On the weak Harnack inequality for fully nonlinear PDEs with unbounded ingredients

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In this walk, we discuss the weak Harnack inequality for L^p-viscosity supersolutions of

$$\mathcal{P}^+(D^2u) + \mu(x)|Du|^m \geq -f(x) \quad \text{in } \Omega,$$

where $\mu \in L^q_+(\Omega)$ $(q > n)$, and $f \in L^p_+(\Omega)$ $(q \geq p > p_0$ for some $p_0 \in [n/2, n))$ are given functions, $\Omega \subset \mathbf{R}^n$ a domain, and $m \geq 1$ a constant. Fixing $0 < \lambda \leq \Lambda$, we use the following Pucci operators:

$$\mathcal{P}^+(X) := \max\{-\mathrm{trace}(AX) \mid \lambda I \leq A \leq \Lambda I\} \quad (X \in S^n)$$

The interior/boundary weak Harnack inequality is a key tool to establish Hölder continuity of L^p-viscosity solutions, strong maximum principle, maximum principle in unbounded domains, and also the local maximum principle.

Motivated by a pioneering work [1] by Caffarelli, the notion of L^p-viscosity solutions was introduced by Caffarelli-Crandall-Kocan-Święch [3] to study fully nonlinear PDEs. Our aim is to establish the weak Harnack inequality even when f belongs to a wider space than L^n, μ is unbounded, and the superlinear growth in Du is considered.

After [3], Fok [4] first studied L^p-viscosity solutions of fully nonlinear PDEs with unbounded ingredients. In [6], we extend some results in [4], e.g. the ABP maximum principle and the strong solvability, by which we mean the existence of L^p-strong solutions. Under some restriction, we also obtain the ABP maximum principle in case when $m > 1$ (see [5], [6]).

In order to prove the weak Harnack inequality, we follow Caffarelli's argument (cf. [2]). However, to this end, we need some modifications because we deal with unbounded coefficients. For instance, we cannot use "explicite" fundamental solutions associated with Pucci operators.

We note that Sirakov [10] obtained the Hölder continuity of L^p-viscosity solutions without the weak Harnack inequality when $m = 1$, $q > n$ and $p \geq n$.

Moreover, to establish the weak Harnack inequality in the superlinear case (i.e. $m > 1$), we obtain the strong solvability of Pucci extremal PDEs with superlinear terms in [8]. We will mention the local maximum principle in [9].

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