  $H_{\text{av}}^{-1}(\mathbf{T}^n)$ . (The function  $\Phi_\pi^q$  also depends on  $\mu$ ; we suppress this dependence, for notational simplicity.) Therefore

$$\frac{du}{dt}(t) \in -\partial_{H^{-1}} \Phi_\pi^q(u(t)) \text{ for a.e. } t > 0, \text{ with } u|_{t=0} = u_0 \in H_{\text{av}}^{-1}(\mathbf{T}^n) \quad (4.3)$$

has a unique solution. This is the meaning of (4.1).

Our analysis of finite-time extinction will be entirely parallel to that of Section 3; the ‘‘extra term’’  $\frac{\mu}{q} |\nabla u|^q$  doesn’t get in the way, but it doesn’t help much either. The analogue of Lemma 3.2 is

**Lemma 4.1.** *Let  $u$  be the solution of (4.3) with initial data  $u_0 \in H_{\text{av}}^{-1}(\mathbf{T}^n)$ . Then*

$$\frac{1}{2} \frac{d}{dt} \|u\|_{H^{-1}}^2(t) = - \int_{\mathbf{T}^n} |\nabla u(\cdot, t)| - \mu \int_{\mathbf{T}^n} |\nabla u(\cdot, t)|^q \, dx \quad \text{for a.e. } t > 0. \quad (4.4)$$

*Proof.* This is an immediate consequence of Lemma 2.2.  $\square$

For  $1 \leq n \leq 4$  we get the same extinction time estimates as in Theorem 3.3 and Theorem 3.11, since (4.4) implies an inequality similar to (3.6).

To obtain an estimate similar to that of Theorem 3.8 we need a growth estimate for  $\|(-\Delta)^{-1}u\|_{\dot{W}^{-1,p}}$  analogous to the one in Lemma 3.7. We will show

**Lemma 4.2.** *Let  $u$  be the solution of (4.3) with initial data  $u_0 \in H_{\text{av}}^{-1}(\mathbf{T}^n)$ . Assume that  $1 \leq p \leq q/(q-1)$ . Then*

$$\|(-\Delta)^{-1}u\|_{\dot{W}^{-1,p}}(t) \leq at + \|(-\Delta)^{-1}u_0\|_{\dot{W}^{-1,p}}$$

for all  $t > 0$ , where

$$a = |\mathbf{T}^n|^{1/p} + \mu^{1/q} |\mathbf{T}^n|^{1/r} (q\Phi_\pi^q(u_0))^{1-1/q}, \quad \text{with } 1/p + 1/q - 1 = 1/r. \quad (4.5)$$

Given this result, the argument used to prove Theorem 3.8 works just as well in the present setting. The outcome is:

**Theorem 4.3.** *Suppose  $1 \leq n \leq 4$ ,  $1 \leq p \leq q/(q-1)$ , and  $1/2 < \theta \leq 1$  are related by  $1 + \frac{n}{2} = \theta(n-1) + (1-\theta)(3 + \frac{n}{p})$ . Let  $u$  be the solution of (4.3) with initial data  $u_0 \in H_{\text{av}}^{-1}(\mathbf{T}^n)$ . Then*

$$\|u\|_{H^{-1}}(t)^{2-(1/\theta)} \leq \|u_0\|_{H^{-1}}^{2-(1/\theta)} - \left(2 - \frac{1}{\theta}\right) C_*^{-1/\theta} \int_0^t A_q(s)^{1-(1/\theta)} ds \quad (4.6)$$

for  $t < T^*(u_0)$ , with

$$A_q(t) = at + \|(-\Delta)^{-1}u_0\|_{\dot{W}^{-1,p}},$$

where  $a$  is given by (4.5). As a consequence, we have

$$T^*(u_0) \leq \frac{1}{2 - (1/\theta)} C_*^{1/\theta} \|(-\Delta)^{-1}u_0\|_{\dot{W}^{-1,p}}^{(1/\theta)-1} \|u_0\|_{H^{-1}}^{2-(1/\theta)}. \quad (4.7)$$

The constant  $C_*$  comes from the interpolation inequality of Lemma 3.4; in particular it is scale-invariant.

**Comment.** Unlike the analogous results in Section 3, the right hand side our estimate (4.6) can be infinite if  $u_0 \in H_{\text{av}}^{-1}(\mathbf{T}^n)$  has  $\Phi_\pi^q(u_0) = \infty$ . But our bound on the extinction time (4.7) does not depend on the initial value of  $\Phi_\pi^q$ ; therefore it holds even when  $\Phi_\pi^q(u_0) = \infty$ . To see this, observe that if  $\Phi_\pi^q(u_0) = \infty$ , we can approximate  $u_0$  in the  $H_{\text{av}}^{-1}(\mathbf{T})$  norm by perturbed initial data  $u_0^\epsilon \in D(\Phi_\pi^q)$ . The associated solutions remain close:

$$\|u - u^\epsilon\|_{H^{-1}}(t) \leq \|u_0 - u_0^\epsilon\|_{H^{-1}},$$

since the semigroup generated by  $\partial_{H^{-1}}\Phi_\pi^q$  is a contraction semigroup. Our extinction time estimates for the solutions starting from  $u_0^\epsilon$  are uniform as  $\epsilon \rightarrow 0$ . Passing to the limit, the solution starting from  $u_0$  must satisfy the same extinction time estimate.

