Global existence for supercritical wave equations with random initial data

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In this talk, we will consider about the following nonlinear wave equations

$$\begin{cases} \partial_t^2 v - \triangle v + |x|^2 v + |v|^\alpha v = 0, \\ v(0) = f_1,\ \partial_t v(0) = f_2, \end{cases} \tag{0.1}$$

here $v : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$.

Our main result is

Theorem 0.1. *Suppose that $\alpha < \frac{4d}{(d+1)(d-2)}$ is positive. Let us fix a real number p such that $\max\{\frac{2(2d+3)\alpha}{12-(d-2)\alpha}, \frac{2(d+1)}{d-1}\} < p < \frac{2d}{d-2}$. Let $(h_n(w), l_n(w))_{n=0}^\infty$ be a sequence of independent random variables on a probability space $(\Omega, \mathcal{A}, p)$, in which h_n and l_n are standard Gaussian random variables. Consider (0.1) with radial initial data*

$$f_1^w = \sum_{n=1}^\infty \frac{h_n(w)}{\lambda_n} e_n,\ \ f_2^w = \sum_{n=1}^\infty l_n(w) e_n, \tag{0.2}$$

where (λ_n^2) is the eigenvalues of the harmonic oscillator $H = -\triangle + |x|^2$, $\lambda_n = \sqrt{2n+d}$, and $(e_n)_{n=0}^\infty$ is the orthonormal basis associated to λ_n^2. Then for every $s < 0$, almost surely in $w \in \Omega$, the problem (0.1) has a unique global solution

$$v^w \in C(\mathbb{R}_t, \mathcal{H}^s(\mathbb{R}^d)) \bigcap L^p(< t >^{-1} dt, \mathcal{W}^{\theta(p)-,p}(\mathbb{R}^d)),$$

with $\theta(p) = \frac{1}{3} - \frac{d}{3}(\frac{1}{2} - \frac{1}{p})$. $\mathcal{H}^s$ and $\mathcal{W}^{\theta(p)-,p}$ will be defined later.

Furthermore, the solution is a perturbation of the linear solution

$$v^w(t) = \cos(t\sqrt{H}) f_1^w + \frac{\sin(t\sqrt{H})}{\sqrt{H}} f_2^w + \tilde{v}^w(t),$$

where $\tilde{v}^w \in C(\mathbb{R}_t, \mathcal{H}^\sigma(\mathbb{R}^d))$ for some $0 < \sigma = \frac{1}{3} + \frac{d}{3} - \frac{2d+3}{3p} -$. Moreover

$$\|v^w\|_{\mathcal{H}^s(\mathbb{R}^d)} \le C(w, s) \ln(2 + |t|)^{\frac{1}{2}}. \tag{0.3}$$

Remark 0.2. *By the result of this Theorem, we can see that, for $s < 0$, the critical α is smaller than $\frac{4}{d}$ which is strictly smaller than $\frac{4d}{(d+1)(d-2)}$. So for $\frac{4}{d} < \alpha < \frac{4d}{(d+1)(d-2)}$, it is supercritical, which means when we choose some special kind of the initial data, the result would be better. In particular, for $d = 2$, the theorem holds for any $\alpha > 0$.*

Remark 0.3. *By the same idea of [5] Lemma 3.2, (please also refer to Lemma of our paper), we can see that almost surely,*

$$(f_1^w, f_2^w) \in \bigcap_{s<0} (\mathcal{H}^s(\mathbb{R}^d) \times \mathcal{H}^{s-1}(\mathbb{R}^d)),$$

but the probability of the event $\{(f_1^w, f_2^w) \in \mathcal{H}^0(\mathbb{R}^d) \times \mathcal{H}^{-1}(\mathbb{R}^d)\}$ is zero. Thus the randomization process has no smoothing property in the scale of $\mathcal{H}^s$ regularity, and in the above statement we obtain global solutions for data which are not in $\mathcal{H}^0(\mathbb{R}^d) \times \mathcal{H}^{-1}(\mathbb{R}^d)$. On the other hand, our result is not a "small data result".

