Congruences for Hecke Eigenvalues of Siegel Modular Forms

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1 Introduction

For $r \in \mathbf{Z}_{>0}$ and $k \in 2\mathbf{Z}_{>0}$ with $k > r$ let f be a holomorphic Siegel cusp form of weight k for $Sp_{2r}(\mathbf{Z})$. Suppose that f is a Hecke eigenform, i.e. a nonzero common eigenfunction of the Hecke algebra. Let $\mathbf{Q}(f)$ be the number field generated over $\mathbf{Q}$ by the eigenvalues of the Hecke operators over $\mathbf{Q}$ on f. Let $L(s, f, \mathrm{St})$ be the standard L-function attached to f. Suppose that the Fourier coefficients of f belong to $\mathbf{Q}(f)$. Then the value

$$L^*(k-r, f, \mathrm{St}) := \frac{L(k-r, f, \mathrm{St})}{\pi^{(2r+1)k-\frac{3r(r+1)}{2}}(f, f)}$$

belongs to $\mathbf{Q}(f)$, where (f, f) is the square of the Petersson norm [10][44][6][32].

Let $n \in \mathbf{Z}_{>0}$ with $n > r$ and assume $k \geq \frac{3}{2}(n+1)$. Let $[f]_r^n$ be the Klingen-Eisenstein series of degree n attached to f. Then $[f]_r^n$ is also a Hecke eigenform and its Fourier coefficients belong to $\mathbf{Q}(f)$ by [26][35]. Let

$$[f]_r^n(Z) = \sum_{N \geq 0} a(N, [f]_r^n)\mathbf{e}(\sigma(NZ))$$

be the Fourier expansion of $[f]_r^n$. Here $\mathbf{e}(x) := e^{2\pi i x}$ for $x \in \mathbf{C}$, N runs over all symmetric positive semidefinite semi-integral matrices of size n, and σ is the trace for matrices.

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The main result of this paper (Theorem 3.1 below) tells that there exists a finite set S of prime ideals of $\mathbf{Q}(f)$, which is explicitly described in Sect. 3 below, with the following property: Suppose a prime ideal $\mathfrak{p}$ of $\mathbf{Q}(f)$ satisfies $\mathfrak{p} \notin S$ and

$$\mathrm{ord}_{\mathfrak{p}} \left(\prod_{j=[\frac{n+1}{2}]}^{[\frac{n+r}{2}]} \zeta(1+2j-2k) \cdot L^*(k-r, f, \mathrm{St}) a(N_0, [f]_r^n)^2 \right) =: -\alpha < 0$$

for some $N_0 > 0$, where $[a]$ for $a \in \mathbf{Q}$ is the largest integer $\leq a$ and $\mathrm{ord}_{\mathfrak{p}}$ is the $\mathfrak{p}$-order. Then there exists a Hecke eigenform F of weight k for $Sp_{2n}(\mathbf{Z})$ such that

$$N_{\mathbf{Q}(F,f)/\mathbf{Q}(f)} \left(\lambda(T, F) - \lambda(T, [f]_r^n)\right) \equiv 0 \pmod{\mathfrak{p}^\alpha} \quad \text{for all} \quad T \in \mathcal{H}_{\mathbf{Z}}^n.$$

Here $\mathcal{H}_{\mathbf{Z}}^n$ is the Hecke algebra over $\mathbf{Z}$, $\lambda(T, F)$ and $\lambda(T, [f]_r^n)$ are the eigenvalues of T on F and on $[f]_r^n$ respectively, $\mathbf{Q}(F, f) := \mathbf{Q}(F)\mathbf{Q}(f)$ is the composite field, and $N_{\mathbf{Q}(F,f)/\mathbf{Q}(f)}$ is the norm map for $\mathbf{Q}(F, f)/\mathbf{Q}(f)$.

Such congruences were first discovered by Kurokawa [23] as numerical examples in case $n = 2$, $r = 1$, and $k \leq 20$; he also posed a general conjecture predicting the existence of similar types of congruences modulo special values of L-functions. After that we proved the above assertion for $n = 2$ in [31] under a "multiplicity one condition" which was quite restrictive.

Meanwhile some related topics have been discussed by several authors [7][46].

Now in this paper we prove the congruences for general degree, and moreover, without assuming the multiplicity one condition. Thus our result settles Kurokawa's conjecture (under some additional conditions defining the set S of exceptional prime ideals).

Our result may also be considered as a kind of results which characterize the prime ideals giving congruences between lifted and nonlifted modular forms as special values of automorphic L-functions; this theme has been pursued by [17][18].

The fundamental difference between the method of this paper and that of [31] lies in the following two points:

(1) In [31], Lemma 4, we used a modular form of the form

$$\varphi_4^a \varphi_6^b \tag{1.1}$$

where φ_l is the degree two Siegel-Eisenstein series of weight l and a and b are some non-negative integers satisfying $4a+6b=k$. This modular form worked as a kind of "bridge" between $[f]_1^2$ and the space of cusp forms of degree two. Instead of (1.1) we take in this paper the pullback of Siegel-Eisenstein series of degree $2n$ and use the formula of Garrett [9] and Böcherer [5]. This enables us to generalize the argument in [31] to the case of degree n.
(2) We use the integrality lemma in its general form as in [37] which is valid for general n and does not require the multiplicity one condition. In [31] the lemma was proved only for $n \leq 2$ under this last condition.

The paper is organized as follows. Sect. 2 is preparatory; we summarize what we need later to state our result precisely. In Sect. 3 we state our main theorem and in Sect. 4 we give the proof. In Sect. 5 we give some numerical examples in case of degree three; the method of computation is the one developed in [18].