The rest of this section is devoted to proving Lemma 4.2. As in Section 3.6, the main issue is an adequate understanding of the subdifferential  $\partial\Phi_\pi^q$ . This is nontrivial, because the subdifferential of a sum is not always equal to the sum of the subdifferentials.

**Lemma 4.4** (Characterization of  $\partial_{H^{-1}}\Phi_\pi^q(v)$ ). *Suppose  $1 \leq n \leq 4$ , and assume that  $v \in D(\Phi_\pi^q) \subset H_{\text{av}}^{-1}(\mathbf{T}^n)$ . Then  $f \in H_{\text{av}}^{-1}(\mathbf{T}^n)$  belongs to  $\partial_{H^{-1}}\Phi_\pi^q(v)$  if and only if*

$$(-\Delta)^{-1}f = \operatorname{div}(z + \mu|\nabla v|^{q-2}\nabla v)$$

for some  $z \in L^\infty(\mathbf{T}^n, \mathbf{R}^n)$  satisfying  $\|z\|_{L^\infty} \leq 1$ ,  $\operatorname{div} z \in L_{\text{av}}^2(\mathbf{T}^n)$  and  $z(x) = \nabla v(x)/|\nabla v(x)|$  for a.e.  $x$  such that  $|\nabla v(x)| \neq 0$ .

*Proof.* This was essentially proved by Kashima in Corollary 3.12 of [Ka]. In fact, he characterized  $\partial\Phi_D^q$  (with a Dirichlet condition) in  $H^{-1}(\Omega)$  when  $\Omega$  is a bounded domain with piecewise smooth boundary in  $\mathbf{R}^n$  for  $1 \leq n \leq 4$ , but his arguments also work in the periodic setting.

When  $1/2 + 1/n > 1/q$  we can also offer the following alternative argument. Recall that  $f \in \partial_{H^{-1}}\Phi_\pi^q(v)$  implies  $(-\Delta)^{-1}f \in \partial_{L^2}\Phi_\pi^q(v)$  provided that  $v \in L_{\text{av}}^2(\mathbf{T}^n)$ . If  $1/2 + 1/n > 1/q$  then  $D(\Phi_\pi^q) \subset L_{\text{av}}^2(\mathbf{T}^n)$ , so the condition that  $v \in L_{\text{av}}^2$  is redundant. Now, the  $L^2$  subdifferential was characterized by Attouch and Damlamian in [AD] (they used Dirichlet boundary condition, but their argument works just as well in the periodic setting.) Their result gives the desired conclusion.  $\square$

*Proof of Lemma 4.2.* We may assume that  $\nabla u_0 \in L^q(\mathbf{T})$ . The basic idea is the same as the proof of Lemma 3.7. By the characterization of  $\partial_{H^{-1}}\Phi_\pi^q$  in Lemma 4.2 we have

$$\frac{d}{dt}\|(-\Delta)^{-1}u\|_{\dot{W}^{-1,p}}(t) \leq \|\operatorname{div}(z + \mu|\nabla u|^{q-2}\nabla u)\|_{\dot{W}^{-1,p}}(t)$$

for some  $z \in L^\infty(\mathbf{T}^n, \mathbf{R}^n)$  satisfying  $\|z\|_{L^\infty} \leq 1$ . By definition of the norm we have

$$\|\operatorname{div}(z + \mu|\nabla u|^{q-2}\nabla u)\|_{\dot{W}^{-1,p}} \leq \|(z + \mu|\nabla u|^{q-2}\nabla u)\|_{L^p}.$$

Applying Hölder's inequality, we conclude that

$$\|(z + \mu|\nabla u|^{q-2}\nabla u)\|_{L^p} \leq |\mathbf{T}^n|^{1/p} + \mu|\mathbf{T}^n|^{1/r}\|\nabla u\|_q^{q-1}.$$

Since  $u$  is the solution of the gradient flow associated with  $\Phi_\pi^q$ , the value of  $\Phi_\pi^q(u(t))$  is nonincreasing, so

$$\mu\|\nabla u(t)\|_q^q \leq q\Phi_\pi^q(u(t)) \leq q\Phi_\pi^q(u_0),$$

or in other words

$$\mu\|\nabla u(t)\|_q^{q-1} \leq \mu^{1/q}(q\Phi_\pi^q(u_0))^{1-1/q}.$$

Combining these estimates, we conclude that

$$\frac{d}{dt}\|(-\Delta)^{-1}u\|_{\dot{W}^{-1,p}}(t) \leq |\mathbf{T}^n|^{1/p} + |\mathbf{T}^n|^{1/r}\mu^{1/q}(q\Phi_\pi^q(u_0))^{1-1/q},$$

which yields Lemma 4.2.  $\square$

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