Family of two-dimensional ideal fluid dynamics related to surface quasi-geostrophic equation

Koji Ohkitani (University of Sheffield)

Abstract

We study 2D surface quasi-geostrophic (SQG) equation numerically and theoretically. After reviewing recent results, we consider a generalised class of equations of ideal fluid, where the active scalar is a fractional power α of Laplacian applied to the stream function. This includes 2D SQG and 2D Euler equations as special cases. We present some numerical results of the generalised system and compare them for some different values of α. In an attempt to unify the whole family systematically, a successive approximation is introduced to treat the SQG equation.

I. INTRODUCTION

Mathematical study on the SQG equation was initiated in [1, 2]. Since then many papers have been published regarding the analyses of this equation, which are too numerous to cite here. Numerical studies have been done, e.g. in [1–6]. Mathematically, the following is the best result known for its regularity. We consider the SQG equation with hypo-viscous dissipativity either in $\mathbb{R}^2$ or in $\mathbb{T}^2$

$$\frac{\partial \theta}{\partial t} + (\boldsymbol{u} \cdot \nabla)\theta = -\nu(-\triangle)^{\gamma}\theta \quad (0 \le \gamma \le 1),$$

with an initial datum $\theta(\boldsymbol{x}, 0) = \theta_0(\boldsymbol{x})$. The velocity $\boldsymbol{u} = -\nabla^{\perp}(-\triangle)^{-1/2}\theta$ is a skewed Riesz transform of θ, where $\nabla^{\perp} = (\partial_y, -\partial_x)$. It has been proved that when $\gamma \ge \frac{1}{2}$ we have no blow-up [7, 8]. The hypo-viscous equation has been studied numerically in [9]. See also [10] for more related works.

II. GENERALISED SQG EQUATION FOR INVISCID FLUIDS

We consider a generalised version of SQG equation [3, 11] for inviscid fluids

$$\frac{\partial \theta}{\partial t} + (\boldsymbol{u} \cdot \nabla)\theta = 0, \tag{1}$$

with $\theta(\boldsymbol{x},0)=\theta_0(\boldsymbol{x})$. Here the velocity $\boldsymbol{u}$ is given by

$$\boldsymbol{u}=\nabla^{\perp}\psi, \quad \Lambda^{\alpha}\psi=\theta \quad (0\leq\alpha\leq 2).$$

Here $\Lambda\equiv(-\triangle)^{1/2}$ is Zygmund operator defined by Fourier transform $\tilde{\Lambda}=|\boldsymbol{k}|$. The system reduces to the 2D Euler equations if $\alpha=2$, to the 2D SQG equation if $\alpha=1$, and to a trivially steady state if $\alpha=0$.

III. PERTURBATION THEORY: ODE ANALOGY

We recall a perturbation theory à la Poincaré of an ordinary differential equation (ODE) which depends upon a parameter μ, see e.g. [12, 13]. (We note that notations used in this section are independent from those in the rest of the extended abstract.)

Consider an ODE

$$\frac{dy}{dx}=f(x,y,\mu), \text{ with an initial datum } y(x_0,\mu)=y_0,$$

which is assumed to be solvable for $\mu=\mu_0$. If we consider a variation

$$z(x,\mu)=\frac{\partial y(x,\mu)}{\partial\mu}, \text{ with an initial datum } z(x_0,\mu)=0,$$

it satisfies

$$\frac{dz}{dx}=\left.\frac{\partial f(x,Y,\mu)}{\partial Y}\right|_{Y=y(x,\mu)}z+\left.\frac{\partial f(x,Y,\mu)}{\partial\mu}\right|_{Y=y(x,\mu)},$$

which is called an equation of variation.

An approximation for $y(x,\mu)$ for small $|\mu-\mu_0|$ may be written

$$y(x,\mu)-y(x,\mu_0)=\sum_{n=1}^{\infty}(\mu-\mu_0)^n C_n(x),$$

where $C_n(x)$ are suitable coefficients, e.g.

$$z(x,\mu_0)=\lim_{\mu\to\mu_0}\frac{y(x,\mu)-y(x,\mu_0)}{\mu-\mu_0}=C_1(x).$$

IV. SUCCESSIVE APPROXIMATIONS

We apply the above idea to the generalised SQG equation. We illustrate how this is done for the first variation. If we take the variation of (1) with respect to α, we find

$$\frac{D}{Dt}\frac{\partial\theta}{\partial\alpha}\equiv\frac{\partial}{\partial t}\frac{\partial\theta}{\partial\alpha}+\boldsymbol{u}\cdot\nabla\frac{\partial\theta}{\partial\alpha}=-\frac{\partial\boldsymbol{u}}{\partial\alpha}\cdot\nabla\theta.$$

In $\mathbb{R}^2$, we find more explicitly after straightforward manipulations [14]

$$\frac{D}{Dt}\frac{\partial\theta}{\partial\alpha} = \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{(\boldsymbol{x}-\boldsymbol{y})^\perp}{|\boldsymbol{x}-\boldsymbol{y}|^2}\frac{\partial\theta(\boldsymbol{y})}{\partial\alpha}d\boldsymbol{y}\cdot\nabla\theta(\boldsymbol{x}) + \frac{1}{4\pi}\int_{\mathbb{R}^2}(\log|\boldsymbol{x}-\boldsymbol{y}|)^2\,\nabla^\perp\theta(\boldsymbol{y})d\boldsymbol{y}\cdot\nabla\theta(\boldsymbol{x}).$$

In principle, the equations for higher-order variations may be obtained by successive differentiations. Given these, we may write, for example, near the 2D Euler limit $\alpha = 2$

$$\theta(\boldsymbol{x},t,\alpha) = \theta(\boldsymbol{x},t,2) + \sum_{n=1}^{\infty}(\alpha-2)^n\theta_n(\boldsymbol{x},t),$$

where $\theta_n(\boldsymbol{x},t) \equiv \frac{\partial^n\theta}{\partial\alpha^n}(\boldsymbol{x},t)$.

V. CONCLUSION

In fact, under periodic boundary conditions we can carry out the analyses more systematically. A formal analysis in this case indicates that all the members in the family behave similarly with respect to a 'new time variable' $\xi = \alpha t$. We discuss the implications of this scaling, in connection with numerical simulations. These are to be reported in detail in [14].

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