2 Preliminaries

2.1 Notation

For a subring R of $\mathbf{C}$ the group of units in R is denoted by $R^{\times}$. For a prime ideal $\mathfrak{l}$ of R, $R_{\mathfrak{l}}$ is the localization of R at $\mathfrak{l}$. If A is an $m \times n$ matrix with $m, n \in \mathbf{Z}_{>0}$, we write it also as $A^{(m,n)}$ and as $A^{(m)}$ if $m=n$. For two matrices A and B we write $A[B] := {}^tBAB$ if the right-hand side is defined. The set of all $m \times n$ matrices with entries in R is denoted by $R^{(m,n)}$ and by $R^{(m)}$ if $m=n$. For a number field K the ring of integers in K is denoted by $\mathcal{O}_K$.

2.2 Siegel modular forms

For $n, k \in \mathbf{Z}_{>0}$ the $\mathbf{C}$-vector space of holomorphic modular (resp. cusp) forms of weight k for $\Gamma_n := Sp_{2n}(\mathbf{Z})$ is denoted by M_k^n (resp. S_k^n).

Let H_n be the Siegel upper half space of degree n. For φ and $\psi \in M_k^n$ such that $\varphi\psi$ is a cusp form, the Petersson inner product is defined by

$$(\varphi, \psi) := \int_{\Gamma_n \backslash H_n} \varphi(Z)\overline{\psi(Z)} \det(Y)^{k-n-1} dX dY.$$

Here $Z = X + iY$ with real matrices $X = (x_{jl})$ and $Y = (y_{jl})$; $dX :=$

$\prod_{j\le l} dx_{jl}$, $dY := \prod_{j\le l} dy_{jl}$; the integral is taken over a fundamental domain of $\Gamma_n\backslash H_n$.

Let A_n (resp. A_n^+) be the set of all symmetric positive semidefinite (resp. positive definite) semi-integral matrices of size n. Every $\varphi \in M_k^n$ has a Fourier expansion of the form

$$\varphi(Z) = \sum_{N\in A_n} a(N,\varphi)\mathbf{e}(\sigma(NZ)).$$

For a subring R of $\mathbf{C}$ we denote by $M_k^n(R)$ the R-module consisting of all $\varphi \in M_k^n$ such that $a(N,\varphi) \in R$ for all $N \in A_n$. We also put

$$W(R) := W \cap M_k^n(R)$$

for a subspace W of M_k^n. Since

$$M_k^n = M_k^n(\mathbf{Q})\otimes_{\mathbf{Q}}\mathbf{C}$$

by Shimura [40], the group $\mathrm{Aut}(\mathbf{C})$ of automorphisms of $\mathbf{C}$ acts on this space via

$$\varphi^\tau(Z) := \sum_{N\in A_n} a(N,\varphi)^\tau \mathbf{e}(\sigma(NZ)) \quad \text{for} \quad \tau \in \mathrm{Aut}(\mathbf{C}).$$

2.3 The Hecke algebra

For a subfield L of $\mathbf{R}$ let

$$GSp_{2n}^+(L) := \left\{ g \in L^{(2n)} \,\middle|\, J_n[g] = \nu(g)J_n \quad \text{with} \quad \nu(g) > 0 \right\}$$

be the group of symplectic similitudes over L where $J_n := \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$ with 1_n being the identity matrix of size n. The action of an element

$$g = \begin{pmatrix} A^{(n)} & B^{(n)} \\ C^{(n)} & D^{(n)} \end{pmatrix} \in GSp_{2n}^+(\mathbf{R}) \tag{2.1}$$

on H_n is given by

$$g\langle Z\rangle := (AZ+B)(CZ+D)^{-1}.$$

Let

$$\varphi : H_n \longrightarrow \mathbf{C}$$

be an arbitrary function. For a given $k \in \mathbf{Z}$, an element (2.1) with

$$J_n[g] = \nu(g) J_n$$

acts on F via

$$(\varphi|_k\, g)(Z) := \nu(g)^{nk - \frac{n(n+1)}{2}} \det(CZ+D)^{-k} \varphi(g\langle Z\rangle).$$

For a subring R of $\mathbf{C}$ let $\mathcal{H}_R^n$ be the Hecke algebra over R associated to the Hecke pair

$$(\Gamma_n, GSp_{2n}^+(\mathbf{Q}) \cap \mathbf{Z}^{(2n)})$$

in the sense of Andrianov [1]. (The definition of $\mathcal{H}_R^n$ in Mizumoto [37], p. 115 is false; it should be corrected as above.) By definition

$$\mathcal{H}_R^n = \mathcal{H}_{\mathbf{Z}}^n \otimes_{\mathbf{Z}} R.$$

For $g \in GSp_{2n}^+(\mathbf{Q})$ the double coset $\Gamma_n g \Gamma_n$ splits into a disjoint union of left cosets:

$$\Gamma_n g \Gamma_n = \bigsqcup_{j=1}^{m} \Gamma_n g_j.$$

For $\varphi \in M_k^n$ we put

$$\varphi|\,\Gamma_n g \Gamma_n := \sum_{j=1}^{m} \varphi|_k\, g_j.$$

Extending the action by $\mathbf{C}$-linearity, we have a representation of $\mathcal{H}_{\mathbf{C}}^n$ on M_k^n.

