ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO NONLINEAR SCHRÖDINGER EQUATIONS

NAKAO HAYASHI

1. Cubic nonlinear Schrödinger equation

We consider the nonlinear Schrödinger equation

$$i\partial_t u + \frac{1}{2m}\Delta u = \lambda |u|^2 u \tag{1.1}$$

in $\mathbf{R}$, where $\Delta = \partial_x^2, m$ is mass of particle, $\lambda \in \mathbf{C}$. Equation (1.1) is non relativistic version of nonlinear Klein-Gordon equation

$$\frac{1}{2c^2 m}\partial_t^2 v - \frac{1}{2m}\Delta v + \frac{mc^2}{2} v = -\lambda |v|^2 v, \tag{1.2}$$

where c is the speed of light. Indeed we change $v = e^{-itmc^2} u$ to get

$$\frac{1}{2c^2 m}\partial_t^2 u - i\partial_t u - \frac{1}{2m}\Delta u = -\lambda |u|^2 u.$$

If we let $c \to \infty$, then we can obtain (1.1). We survay results on asymptotic behavior of small solutions to the initial value problem and the final value problem for (1.1). We use the following factorization formula for the free Schrödinger evolution group $\mathcal{U}\left(\frac{t}{m}\right) = \exp\left(\frac{i}{2m}\Delta\right)$such that

$$\mathcal{U}\left(\frac{t}{m}\right) = M^m(t)\,\mathcal{D}\left(\frac{t}{m}\right)\mathcal{V}\left(\frac{t}{m}\right)\mathcal{F}.$$

This formula is useful to study asymptotic behavior of solutions and used in paper [8]. We have from the above

$$\mathcal{F}\mathcal{U}\left(-\frac{t}{m}\right) = \mathcal{V}\left(-\frac{t}{m}\right) i\overline{E}^{\frac{1}{m}}(t)\,\mathcal{D}\left(\frac{m}{t}\right),$$

where we denote

$$M(t) = e^{\frac{i}{2t}|x|^2}, E(t) = e^{\frac{it}{2}|\xi|^2},$$

a dilation operator

$$(\mathcal{D}(t)\,\phi)(x) = \frac{1}{it}\phi(\frac{x}{t})$$

and

$$\mathcal{V}\left(\frac{t}{m}\right) = \mathcal{F}M^m(t)\,\mathcal{F}^{-1}.$$

Note that

$$\mathcal{D}\left(\frac{m}{t}\right)M^m(t) = E^{\frac{1}{m}}(t)\,\mathcal{D}\left(\frac{m}{t}\right).$$

Key words and phrases. Modified wave operator, Nonlinear Schrödinger equations.

Multiplying both sides of (1.1) by $\mathcal{FU}\left(-\frac{t}{m}\right)$ and putting $w=\mathcal{FU}\left(-\frac{t}{m}\right)u$, we obtain

$$i\partial_t w=\lambda\mathcal{FU}\left(-\frac{t}{m}\right)|u|^2 u.$$

Asymptotic behavior of small solutions to the Cauchy problem for (1.1) is obtained by showing the right hand side of the above is decomposed into two terms:

$$\lambda\mathcal{FU}\left(-\frac{t}{m}\right)|u|^2 u=\lambda\frac{m}{t}|w|^2 w+R,$$

where R is considered as a remainder term. Therefore asymptotic behavior of solutions for (1.1) is determined by the ordinary differential equation

$$i\partial_t\widehat{w}=\lambda\frac{m}{t}|\widehat{w}|^2\widehat{w}.$$

Indeed, for the final value problem, we can find a solution in the neighborhood of solutions of the ordinary differential equations. We let $\widehat{w}=re^{i\psi}$, then we have

$$i\partial_t r-r\partial_t\psi=\lambda\frac{m}{t}|r|^2 r$$

from which it follows that if $\lambda=\lambda_1+i\lambda_2,\lambda_1,\lambda_2\in\mathbf{R}$

$$\partial_t r=\lambda_2\frac{m}{t}r^3,-\partial_t\psi=\lambda_1\frac{m}{t}|r|^2.$$

