# A few problems for Hamilton-Jacobi equations arising from step motions of crystal growth

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The growth of crystal is often explained by step motions of crystal surfaces. This idea goes back to W. K. Burton, N. Cabrera and F. C. Frank [BCF]. Several Hamilton-Jacobi equations for heights of crystals are derived as a continuum limit as each step height goes to zero [EY].

A typical problem in the theory of crystal growth is stability of a facet (flat portion). This is qualitatively studied by [C] and more quantatively by [KIO]. The issue is whether perfectly flat crystal surfaces grow keeping its flatness. It seems that there are two kinds of facets—facet due to interfacial energy and facet due to kinetics. The first one is found in an equilibrium shape and it is explained as singularity of surface energy. Its evolution and stability is also studied, see eg. [GR]. The second one is a facet due to kinetics and studied in crystal growth literature [C], [KIO]. It is formulated as follows. We consider an evolution of height $h(= h^\varepsilon)$ of a crystal surface at $x \in \mathbf{R}^2$ and at time $t$ which is determined by

$$h_t - \sigma(x) m(\frac{|\nabla h|}{\varepsilon}) \sqrt{|\nabla h|^2 + 1} = 0$$

Here $m(p) = p\ \tanh(1/p)$ and $\varepsilon$ is a criterion of (microscopic) local slope which is very small. The function $\sigma \geq 0$ is concentration of adatom at the crystal surface. We consider microscopic time approximation proposed in [YGR] by introducing microscopic time $\tau$ so that

$$h^\varepsilon(x, \varepsilon\tau) = \varepsilon u(x, \tau) + o(\varepsilon) \quad \text{as} \quad \varepsilon \to 0$$

to get

$$u_t - \sigma(x)m(|\nabla u|) = 0 \tag{1}$$

Stability of facets in kinetic sense corresponds to the large time asymptotics of the equation (1) when initial data equals zero which corresponds to a flat surface. However, if the initial data equals zero, $u \equiv 0$ is the solution so it is not interesting. This is because we did not consider step source at a crystal surface so it is natural to see that crystal surface does not move at all.

Instead, we consider the one-dimensional Dirichlet problem for (1) in $(0, \infty)$ by assigning the speed at zero. A typical problem is to consider $\sigma(x) = \sigma_0(1 - x^2)_+$, $\sigma_0 > 0$ and $u(0,t) = ct$ with $0 < c < \sigma_0$. Since the Hamiltonian is non coercive, conventional theory for the large time asymptotics for the Hamiltonian does not apply. As printed out in [YGR] when $u = 0$ initially, one observes that

$$u(x,t) \sim ct \quad \text{in} \quad (0, x_c) \quad \text{as} \quad t \to \infty,$$

where $x_c > 0$ is the point such that $\sigma_0(1 - x_c^2)_+ = c$. Outside $(0, x_c), u(x,t)$ grows slowly with respect to $O(t)$. Physically, the region $(0, x_c)$ is the stable region of the facet. The conventional theory for coercive Hamiltonians yields a uniform asymptotics for all domain.

This result is generalized by Q. Liu, H. Mitake and the author [GLM1] by extending a notion of viscosity solutions defined in a part of a domain for more general setting. The asymptotics of the Cauchy problem (1) with slightly different $m$ satisfying $m(0) > 0$ is studied by Q. Liu, H. Mitake and the author [GLM2] by introducing a singular Neumann problem. In this talk we shall explain some of these results.

Finally, we mention the issue of step source. In [SK] an explicit 'solution' of evolution with step source is given without defining the notion of solutions of the equation. A typical example is

$$h_t - |\nabla h| = \sum_{j=1}^{m} v_j I(x - a_j), \tag{2}$$

where $v_j > 0$ and $I(x) = 1$ for $x = 0$, $I(x) = 0$ otherwise and $a_j \in \mathbf{R}^2$ is $j$-th step source place. A suitable notion of the solution was not known. In a work with progress the author with his student N. Hamamuki introduced a new notion for the solution which leads a unique global existence of Lipschitz solutions for Lipschitz initial data. One important observation is that we interpret (2) as

$$h_t - |\nabla h| = \sum_{j=1} (v_j - |\nabla h|)_+ I(x - a_j).$$

In this talk we also plan to mention these topics.

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