We call $\varphi \in M_k^n$ a *Hecke eigenform* if it is a nonzero common eigenfunction of $\mathcal{H}_{\mathbf{C}}^n$. If $\varphi \in M_k^n$ is a Hecke eigenform, we write the eigenvalue of $T \in \mathcal{H}_{\mathbf{C}}^n$ on φ as $\lambda(T, \varphi)$. For a Hecke eigenform $\varphi \in M_k^n$ we put

$$\mathbf{Q}(\varphi) := \mathbf{Q}(\lambda(T,\varphi) \,|\, T \in \mathcal{H}_{\mathbf{Q}}^n).$$

The field $\mathbf{Q}(\varphi)$ is a totally real finite extension of $\mathbf{Q}$ and there exists a basis $\{\varphi_1, \ldots, \varphi_d\}$ of S_k^n such that each φ_j is a Hecke eigenform whose Fourier coefficients lie in $\mathbf{Q}(\varphi_j)$ by Kurokawa [25]. Moreover if $k \geq \frac{3}{2}(n+1)$, the above basis can be taken so that the elements φ_j are mutually orthogonal and permuted under $\mathrm{Aut}(\mathbf{C})$ by [32], Appendix A.

2.4 L-functions and Eisenstein series

For a Hecke eigenform $\varphi \in S_k^n$ let

$$L(s,\varphi,\mathrm{St}) := \prod_{p\,\text{prime}} \left\{ (1-p^{-s}) \prod_{j=1}^{n} (1-\alpha_j(p)p^{-s})(1-\alpha_j(p)^{-1}p^{-s}) \right\}^{-1} \tag{2.2}$$

be the standard L-function attached to φ. Here $(\alpha_1(p),\cdots,\alpha_n(p)) \in (\mathbf{C}^\times)^n$ is one set of the Satake p-parameters of φ for a prime number p and s is a complex variable. By Shimura [41] the right-hand side of (2.2) converges absolutely and uniformly for $\mathrm{Re}(s) \geq \frac{n}{2}+1+\delta$ for any $\delta > 0$. The L-function $L(s,\varphi,\mathrm{St})$ has meromorphic continuation to the whole s-plane [2][5].

Let $n \in \mathbf{Z}_{>0}$ and $r \in \mathbf{Z}$ such that $0 \leq r \leq n$. We put

$$M_k^0 = S_k^0 = \mathbf{C} \quad \text{(constant functions)}.$$

Let $\Delta_{n,r}$ be the subgroup of Γ_n defined by

$$\Delta_{n,r} := \left\{ \begin{pmatrix} * & * \\ 0^{(n-r,n+r)} & * \end{pmatrix} \in \Gamma_n \right\}.$$

For $f \in S_k^r$ with $k \in 2\mathbf{Z}_{>0}$ the nonholomorphic Eisenstein series for Γ_n attached to f in the sense of Langlands [28] and Klingen [20] is defined by

$$[f]_r^n(Z,s) := \sum_{M \in \Delta_{n,r}\backslash\Gamma_n} \left(\frac{\det(\mathrm{Im}(M\langle Z\rangle))}{\det(\mathrm{Im}(M\langle Z\rangle^*))} \right)^s f(M\langle Z\rangle^*)\det(CZ+D)^{-k}. \tag{2.3}$$

Here $s \in \mathbf{C}$, Z is a variable on H_n, $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with A,B,C,D being $n \times n$ blocks runs over a complete set of representatives of $\Delta_{n,r}\backslash\Gamma_n$, and $M\langle Z\rangle^*$ is the upper left $r \times r$ block of $M\langle Z\rangle$. By [20] the right-hand side of (2.3) converges absolutely and uniformly on

$$\left\{ (Z,s) \in H_n \times \mathbf{C} \,\middle|\, \sigma(X^2) \leq \delta^{-1},\, Y \geq \delta 1_n,\, \mathrm{Re}(s) \geq \frac{n+r+1-k}{2} + \delta \right\}$$

for any $\delta > 0$. If $r = 0$, we write

$$\begin{aligned} E_k^{(n)}(Z,s) &:= [1]_0^n(Z,s) \\ &= \det(Y)^s \sum_{M \in \Delta_{n,0}\backslash\Gamma_n} \det(CZ+D)^{-k}|\det(CZ+D)|^{-2s} \end{aligned}$$

for $\mathrm{Re}(s) > (n+1-k)/2$. As a function in s, (2.3) has meromorphic continuation to the whole s-plane [5][28][33]. Hereafter we assume

$$k \geq \frac{3}{2}(n+1). \tag{2.4}$$

Then by WEISSAUER [47] and HARUKI [11], $E_k^{(m)}(Z,s)$ for every $1 \leq m \leq 2n$ is holomorphic in s at $s=0$ and

$$E_k^{(m)}(Z) := E_k^{(m)}(Z,0)$$

belongs to $M_k^m(\mathbf{Q})$.

Let $f \in S_k^r$ be a Hecke eigenform with $1 \leq r \leq n$. By GARRETT [9] and BÖCHERER [5]

$$\left(f, E_k^{(n+r)}\begin{pmatrix} -\bar{Z}^{(n)} & 0 \\ 0 & * \end{pmatrix}\right) = c(f)[f]_r^n(Z) \tag{2.5}$$

with

$$[f]_r^n(Z) := [f]_r^n(Z,0) \in M_k^n$$

and

$$\begin{aligned} c(f) :=& (-1)^{\frac{rk}{2}} 2^{\frac{r(r+3)}{2}+1-kr} \pi^{\frac{r(r+1)}{2}} \cdot \frac{\Gamma_r(k-\frac{r+1}{2})}{\Gamma_r(k)} \\ & \cdot \frac{L(k-r,f,\mathrm{St})}{\zeta(k)\prod_{j=1}^r \zeta(2k-2j)}; \end{aligned}$$

here

$$\Gamma_r(s) := \pi^{\frac{r(r-1)}{4}} \prod_{j=1}^{r} \Gamma(s - \frac{j-1}{2}).$$

By (2.4) we have $c(f) \neq 0$. We put

$$c^*(f) := \frac{c(f)}{(f,f)}. \tag{2.6}$$

If the Fourier coefficients of f belong to $\mathbf{Q}(f)$, then $c^*(f)$ and the Fourier coefficients of $[f]_r^n$ belong to $\mathbf{Q}(f)$ by [32], Appendix A.