By a given function ϕ, we have for $\lambda_2\le 0$

$$r(t)=\frac{\left|\widehat{\phi}\right|}{\left(1-2m\lambda_2\left|\widehat{\phi}\right|^2\log t\right)^{\frac{1}{2}}},$$

and for $\lambda_2<0$

$$\begin{aligned}\psi(t)&=\frac{\lambda_1}{2\lambda_2}\int^t\frac{d\left(-2\lambda_2 m\left|\widehat{\phi}\right|^2\log t\right)}{\left(1-2m\lambda_2\left|\widehat{\phi}\right|^2\log t\right)}\\&=\frac{\lambda_1}{2\lambda_2}\log\left(1-2m\lambda_2\left|\widehat{\phi}\right|^2\log t\right)\end{aligned}$$

for $\lambda_2=0$

$$\psi(t)=-\lambda_1 m\left|\widehat{\phi}\right|^2\log t.$$

Thus we have the solution of ordinary differential equation such that

$$\begin{aligned}\widehat{w}&=re^{i\psi}\\&=\frac{\left|\widehat{\phi}\right|\exp\left(i\frac{\lambda_1}{2\lambda_2}\log\left(1-2m\lambda_2\left|\widehat{\phi}\right|^2\log t\right)\right)}{\left(1-2m\lambda_2\left|\widehat{\phi}\right|^2\log t\right)^{\frac{1}{2}}},\end{aligned}$$

for $\lambda_2<0$ and

$$\begin{aligned}\widehat{w}&=re^{i\psi}\\&=\left|\widehat{\phi}\right|\exp\left(-i\lambda_1 m\left|\widehat{\phi}\right|^2\log t\right),\end{aligned}$$

for $\lambda_2 = 0$. We make a changing of variable $\widehat{\varphi}(t) = \left|\widehat{\phi}\right| \exp\left(i\frac{\lambda_1}{2\lambda_2}\log\left(1 - 2m\lambda_2\left|\widehat{\phi}\right|^2 \log t\right)\right)$ or $\widehat{\varphi}(t) = \left|\widehat{\phi}\right| \exp\left(-i\lambda_1 m\left|\widehat{\phi}\right|^2 \log t\right)$. Then

$$\widehat{w} = \frac{\widehat{\varphi}(t)}{\left(1 - 2m\lambda_2 \left|\widehat{\varphi}(t)\right|^2 \log t\right)^{\frac{1}{2}}}, \lambda_2 < 0$$

$$\widehat{w} = \widehat{\varphi}(t) \exp\left(-i\lambda_1 m \left|\widehat{\varphi}(t)\right|^2 \log t\right), \lambda_2 = 0.$$

2. System of NLS in 2D

In this section we report the recent results obtained in [5]. We consider a system of nonlinear Schrödinger equations

$$\left\{\begin{array}{l} i\partial_t v_1 + \frac{1}{2m_1}\Delta v_1 = \lambda \overline{v_1} v_2 \\ i\partial_t v_2 + \frac{1}{2m_2}\Delta v_2 = \mu v_1^2 \end{array}\right. \tag{2.1}$$

in $\mathbf{R}^2$, where $\Delta = \sum_{j=1}^2 \partial_j^2, \partial_j = \partial/\partial x_j, m_1, m_2$ are masses of particles and $\lambda, \mu \in \mathbf{C}$. We make the scaling $v_1 = \frac{1}{\sqrt{|\lambda\mu|}}u_1$ and $v_2 = \frac{\mu}{|\lambda\mu|}u_2$, to exclude the constants λ and μ from system (2.1) to get

$$\left\{\begin{array}{l} i\partial_t u_1 + \frac{1}{2m_1}\Delta u_1 = \gamma \overline{u_1} u_2, \\ i\partial_t u_2 + \frac{1}{2m_2}\Delta u_2 = u_1^2, \end{array}\right. \tag{2.2}$$

where $\gamma = \frac{\lambda\mu}{|\lambda\mu|} \in \mathbf{C}$, $|\gamma| = 1$. We assume the mass condition

$$2m_1 = m_2 \tag{2.3}$$

which is called the resonance condition. We also consider the case

$$2m_1 \neq m_2, m_1 \neq m_2 \tag{2.4}$$

which is call the non resonance condition.

