Behavior of solutions to an area-preserving crystalline motion

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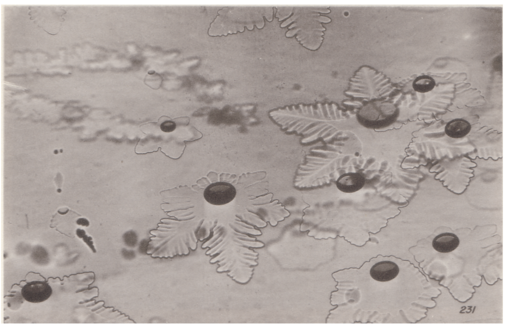
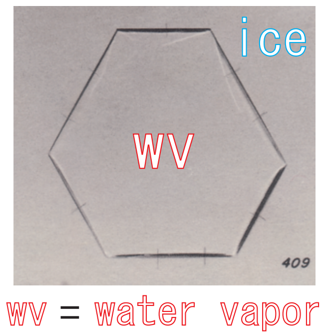
1 Introduction

When a block of ice crystal is illuminated by strong beams, the ice crystal starts to melt inside of the crystal as well as the surface and each water region forms a snowflake-like-pattern which has six petals, called "Tyndall figure" (see Figure 1 (a)). This figure has a vapor bubble in water region and when this figure is refrozen, the vapor bubble remains in the ice as a hexagonal disk (see Figure 1 (b)). This hexagonal disk is a kind of negative crystals and the interior region is filled with water vapor saturated at that temperature. McConnel([6]) found these disks in the ice of Davos lake. Nakaya called this hexagonal disk "Kuuzou(空像)" in Japanese and investigated its properties [7].



(a)



(b)

Figure 1: (a) Tyndall figures (seen from 45° to the c-axis) and (b) a negative crystal (by U. Nakaya).

In [5], we proposed a motion equation for a polygonal curve in the plane as a simple model of the formation process of negative crystals after the water region in a Tyndall figure is completely refrozen. This model equation is obtained by a gradient flow of total surface energy under an area-preserving constraint:

$$V_i = \overline{H} - H_i.$$

Here V_i is the outward normal velocity on the i-th facet $\mathcal{F}_i$ of vapor region $\Omega(t)$ (enclosed region by a polygon), H_i is the crystalline curvature of $\mathcal{F}_i$ and $\overline{H}$ is the average of all

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crystalline curvatures. This equation is called area-preserving crystalline motion or area-preserving crystalline curvature flow. Crystalline motion is a singular weighted curvature flow with non-smooth surface energy γ and J. Taylor[8] and Angenent and Gurtin [1] proposed the framework of crystalline motions. In this framework, the interfaces are restricted in the class of polygonal curves (two dimensional case) which satisfy an admissibility condition based on the equilibrium shape of the crystal. This equilibrium shape is called the *Wulff shape* and plays important roles for not only the definition of the crystalline curvature and admissibility condition, but also the asymptotic behavior of the solution polygons. The detailed formulations will be mentioned in next section.

In the case that an initial shape Ω_0 is convex, the solution polygon $\Omega(t)$ keeps its convexity. S. Yazaki [9, Part I] show that no facets disappear globally in time and the solution polygon converges to the rescaled Wulff shape whose area is equal to that of Ω_0 in the Hausdorff metric. However, when the vapor region is surrounded by the ice region in refreezing process, many fine facets appear on the interface and the shape of the vapor region is not convex in general. Thus, in this talk, we consider the case that Ω_0 is not convex. In this case, there is a possibility that the solution has some singularities in finite time, for example, facet-extinction and self-intersection of the interface. We show the sufficient conditions on the Wulff shape and an initial polygon to keep admissibility of the solution polygons. Moreover, we also show that the solution polygon from non-convex initial polygon becomes convex in finite time.

2 Area-preserving crystalline motion

Crystalline energy and the Wulff shape. Let $\gamma = \gamma(\boldsymbol{n})$ be a positive continuous function defined on S^1 and describe interfacial energy density for the direction $\boldsymbol{n}$. In this note, we consider the case where the Wulff shape of γ, $\mathcal{W}_\gamma = \{x \in \mathbb{R}^2 | x \cdot \boldsymbol{n} \leq \gamma(\boldsymbol{n}) \text{ for all } \boldsymbol{n} \in S^1\}$, is a convex polygon. Such γ is called *crystalline energy.* If $\mathcal{W}_\gamma$ is a J-sided convex polygon ($J \geq 3$), then $\mathcal{W}_\gamma$ is expressed as