2.5 The ideal $\mathfrak{A}(\varphi)$

We assume (2.4). Let $\varphi \in S_k^n$ be a Hecke eigenform with Fourier coefficients in $\mathbf{Q}(\varphi)$. Put

$$V := \bigoplus_{\tau} \mathbf{C}\varphi^{\tau},$$

where τ runs over all embeddings of $\mathbf{Q}(\varphi)$ into $\mathbf{C}$. By [34], p. 221,

$$V(\mathbf{Q})\otimes_{\mathbf{Q}}\mathbf{C} = V$$

and

$$V^{\perp}(\mathbf{Q})\otimes_{\mathbf{Q}}\mathbf{C} = V^{\perp},$$

where $V^{\perp}$ is the orthogonal complement of V in S_k^n. Hence $V(\mathbf{Z}) \oplus V^{\perp}(\mathbf{Z})$ is a sublattice of maximal rank in $S_k^n(\mathbf{Z})$. Let $\nu(\varphi)$ be the exponent of (i.e. the minimal positive integer that annihilates) the finite abelian group

$$S_k^n(\mathbf{Z})/(V(\mathbf{Z}) \oplus V^{\perp}(\mathbf{Z})).$$

By [34], $\lambda(T,\varphi) \in \mathcal{O}_{\mathbf{Q}(\varphi)}$ for all $T \in \mathcal{H}_{\mathbf{Z}}^n$ since $k > n$. We define $\kappa(\varphi) \in \mathbf{Z}_{>0}$ to be the exponent of the finite abelian group

$$\mathcal{O}_{\mathbf{Q}(\varphi)}/\mathbf{Z}[\lambda(T,\varphi) | T \in \mathcal{H}_{\mathbf{Z}}^n].$$

Let $\mathfrak{D}(\mathbf{Q}(\varphi))$ be the different of $\mathbf{Q}(\varphi)/\mathbf{Q}$. We put

$$\mathfrak{A}(\varphi) := \kappa(\varphi)\nu(\varphi)\mathfrak{D}(\mathbf{Q}(\varphi))$$

which is an integral ideal of $\mathbf{Q}(\varphi)$.

If $\varphi \in S_k^n(\mathbf{Q})$ and $\mathcal{H}_{\mathbf{Q}}^n$ acts irreducibly on $S_k^n(\mathbf{Q})$, then $\mathfrak{A}(\varphi) = (1)$.

3 Statement of Results

To state our main theorem (Theorem 3.1 below), we assume the following conditions (i)–(vii):

(i) $n \in \mathbf{Z}_{>0}$, k even, $\quad k \geq \frac{3}{2}(n+1), \quad 1 \leq r \leq n-1$.

(ii) The cusp form $f \in S_k^r$ is a Hecke eigenform whose Fourier coefficients belong to $\mathbf{Q}(f)$.

(iii) There exists a prime ideal $\mathfrak{p}$ of $\mathbf{Q}(f)$ such that

$$\mathrm{ord}_{\mathfrak{p}}\left(\prod_{j=[\frac{n+1}{2}]}^{[\frac{n+r}{2}]}\zeta(1+2j-2k)\cdot L^*(k-r,f,\mathrm{St})a(N_0,[f]_r^n)^2\right)=:-\alpha<0 \tag{3.1}$$

for some $N_0 \in A_n^+$. Observe that the left hand side of (3.1) remains unchanged if we replace f by γf with any $\gamma \in \mathbf{Q}(f)^\times$.

(iv) Let p be the prime number lying under $\mathfrak{p}$. Then $p \geq 2k+1$ and p does not divide

$$\prod_{i=1}^{[\frac{n-1}{2}]}\zeta(1+2i-2k)\cdot\prod_{j=[\frac{n+r}{2}]+1}^{n}\zeta(1+2j-2k)\in\mathbf{Z}_{(p)}.$$

For every integer $1 \leq \nu \leq n$ such that $S_k^\nu \neq \{0\}$, there exists a basis $\{f_1^{(\nu)}, \ldots, f_{d_\nu}^{(\nu)}\}$ of S_k^ν satisfying the following (a)–(e) by [32], Appendix A:

(a) The Fourier coefficients of $f_j^{(\nu)}$ belong to $\mathbf{Q}(f_j^{(\nu)})$.

(b) If $i \neq j$, then $(f_i^{(\nu)}, f_j^{(\nu)}) = 0$.

(c) The cusp forms $f_1^{(\nu)}, \ldots, f_{d_\nu}^{(\nu)}$ are permuted under the action of $\mathrm{Aut}(\mathbf{C})$.

(d) Each $f_j^{(\nu)}$ has one Fourier coefficient that is equal to 1.

(e) The $f_1^{(r)}$ is a constant multiple of f.

On these bases we assume:

(v) If $1 \leq \nu \leq n-1$, $1 \leq j \leq d_\nu$, and $(\nu, j) \neq (r, 1)$, then

$$\mathrm{ord}_{\mathfrak{q}}\left(\prod_{j=\nu+1}^{[\frac{n+\nu}{2}]}\zeta(1+2j-2k)\cdot L^*(k-\nu,f_j^{(\nu)},\mathrm{St})\right)=0$$

for every prime ideal $\mathfrak{q}$ of $\mathbf{Q}(f, f_j^{(\nu)})$ lying above $\mathfrak{p}$; we understand that an empty product is equal to 1.