The system (2.2) is non relativistic version of a system of nonlinear Klein-Gordon equations

$$\left\{\begin{array}{l} \frac{1}{2c^2 m_1}\partial_t^2 v_1 - \frac{1}{2m_1}\Delta v_1 + \frac{m_1 c^2}{2} v_1 = -\gamma \overline{v_1} v_2, \\ \frac{1}{2c^2 m_2}\partial_t^2 v_2 - \frac{1}{2m_2}\Delta v_2 + \frac{m_2 c^2}{2} v_2 = -v_1^2, \end{array}\right. \tag{2.5}$$

where c is the speed of light.

We introduce the weighted Sobolev space

$$\mathbf{H}^{m,s} = \left\{ f = (f_1, f_2) \in \mathbf{L}^2; \|f\|_{\mathbf{H}^{m,s}} = \sum_{j=1}^{2} \|f_j\|_{\mathbf{H}^{m,s}} < \infty \right\},$$

where

$$\|f\|_{\mathbf{H}^{m,s}} = \left\| (1-\Delta)^{\frac{m}{2}} \left(1 + |x|^2\right)^{\frac{s}{2}} f \right\|_{\mathbf{L}^2}.$$

We write $\mathbf{H}^m = \mathbf{H}^{m,0}$ for simplicity.

Under the resonance condition (2.3) we prove

Theorem 1. *Let* $2m_1 = m_2$, $\gamma < 0, t \geq 1, \widehat{w_1} \in \mathbf{H}^{2,0}$ *and* $|\widehat{w_1}(\xi)| \geq \delta > 0$. *Then there exists an* $\varepsilon > 0$ *such that (2.2) has a unique global solution*

$$(u_1(t), u_2(t)) \in \mathbf{C}\left([1,\infty); \mathbf{L}^2 \times \mathbf{L}^2\right)$$

satisfying the asymptotics

$$\left\| u_1(t) - \frac{1}{m_1}\mathcal{U}\left(\frac{t}{m_1}\right)\mathcal{F}^{-1}\psi_{1+}\left(t, \frac{\cdot}{m_1}\right)\right\|_{\mathbf{L}^2} + \left\| u_2(t) - \frac{1}{m_2}\mathcal{U}\left(\frac{t}{m_2}\right)\mathcal{F}^{-1}\psi_{2+}\left(t, \frac{\cdot}{m_2}\right)\right\|_{\mathbf{L}^2} \leq Ct^{-b}$$

for $t \geq 1$ *and any* $\widehat{w_1}$ *such that* $\|\widehat{w_1}\|_{\mathbf{H}^{s,0}} \leq \varepsilon$, *where* $\frac{1}{2} < b < 1$,

$$\psi_{1+}(t,\xi) = \frac{\widehat{w_1}(\xi)}{1 + \sqrt{|\gamma|}\,|\widehat{w_1}(\xi)|\log t}$$

and

$$\begin{aligned}\psi_{2+}(t,\xi) &= \frac{1}{\sqrt{|\gamma|}}\frac{1}{|\widehat{w_1}(\xi)|}\frac{\widehat{w_1}(\xi)^2}{1 + \sqrt{|\gamma|}\,|\widehat{w_1}(\xi)|\log t} \\ &= \frac{1}{\sqrt{|\gamma|}}\frac{\psi_{1+}(t,\xi)^2}{|\psi_{1+}(t,\xi)|}.\end{aligned}$$

Theorem 2. *Let* $2m_1 = m_2$, $\gamma > 0, t \geq 1$, $\widehat{w_1} \in \mathbf{H}^{2,0}$ *and* $|\widehat{w_1}(\xi)| > \delta$. *Then the same result as in Theorem 1 holds for*

$$\psi_{1+}(t) = i\widehat{w_1}(\xi)\, e^{i\sqrt{\frac{\gamma}{2}}|\widehat{w_1}(\xi)|\log t}$$

and

$$\psi_{2+}(t) = -i\frac{1}{\sqrt{2\gamma}}\frac{\widehat{w_1}(\xi)^2}{|\widehat{w_1}(\xi)|}e^{2i\sqrt{\frac{\gamma}{2}}|\widehat{w_1}(\xi)|\log t} = i\frac{1}{\sqrt{2\gamma}}\frac{\psi_{1+}(t,\xi)^2}{|\psi_{1+}(t,\xi)|}.$$

for $t \geq 1$, *where* $\frac{1}{2} < b < 1$.