$$\mathcal{W}_\gamma = \bigcap_{i=1}^{J} \left\{ \boldsymbol{x} \in \mathbb{R}^2;\ \boldsymbol{x} \cdot \boldsymbol{\nu}_i \leq \gamma(\boldsymbol{\nu}_i) \right\},$$

where $\boldsymbol{\nu}_i = \boldsymbol{n}(\phi_i)$ and ϕ_i is the exterior normal angle of the i-th facet with $\phi_i \in (\phi_{i-1}, \phi_{i-1} + \pi)$ for all i ($\phi_0 = \phi_J$, $\phi_{J+1} = \phi_1$). We define a set of normal vectors of $\mathcal{W}_\gamma$ by $\mathcal{N}_\gamma = \{\boldsymbol{\nu}_1, \boldsymbol{\nu}_2, \dots, \boldsymbol{\nu}_J\}$.

Polygons and polygonal curves. Let Ω be N-sided polygon in the plane $\mathbb{R}^2$, $\mathcal{P}$ its boundary, that is, $\mathcal{P} = \partial\Omega$ and label the position vector of vertices $\boldsymbol{p}_i$ ($i = 1, 2, \dots, N$) in an anticlockwise order: $\mathcal{P} = \bigcup_{i=1}^{N} \mathcal{F}_i$, where $\mathcal{F}_i = \{(1-t)\boldsymbol{p}_i + t\boldsymbol{p}_{i+1};\ t \in [0,1]\}$ is the i-th facet ($\boldsymbol{p}_0 = \boldsymbol{p}_N$, $\boldsymbol{p}_{N+1} = \boldsymbol{p}_1$). The length of $\mathcal{F}_i$ is $d_i = |\boldsymbol{p}_{i+1} - \boldsymbol{p}_i|$, and then the i-th unit tangent vector is $\boldsymbol{t}_i = (\boldsymbol{p}_{i+1} - \boldsymbol{p}_i)/d_i$ and the i-th unit outward normal vector is $\boldsymbol{n}_i = -\boldsymbol{t}_i^{\perp}$, where $(a, b)^{\perp} = (-b, a)$. We define a set of normal vectors of $\mathcal{P}$ by $\mathcal{N} = \{\boldsymbol{n}_1, \boldsymbol{n}_2, \dots, \boldsymbol{n}_N\}$. Let θ_i be the exterior normal angle of $\mathcal{F}_i$. Then $\boldsymbol{n}_i = \boldsymbol{n}(\theta_i)$ and $\boldsymbol{t}_i = \boldsymbol{t}(\theta_i)$ hold ($\theta_0 = \theta_N$, $\theta_{N+1} = \theta_1$), where $\boldsymbol{t}(\theta) = (-\sin\theta, \cos\theta)$.

We define the i-th hight function $h_i = \boldsymbol{p}_i \cdot \boldsymbol{n}_i = \boldsymbol{p}_{i+1} \cdot \boldsymbol{n}_i$ ($h_0 = h_N$, $h_{N+1} = h_1$). By using $\{h_{i-1}, h_i, h_{i+1}\}$ and $\{\boldsymbol{n}_{i-1}, \boldsymbol{n}_i, \boldsymbol{n}_{i+1}\}$, the length of i-th facet d_i is described as follows:

$$d_i = \frac{\chi_{i-1,i}(h_{i-1} - (\boldsymbol{n}_{i-1} \cdot \boldsymbol{n}_i)h_i)}{\sqrt{1 - (\boldsymbol{n}_{i-1} \cdot \boldsymbol{n}_i)^2}} + \frac{\chi_{i,i+1}(h_{i+1} - (\boldsymbol{n}_i \cdot \boldsymbol{n}_{i+1})h_i)}{\sqrt{1 - (\boldsymbol{n}_i \cdot \boldsymbol{n}_{i+1})^2}}, \quad i = 1, 2, \dots, N,$$

where $\chi_{i,j} = \mathrm{sgn}(\boldsymbol{n}_i \wedge \boldsymbol{n}_j)$ and $\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 = \det(\boldsymbol{a}_1, \boldsymbol{a}_2)$ is the determinant of the 2×2 matrix with column vectors $\boldsymbol{a}_1$, $\boldsymbol{a}_2$. Since $\boldsymbol{n}_i \cdot \boldsymbol{n}_j = \cos(\theta_i - \theta_j)$, we have another expression:

$$d_i = -(\cot\vartheta_i + \cot\vartheta_{i+1})h_i + h_{i-1}\,\mathrm{cosec}\,\vartheta_i + h_{i+1}\,\mathrm{cosec}\,\vartheta_{i+1}, \quad i = 1, 2, \dots, N, \tag{1}$$