(vi) The ideals $\mathfrak{A}(f_j^{(\nu)})$ $(1 \leq \nu \leq n-1, 1 \leq j \leq d_\nu)$ are coprime with p.

For $1 \le \nu \le n-1$ we define $\mu_k(\nu) \in \mathbf{Z}_{>0}$ as in [34], p. 225: for $0 \le m \le \nu - 1$ let $e_k(\nu, m)$ be the exponent of the finite abelian group $[S_k^m(\mathbf{Z})]_m^\nu / [S_k^m]_m^\nu(\mathbf{Z})$; we understand that $e_k(\nu, m) = 1$ if $S_k^m = \{0\}$.

Then we put

$$\mu_k(\nu) := \prod_{m=0}^{\nu-1} e_k(\nu, m).$$

On this we assume:

(vii) p does not divide $\mu_k(\nu)$ $(1 \le \nu \le n-1)$.

Now we state our main result:

Theorem 3.1. *Under the above assumptions* (i)–(vii), *there exists a Hecke eigenform $F \in S_k^n$ such that*

$$N_{\mathbf{Q}(F,f)/\mathbf{Q}(f)}(\lambda(T, F) - \lambda(T, [f]_r^n)) \equiv 0 \pmod{\mathfrak{p}} \quad \textit{for all} \quad T \in \mathcal{H}_{\mathbf{Z}}^n.$$

If moreover p is coprime with every $\mathfrak{A}(f_j^{(n)})$ $(1 \le j \le d_n)$, then there exists a Hecke eigenform $G \in S_k^n$ such that

$$N_{\mathbf{Q}(G,f)/\mathbf{Q}(f)}(\lambda(T, G) - \lambda(T, [f]_r^n)) \equiv 0 \pmod{\mathfrak{p}^\alpha} \quad \textit{for all} \quad T \in \mathcal{H}_{\mathbf{Z}}^n.$$

Remark 3.2. (1) We do not assume the multiplicity one condition which we assumed in [31]. We needed that condition in [31] in order to use the integrality lemma for $n = 2$ (Lemma 3 there) since we did not know the detailed structure of S_k^n as a module over $\mathcal{H}_{\mathbf{C}}^n$ which we found later in [34], pp. 211–222.

(2) We call an $N \in A_n^+$ a *kernel form* if every nonsingular $G \in \mathbf{Z}^{(n)}$ such that $N[G^{-1}] \in A_n^+$ satisfies $\det(G) = \pm 1$. For every kernel form $N \in A_n^+$ we have

$$\begin{aligned} a(N, [f]_r^n) = &C_{r,k}^n \zeta(k) \prod_{j=1}^{r} \zeta(2k - 2j) \\ &\cdot a(N, E_k^{(n)}) \cdot \frac{\tilde{D}(k - \frac{r+1}{2}, f, \vartheta_N^{(r)})}{L(k - r, f, \mathrm{St})} \end{aligned}$$

under the assumption of Theorem 3.1 ([3]; see also [36, p. 202]). Here

$$C_{r,k}^n := (-1)^{\frac{rk}{2}} 2^{\frac{r(r-1)}{2} - rk} \pi^{-rk + \frac{r^2}{2} - \frac{rn}{2}} \frac{\Gamma_n(k)}{\Gamma_{n-r}(k - \frac{r}{2})}$$

and

$$\vartheta_N^{(r)}(Z) := \sum_{G \in \mathbf{Z}^{(n,r)}} \mathbf{e}\left(\sigma(N[G]Z)\right)$$

is the theta function of degree r associated with N which is a modular form of weight $n/2$ for some congruence subgroup of Γ_r. The function $\tilde{D}(s, f, \vartheta_N^{(r)})$ is defined for sufficiently large $\mathrm{Re}(s)$ by

$$\tilde{D}(s, f, \vartheta_N^{(r)}) := \sum_{T \in A_r^+/GL_r(\mathbf{Z})} \frac{a(T, f)a(T, \vartheta_N^{(r)})}{|\mathrm{Aut}(T)|} \det(T)^{-s}$$

and after that by analytic continuation [36]. Here $U \in GL_r(\mathbf{Z})$ acts on $T \in A_r^+$ via $T \mapsto T[U]$ and $|\mathrm{Aut}(T)|$ stands for the order of the group

$$\mathrm{Aut}(T) := \{U \in GL_r(\mathbf{Z}) \,|\, T[U] = T\}.$$

By [36],

$$\frac{\tilde{D}(k - \frac{r+1}{2}, f, \vartheta_N^{(r)})}{\pi^{rk - \frac{3}{4}r^2 - \frac{r}{4} - \frac{1}{2}[\frac{r}{2}]}(f, f)} \in \mathbf{Q}(f).$$

The explicit formula of $a(N, E_k^{(n)})$ for all $N \in A_n$ is given in [15].

4 Proofs

Let $n \in \mathbf{Z}$ and $k \in 2\mathbf{Z}$ such that $k \geq \frac{3}{2}(n+1)$. Put

$$Z(n,k) := \zeta(1-k) \prod_{j=1}^{[\frac{n}{2}]} \zeta(1 + 2j - 2k) \tag{4.1}$$

and

$$\tilde{E}_k^{(n)}(Z) := Z(n,k) E_k^{(n)}(Z). \tag{4.2}$$

By the Siegel-Böcherer theorem ([42], [4]; cf. [34], p. 223 for low weights) we have

Lemma 4.1. *Under the assumptions* (i) *and* (iv) *in Sect.* 3*, the Fourier coefficients of* $\tilde{E}_k^{(m)}$ *are* p*-integral for* $1 \leq m \leq 2n$*.*

Remark 4.2. For every Hecke eigenform $\varphi \in S_k^\nu$ for $1 \le \nu \le n$ let $c^*(\varphi)$ be as in (2.6). Then there exists a $u \in \mathbf{Z}_{(p)}^\times$ such that

$$c^*(\varphi) = u \cdot \frac{L^*(k-\nu, \varphi, \mathrm{St})}{Z(2\nu, k)}$$

for every prime number $p \ge 2k+1$.