Remark 0.4. *By the result of Koch and Tatăru [9], this theorem might hold for any $V(x)$ that is radial and behaves like $|x|^2$ for $|x| \to \infty$, for example $< x >^2$. For the sake of conciseness, we just state the special case of $V(x) = |x|^2$.*

By the previous work [6], Burq and Tzvetkov have developed a general theory for constructing local strong solutions to nonlinear wave equations, posed on compact Riemannian manifolds with supercritical random initial data. Then in [7], they showed that in a particular case, which is the nonlinear wave equation with Dirichlet boundary condition posed on the unit ball of $\mathbb{R}^3$, there would be global solutions by combining the local theory with some invariant measure arguments in [1], [2], [10], [12] and [5]. Thomann in [11] got some local well posedness for the Schrödinger equation with a confining potential on the whole space, and then extended it to the one without the potential. Then recently, Burq, Thomann and Tzvetkov in [4] proved the global existence of solutions of Schrödinger equations with random initial data in $\mathbb{R}$. The purpose of our paper is considering global strong solution of the wave equation with the harmonic potential on the whole space. So we will use some idea from [6], [7], [11], [4] and so on. But first of all, we need to prove the Strichartz estimate for (0.1).

Let us consider about the linear wave equation without the potential term first, i.e.

$$\begin{cases} \partial_t^2 v - \triangle v = 0 \\ v(0) = v_0, \ \partial_t v(0) = v_1, \end{cases}$$

then, there is some Strichartz estimate:

$$\|v\|_{L^p((0,T),L^q(\mathbb{R}^d))} \leq C(\|v_0\|_{H^s(\mathbb{R}^d)} + \|v_1\|_{H^{s-1}(\mathbb{R}^d)}), \tag{0.4}$$

where $H^s(\mathbb{R}^d)$ is the usual Sobolev space, and admissible pair (p,q) satisfies $2 \le p \le \infty$, $2 \le q < \infty$ and

$$\frac{1}{p} + \frac{d}{q} = \frac{d}{2} - s, \quad \frac{2}{p} + \frac{d-1}{q} \le \frac{d-1}{2}. \tag{0.5}$$

There are lots of results about Strichartz estimates of the above type on the whole space $\mathbb{R}^d$, compact manifolds with or without boundary, noncompact manifolds and spaces with other geometric conditions.

It is well known that there are some similar properties between the problem on the compact manifolds with the one associated to the harmonic oscillator, so what about our case?

Theorem 0.5. *For $x \in \mathbb{R}^d$, (p_1, q_1), (p_2, q_2) satisfying (0.5), and*

$$\frac{1}{p_1} + \frac{d}{q_1} = \frac{d}{2} - s = \frac{1}{p_2'} + \frac{d}{q_2'} - 2,$$

we have the following estimates for solutions v to (0.1)

$$\|v\|_{L^{p_1}((0,1),L^{q_1}(\mathbb{R}^d))} \le C(\|f_1\|_{\mathcal{H}^s(\mathbb{R}^d)} + \|f_2\|_{\mathcal{H}^{s-1}(\mathbb{R}^d)} + \|F\|_{L^{p_2'}((0,1),L^{q_2'}(\mathbb{R}^d))}), \tag{0.6}$$

here F is the nonlinear term of the equation.

Remark 0.6. *Our result is uniformly with respect to time.*

Remark 0.7. *This result is not only right for $|x|^2$, but also for any $V(x) = \sum_{j=1}^d a_j x_j^2$, with $a_j > 0$ and even some $V(x)$ behaving roughly like $|x|^2$, for example $< x >^2$.*

To prove this Theorem, we will use the idea from [8] and so on. First, we do the dyadic decomposition by the idea of [3], and reduce the problem to a fixed frequency. Then, we try to write out the approximation expression of the operator $e^{-it\sqrt{H}}$ $(H = -\triangle + |x|^2)$. By calculating the dispersion of the operator, the result of Theorem is gained by applying the idea of Keel and Tao [9].

The difference between the proof of (0.4) with (0.6) is that there are cases that the growth of $|x|$ might be much larger than $|\xi|$. Fortunately, for this cases, by estimating the Hessian Matrix, the dispersive effect would even be better.

By the above Theorem, we will prove Theorem 0.1 by the idea of [7]. However, there are some points we should pay attention to. First, without the periodic condition, we show that there is some decaying of time t, i.e. $v^w \in L^p(< t >^{-1} dt, \mathcal{W}^{\theta(p)-,p}(\mathbb{R}^d))$. This would be enough to get the global result and could be applied to more general cases. Secondly, because we are dealing with the whole space case, there are some differences in the interpolation theory.

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