where $\vartheta_i = \theta_i - \theta_{i-1}$. Note that $0 < |\vartheta_i| < \pi$ holds for all i. Furthermore, the i-th vertex $\boldsymbol{p}_i$ $(i = 1, 2, \dots, N)$ is described as follows:

$$\boldsymbol{p}_i = h_i \boldsymbol{n}_i + \frac{h_{i-1} - (\boldsymbol{n}_{i-1} \cdot \boldsymbol{n}_i)h_i}{\boldsymbol{n}_{i-1} \cdot \boldsymbol{t}_i}\boldsymbol{t}_i. \quad i = 1, 2, \dots, N. \tag{2}$$

Admissibility and crystalline curvature. We call Ω and $\mathcal{P}$ *admissible* (associated with $\mathcal{W}_\gamma$) if and only if $\mathcal{N} = \mathcal{N}_\gamma$ holds and any adjacent two normal vectors in the set $\mathcal{N}$ are also adjacent in the set $\mathcal{N}_\gamma$, i.e., for any i, there exists j such that $\{\boldsymbol{\nu}_j, \boldsymbol{\nu}_{j+1}\} = \{\boldsymbol{n}_i, \boldsymbol{n}_{i+1}\}$ holds.

Let $\mathcal{P}$ be an admissible polygonal curve. For each facet $\mathcal{F}_i$ a *crystalline curvature* is defined by

$$H(\mathcal{F}_i) = \chi_i \frac{l_\gamma(\boldsymbol{n}_i)}{d_i}, \quad i = 1, 2, \dots, N,$$

where $\chi_i = (\chi_{i-1,i} + \chi_{i,i+1})/2$ is the transition number and it takes $+1$ (resp. -1) if $\mathcal{P}$ is convex (resp. concave) around $\mathcal{F}_i$ in the direction of $-\boldsymbol{n}_i$, otherwise $\chi_i = 0$; and $l_\gamma(\boldsymbol{n}_i)$ is the length of the j-th facet of $\mathcal{W}_\gamma$ if $\boldsymbol{n}_i = \boldsymbol{\nu}_j$. If Ω is an admissible convex polygon, then $\boldsymbol{n}_i = \boldsymbol{\nu}_i$ and $\chi_i = 1$ for all $i = 1, 2, \dots, N = J$; and moreover, if $\Omega = \mathcal{W}_\gamma$, then the crystalline curvature is 1. In this note, we call a facet which zero transition number "inflection facet."

We note that the total interfacial crystalline energy on $\mathcal{P}$ is

$$\mathcal{E}_\gamma = \sum_{i=1}^{N} \gamma(\boldsymbol{n}_i) d_i, \tag{3}$$

and the crystalline curvature $H(\mathcal{F}_i)$ is characterized as the first variation of $\mathcal{E}_\gamma$ on $\mathcal{P}$ at $\mathcal{F}_i$ with a suitable norm. Here and hereafter, we denote $H(\mathcal{F}_i)$ by H_i for short.

Area-preserving crystalline motion. The normal velocity on $\mathcal{F}_i$ in the direction $\boldsymbol{n}_i$ is $V_i = \dot{h}_i$. Here and hereafter, we denote that the derivative of a function $u = u(t)$ with respect to time t by $\dot{u}$. The area-preserving crystalline motion is the gradient flow of $\mathcal{E}_\gamma$ along $\mathcal{P}$ which encloses a fixed area, and it is described as follows:

$$V_i = \overline{H} - H_i, \quad i = 1, 2, \dots, N, \tag{4}$$

where

$$\overline{H} = \frac{\sum_{i=1}^{N} H_i d_i}{\mathcal{L}}$$

is the average of the crystalline curvature, and $\mathcal{L} = \sum_{k=1}^{N} d_k$ is the total length of the curve $\mathcal{P}$. From (1), we have

$$\dot{d}_i = -(\cot\vartheta_i + \cot\vartheta_{i+1})V_i + V_{i-1}\,\mathrm{cosec}\,\vartheta_i + V_{i+1}\,\mathrm{cosec}\,\vartheta_{i+1}, \quad i = 1, 2, \dots, N. \tag{5}$$

Furthermore, by (2) we have

$$\dot{\boldsymbol{p}}_i = V_i \boldsymbol{n}_i + \frac{V_{i-1} - (\boldsymbol{n}_{i-1} \cdot \boldsymbol{n}_i)V_i}{\boldsymbol{n}_{i-1} \cdot \boldsymbol{t}_i}\boldsymbol{t}_i, \quad i = 1, 2, \dots, N. \tag{6}$$

Note that (4), (5) and (6) are equivalent each other. It is easy to check that the enclosed area $\mathcal{A}(t) = \sum_{i=1}^{N} h_i d_i/2$ is preserving in time: $\dot{\mathcal{A}}(t) = \sum_{i=1}^{N} V_i d_i = 0$.