Remark 4.3. If $1 \le m_1 \le m_2 \le 2k-2$, then

$$Z(m_2, k)/Z(m_1, k) \in \mathbf{Z}_{(p)} \quad \text{for} \quad p \ge 2k+1$$

by von-Staudt's theorem.

Define $g_N^{(\nu)} \in M_k^\nu(\mathbf{Q})$ for $N \in A_n$ and $1 \le \nu \le n$ by

$$\tilde{E}_k^{(n+\nu)} \begin{pmatrix} Z^{(n)} & 0 \\ 0 & W^{(\nu)} \end{pmatrix} = \sum_{N \in A_n} g_N^{(\nu)}(W) \mathbf{e}(\sigma(NZ)). \tag{4.3}$$

For $\nu = n$ we have

$$g_N^{(n)}(W) = Z(2n, k) \sum_{\nu=0}^{n} \sum_{j=1}^{d_\nu} c^*(f_j^{(\nu)}) a(N, [f_j^{(\nu)}]_\nu^n) [f_j^{(\nu)}]_\nu^n(W) \tag{4.4}$$

by (2.5). Here we put $d_0 := 1$, $f_1^{(0)} := 1$, and $c^*(1) := 1$. Observe that each term for $\nu \ge 1$ in the right-hand side of (4.4) is invariant if we multiply $f_j^{(\nu)}$ by an element of $\mathbf{Q}(f_j^{(\nu)})^\times$. In particular, (4.4) is still valid if we replace $f_1^{(r)}$ by f.

Lemma 4.4. *For $1 \le \nu \le n$ let $\varphi \in S_k^\nu$ be a Hecke eigenform. Suppose that the Fourier coefficients of φ belong to $\mathbf{Q}(\varphi)$ and that one of them is equal to 1. Then, under the assumptions* (i) *and* (iv)*, for every $N \in A_n$ we have*

$$c^*(\varphi) a(N, [\varphi]_\nu^n) \in \mu_k(\nu)^{-1} Z(n+\nu, k)^{-1} \mathfrak{A}(\varphi)^{-1} \cdot \mathbf{Z}_{(p)}^\times. \tag{4.5}$$

Proof. This follows directly from [34], Theorem 6.5. By Remark 4.3 the assumption $N \in A_n^+$ is not necessary. $\square$

Hereafter in this section we assume that the conditions (i)–(vii) in Sect. 3 are satisfied.

Lemma 4.5. *For $1 \le \nu \le n-1$ and $(\nu, j) \ne (r, 1)$, the Fourier coefficients of*

$$Z(2n,k)c^*(f_j^{(\nu)})a(N_0,[f_j^{(\nu)}]_\nu^n)[f_j^{(\nu)}]_\nu^n(W)$$

are $\mathfrak{p}$-integral, i.e., $\mathfrak{q}$-integral for every prime ideal $\mathfrak{q}$ of $\mathbf{Q}(f, f_j^{(\nu)})$ lying above $\mathfrak{p}$.

Proof. For any $N \in A_n$ we have

$$\begin{aligned}&Z(2n,k)c^*(f_j^{(\nu)})a(N_0,[f_j^{(\nu)}]_\nu^n)a(N,[f_j^{(\nu)}]_\nu^n)\\ =&\frac{Z(2n,k)}{Z(n+\nu,k)}\cdot\frac{v}{c^*(f_j^{(\nu)})Z(n+\nu,k)}\end{aligned}$$

with a $\mathfrak{q}$-integral element $v \in \mathbf{Q}(f, f_j^{(\nu)})$ by Lemma 4.4. Here $c^*(f_j^{(\nu)})Z(n+\nu,k)$ is a $\mathfrak{q}$-unit by the assumption (v) and Remark 4.2. This, together with Remark 4.3, gives the assertion. □

Putting $N = N_0$ in (4.4), we use Lemmas 4.1 and 4.5 to obtain:

$$Z(2n,k)c^*(f)a(N_0,[f]_r^n)[f]_r^n(W) + \sum_{j=1}^{d_n}\gamma_j f_j^{(n)}(W) \equiv 0 \pmod{(\mathcal{O}_{\mathbf{Q}(f)})_\mathfrak{p}} \tag{4.6}$$

with

$$\gamma_j := Z(2n,k)c^*(f_j^{(n)})a(N_0,f_j^{(n)}) \in \mathbf{Q}(f_j^{(n)}).$$

Here the congruence is understood to be the system of congruences for Fourier coefficients. By (4.6) and Remark 4.2 we have

$$\begin{aligned}&Z(2n,k)L^*(k-r,f,\mathrm{St})a(N_0,[f]_r^n)[f]_r^n(W)\\ &+u_1Z(2r,k)\sum_{j=1}^{d_n}\gamma_j f_j^{(n)}(W) \equiv 0 \pmod{(\mathcal{O}_{\mathbf{Q}(f)})_\mathfrak{p}}\end{aligned} \tag{4.7}$$

with a $\mathfrak{p}$-unit $u_1 \in \mathbf{Q}(f)$. In the assumption (vii), $\mu_k(1)$ is equal to the numerator of $2^{-1}\zeta(1-k)$. Hence the assumptions (iii),(iv) and (vii) imply that

$$\mathrm{ord}_\mathfrak{p}\left(Z(2n,k)L^*(k-r,f,\mathrm{St})a(N_0,[f]_r^n)^2\right) = -\alpha. \tag{4.8}$$