It is known that by the above theorems, the identity $\mathcal{U}\left(\frac{t}{m}\right) = M^m(t)\,\mathcal{D}\left(\frac{t}{m}\right)\mathcal{F}M^m(t)$ we see that

$$\begin{aligned}&\left\| u_1(t) - \frac{1}{it}M^{m_1}(t)\,\psi_{1+}\left(t, \frac{\cdot}{t}\right)\right\|_{\mathbf{L}^2} \\ &+ \left\| u_2(t) - \frac{1}{it}M^{m_2}(t)\,\psi_{2+}\left(t, \frac{\cdot}{t}\right)\right\|_{\mathbf{L}^2} \\ \leq\ & Ct^{-b}.\end{aligned}$$

Under the non resonance condition we prove

Theorem 3. *Let* $2m_1 \neq m_2$, $m_1 \neq m_2$, $(\phi_{1+}, \phi_{2+}) \in \mathbf{H}^{0,2} \cap \dot{\mathbf{H}}^{0,-2b}$. *Then there exists an* $\varepsilon > 0$ *such that (2.2) has a unique global solution*

$$(u_1(t), u_2(t)) \in \mathbf{C}\left([1,\infty); \mathbf{L}^2 \times \mathbf{L}^2\right)$$

for any (ϕ_{1+}, ϕ_{2+}) *satisfying* $\sum_{j=1}^{2}\|\phi_{j+}\|_{\mathbf{H}^{s,0}} \leq \varepsilon$. *Furthermore we have the asymptotics*

$$\left\| u_1(t) - \mathcal{U}\left(\frac{t}{m_1}\right)\phi_{1+}\right\|_{\mathbf{L}^2} + \left\| u_2(t) - \mathcal{U}\left(\frac{t}{m_2}\right)\phi_{2+}\right\|_{\mathbf{L}^2} \leq Ct^{-b}$$

for $t \geq 1$, *where* $\frac{1}{2} < b < 1$.

References

[1] J.E.Barab, *Nonexistence of asymptotically free solutions for a nonlinear Schrödinger equation,* J. Math. Phys., **25** (1984), pp. 3270-3273.

[2] M. Colin and T. Colin, *On a quasilinear Zakharov system describing laser-plasma interactions*, Diff. Integral Equations, **17** (2004), 297-330.

[3] J. Ginibre and T. Ozawa, *Long range scattering for nonlinear Schrödinger and Hartree equations in space dimension* $n \geq 2$, Commun. Math. Phys., **151** (1993), pp. 619-645.

[4] N.Hayashi, C.Li and P.I.Naumkin, *On a system of nonlinear Schrödinger equations in 2d*, preprint, to appear in Differental Integral Equations.

[5] N.Hayashi, C.Li and P.I.Naumkin, *Modified wave operator for a system of nonlinear Schrödinger equations in 2d,* preprint

[6] N.Hayashi and P.I.Naumkin, *Asymptotics in large time of solutions to nonlinear Schrödinger and Hartree equations*, Amer. J. Math., **120** (1998), pp.369-389.

[7] N.Hayashi and P.I.Naumkin, *Large time behavior of solutions for derivative cubic nonlinear Schrödinger equations without a self-conjugate property*, Funkcialaj Ekvacioj, **42** (1999), pp.311-324.

[8] N.Hayashi and T.Ozawa, *Scattering theory in the weighted* $L^2(\mathbf{R}^n)$ *spaces for some Schrödinger equations,* Ann. Inst. H. Poincarè Phys., **48** (1988), pp.17-37.

[9] S.Machihara, K.Nakanishi and T.Ozawa, *Nonrelativistic limit in the energy space for nonlinear Klein-Gordon equations*, Math. Ann., **332**(2002), 603-621.

Department of Mathematics, Graduate School of Science, Osaka University, Osaka, Tokyonaka, Japan

E-mail address: nhayashi@math.sci.osaka-u.ac.jp