3 Results

For any given admissible initial polygon Ω_0, we have short time existence and uniqueness result by the standard argument since (5) is the system of ordinary differential equations.

Known results for convex polygons. What might happen to $\Omega(t)$ as t tends to the maximal existence time $T \leq \infty$? For this question, we have the following result.

Theorem 1 Let the crystalline energy be $\gamma > 0$. Assume the initial polygon Ω_0 is an N-sided admissible convex polygon. Then the solution admissible polygon $\Omega(t)$ exists globally in time keeping the area enclosed by the polygon constant $\mathcal{A}$, and $\Omega(t)$ converges to the shape of the boundary of the Wulff shape $\partial\mathcal{W}_{\gamma_*}$ in the Hausdorff metric as t tends to infinity, where $\gamma_*(\boldsymbol{n}_i) = \gamma(\boldsymbol{n}_i)/W$, $W = \sqrt{|\mathcal{W}_\gamma|/\mathcal{A}}$ for all $i = 1, 2, \ldots, N$ and $|\mathcal{W}_\gamma| = \sum_{k=1}^{N} \gamma(\boldsymbol{n}_k) l_\gamma(\boldsymbol{n}_k)/2$ is enclosed area of $\mathcal{W}_\gamma$.

This theorem is proved in Yazaki [9, Part I] by using the anisoperimetric inequality or Brünn and Minkowski's inequality and the theory of dynamical systems.

Our results for non-convex polygons.

In the previous case, the solution polygon keeps its convexity and admissibility, that is, the length of each facet is positive globally in time and the self-intersection of $\mathcal{P}(t)$ never occur. However, if Ω_0 is non-convex, the facet-extinction or the self-touching may occur in finite time. Indeed, we can easily construct the example of the self-intersection of $\mathcal{P}(t)$ and $\Omega(t)$ becomes non-admissible after the singularity. Thus, the admissibility of solution polygons may break down in finite time. To track the motion globally in time in the class of admissible polygons, we prepare the following assumptions:

(A1) $\mathcal{W}_\gamma$ is symmetric with respect to the origin.

(A2) Transition numbers of Ω_0 are all nonnegative: $\chi_i \geq 0$ for any i.

Theorem 2 *Assume the assumptions (A1) and (A2). Let Ω_0 be an N-sided non-convex admissible polygon. Then, there exists $T_1 > 0$ such that the solution polygon is an N-sided admissible polygon for $0 \leq t < T_1$ and there exists at least one inflection facet whose length tends to zero as $t \to T_1$. Moreover, $\Omega(t)$ converges to an admissible polygon Ω^* in the Hausdroff topology as $t \to T_1$ and area of Ω^* is equal to area of Ω_0.*

This theorem means that we can restart the motion with the initial polygon Ω^* and obtain the solution in the class of admissible polygons beyond the singularity. If Ω^* is non-convex, then we can apply Theorem 2 again and again. We finally have a finite sequence of facet-extinction time : $0 < T_1 < T_2 < \cdots < T_m < +\infty$. Then, we obtain the following convexity result.

Theorem 3 *Assume that the same assumption as in Theorem 2. Then, the solution polygon becomes convex at $t = T_m$.*

After the convexity phenomena occurs, we can apply Theorem 1. Therefore, the solution polygon exists globally in time in the class of admissible polygons and the solution polygon finally converges to the rescaled Wulff shape.

4 For negative crystals

For usual crystal case, enclosed region describes the crystal and then normal vector $\boldsymbol{n}$ is direction from the crystal to its outside region. However, for negative crystal case, the outside region describes the crystal. Thus, applying the area-preserving crystalline motion to understand the motion of the boundary of negative crystals, we need to use $\gamma(-\boldsymbol{n})$ as the interfacial energy density. Therefore, we use the figure:

$$\bigcap_{i=1}^{J} \left\{ \boldsymbol{x} \in \mathbb{R}^2;\ \boldsymbol{x} \cdot (-\boldsymbol{\nu}_i) \leq \gamma(\boldsymbol{\nu}_i) \right\},$$

as the Wulff shape for negative crystal case.

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