Lemma 4.6. *Let $n, l \in \mathbf{Z}_{>0}$ with $l > n$. Let $f_1, \ldots, f_d$ be Hecke eigenforms in M_l^n linearly independent over $\mathbf{C}$, and $K := \mathbf{Q}(f_1) \cdots \mathbf{Q}(f_d)$ the composite field. Suppose that the Fourier coefficients of every f_j belong to K. Let $\mathfrak{P}$ be a prime ideal of K and H an element of $M_l((\mathcal{O}_K)_{\mathfrak{P}})$. Assume that there exist $c_1, \ldots, c_d \in K$ such that $\mathrm{ord}_{\mathfrak{P}}(a(N_0, f_1)c_1) < 0$ for some $N_0 \in A_n^+$ and*

$$H(z) = \sum_{i=1}^{d} c_i f_i(z).$$

Then there exists $i \neq 1$ such that

$$\lambda(T, f_i) \equiv \lambda(T, f_1) \pmod{\mathfrak{P}} \quad \textit{for all} \quad T \in \mathcal{H}_{\mathbf{Z}}^n.$$

Proof. This lemma is a slight generalization of [18], Lemma 5.1 and is proved similarly. Here we use the fact that $\mathcal{H}_{\mathbf{Z}}^n$ preserves $M_l((\mathcal{O}_K)_{\mathfrak{P}})$ if $l > n$ ([37], Lemma A.6). □

Thus (4.7), (4.8) and Lemma 4.6 give the former half of the assertion of Theorem 3.1.

To prove the latter half of Theorem 3.1, we proceed as follows. Observe that

$$\sum_{j=1}^{d_n} \gamma_j f_j^{(n)}(W) \in S_k^n(\mathbf{Q}) \tag{4.9}$$

in (4.6) since $\gamma_j f_j^{(n)}(W)$ for $1 \leq j \leq d_n$ are permuted under the action of $\mathrm{Aut}(\mathbf{C})$. Multiplying each $f_j^{(n)}$ by an element of $\mathbf{Q}(f_j^{(n)})^{\times}$ if necessary, we assume that

$$a(N_0, f_j^{(n)}) = \begin{cases} 1 & (1 \leq j \leq s_0) \\ 0 & (s_0 < j \leq d_n) \end{cases}$$

without loss of generality. From the assumption (iii) and (4.7) it follows that $s_0 \geq 1$.

Lemma 4.7. *There exist a number t with $1 \leq t \leq s_0$ and a prime ideal $\mathfrak{P}$ of $\mathbf{Q}(f, f_t^{(n)})$ lying above $\mathfrak{p}$ such that*

$$\mathrm{ord}_{\mathfrak{P}}(Z(2r, k)\gamma_t) \leq -\alpha.$$

Proof. From (4.9) we have $\sum_{j=1}^{s_0} \gamma_j \in \mathbf{Q}$. Thus from (4.7) and (4.8) we obtain

$$\mathrm{ord}_{\mathfrak{p}}(Z(2r,k)\sum_{j=1}^{s_0}\gamma_j) = -\alpha.$$

which gives the assertion. $\square$

For any $T \in \mathcal{H}_{\mathbf{Z}}^n$ we apply $T - \lambda(T, [f]_r^n) \cdot \mathrm{id}$ on the both sides of (4.6), where id is the identity operator. As above, the $\mathfrak{p}$-integrality of the Fourier coefficients are preserved. Hence

$$H(W) := Z(2r,k)\sum_{j=1}^{d_n}\gamma_j(\lambda(T, f_j^{(n)}) - \lambda(T,[f]_r^n))f_j^{(n)}(W) \equiv 0 \pmod{(\mathcal{O}_{\mathbf{Q}(f)})_{\mathfrak{p}}}. \tag{4.10}$$

Lemma 4.8 (Integrality Lemma [37])**.** *Let $\varphi \in S_l^n$ be a Hecke eigenform with $n, l \in \mathbf{Z}_{>0}$ such that $l \geq \frac{3}{2}(n+1)+\varepsilon$, where $\varepsilon = 0$ or 1 according as l is even or odd. Suppose that the Fourier coefficients of φ belong to $\mathbf{Q}(\varphi)$ and φ has a Fourier coefficient which is equal to 1. Let K be an algebraic number field. Then for any $\psi \in S_l^n(\mathcal{O}_K)$ we have*

$$\frac{(\psi,\varphi)}{(\varphi,\varphi)} \in \mathfrak{A}(\varphi)^{-1}\cdot \mathcal{O}_{K\cdot\mathbf{Q}(\varphi)}.$$

By [37, p. 116, Lemma A.3] there exists a $\mathfrak{p}$-unit $u_2 \in \mathcal{O}_{\mathbf{Q}(f)}$ such that

$$u_2 H \in S_k^n(\mathcal{O}_{\mathbf{Q}(f)}).$$

Applying Lemma 4.8 to (4.10) with $\psi = u_2 H$, $\varphi = f_j^{(n)}$, and $K = \mathbf{Q}(f)$, we see

$$u_2 Z(2r,k)\gamma_j(\lambda(T,f_j^{(n)}) - \lambda(T,[f]_r^n)) \in \mathfrak{A}(f_j^{(n)})^{-1}\cdot\mathcal{O}_{\mathbf{Q}(f,f_j^{(n)})} \quad \text{for} \quad 1 \leq j \leq s_0.$$

Here $\mathfrak{A}(f_j^{(n)})$ is coprime with p by the assumption. Therefore Lemma 4.7 gives

$$\lambda(T,F) \equiv \lambda(T,[f]_r^n) \pmod{\mathfrak{P}^\alpha}$$

for $F = f_t^{(n)}$, hence

$$N_{\mathbf{Q}(F,f)/\mathbf{Q}(f)}(\lambda(T,F) - \lambda(T,[f]_r^n)) \equiv 0 \pmod{\mathfrak{p}^\alpha}.$$

This completes the proof of Theorem 3.1.

Remark 4.9. By a similar argument as above we obtain congruences for Hecke eigenvalues of $E_k^{(n)}$ and of some $\varphi \in S_k^n$ modulo every prime factor of the numerator of $\zeta(1+n-2k)$ under some additional conditions if n is even. But in this case a more precise result is easily obtained by applying the Ikeda lift [12] to the Ramanujan type congruences in M_{2k-n}^1; cf. KUROKAWA [24] for the case $n = 2$.

5 Numerical Examples

We compute the standard zeta values and the Fourier coefficients of the Klingen-Eisenstein series with Mathematica, and give some examples of congruence between the Klingen-Eisenstein series and cusp forms of degree three. For examples in degree two case, we refer to [31].

Let p be a prime number. Let $\mathbf{Q}_p$ be the field of p-adic numbers and $\mathbf{Z}_p$ the ring of p-adic integers. Two symmetric matrices A and A' with entries in $\mathbf{Q}_p$ are called equivalent over $\mathbf{Z}_p$ with each other and written $A \sim_{\mathbf{Z}_p} A'$ if there is an element X of $GL_n(\mathbf{Z}_p)$ such that $A' = A[X]$. For square matrices X and Y we write $X \perp Y = \begin{pmatrix} X & O \\ O & Y \end{pmatrix}$. We denote by A_{np} the set of semi-integral matrices of size n over $\mathbf{Z}_p$. To see the Fourier expansion of $E_k^{(n)}(Z)$, for a semi-integral matrix T of size n over $\mathbf{Z}_p$ define the local Siegel series $b_p(T, s)$ as in [15]. We define $\chi_p(a)$ for $a \in \mathbf{Q}_p^\times$ as follows:

$$\chi_p(a) := \begin{cases} +1 & \text{if } \mathbf{Q}_p(\sqrt{a}) = \mathbf{Q}_p, \\ -1 & \text{if } \mathbf{Q}_p(\sqrt{a})/\mathbf{Q}_p \text{ is quadratic unramified,} \\ \ \ 0 & \text{if } \mathbf{Q}_p(\sqrt{a})/\mathbf{Q}_p \text{ is quadratic ramified.} \end{cases}$$

For a semi-integral matrix T of even size n define $\xi_p(T)$ by

$$\xi_p(T) := \chi_p((-1)^{n/2} \det T).$$

Let $T \in A_n^+$ with n even. Then we can write $(-1)^{n/2} 2^n \det T = \mathfrak{d}_T \mathfrak{f}_T^2$ with $\mathfrak{d}_T$ a fundamental discriminant and $\mathfrak{f}_T \in \mathbf{Z}_{>0}$. Furthermore, let $\chi_T = (\frac{\mathfrak{d}_T}{*})$ be the Kronecker character corresponding to $\mathbf{Q}(\sqrt{(-1)^{n/2} \det T})/\mathbf{Q}$. We note that we have $\chi_T(p) = \xi_p(T)$ for any prime p. For a nondegenerate semi-integral matrix T of size n over $\mathbf{Z}_p$ define a polynomial $\gamma_p(T, X)$ in X by

$$\gamma_p(T, X) := \begin{cases} (1 - X) \prod_{i=1}^{n/2} (1 - p^{2i} X^2)(1 - p^{n/2} \xi_p(T) X)^{-1} & \text{if } n \text{ is even,} \\ (1 - X) \prod_{i=1}^{(n-1)/2} (1 - p^{2i} X^2) & \text{if } n \text{ is odd.} \end{cases}$$

Then it is well known that there exists a unique polynomial $F_p(T,X)$ in X over $\mathbf{Q}$ such that

$$b_p(T,s)=\gamma_p(T,p^{-s})F_p(T,p^{-s})$$

(e.g. [19]).

Remark 5.1. For an element $T\in A_n$ of rank $m\geq 1$, there exists an element $\tilde{T}\in A_m^+$ such that $T\sim\tilde{T}\perp O_{n-m}$. We note that $b_p(\tilde{T},s)$ does not depend on the choice of $\tilde{T}$. Thus we write this as $b_p^{(m)}(T,s)$. Furthermore, $F_p(\tilde{T},X)$ does not depend on the choice of $\tilde{T}$. Then we put $F_p^{(m)}(T,X)=F_p(\tilde{T},X)$. Then, $\det\tilde{T}$ does not depend on the choice of T. Thus we put $\det^{(m)}T=\det\tilde{T}$. Similarly, we write $\chi_T^{(m)}=\chi_{\tilde{T}}$ if m is even.

Now for $T\in A_n$ of rank m, we put

$$c_k^{(n)}(T):=2^{[(m+1)/2]}\prod_p F_p^{(m)}(T,p^{k-m-1})$$
$$\cdot\begin{cases}\prod_{i=m/2+1}^{[n/2]}\zeta(1+2i-2k)L(1+m/2-k,\chi_T^{(m)}) & \text{if } m \text{ is even,}\\ \prod_{i=(m+1)/2}^{[n/2]}\zeta(1+2i-2k) & \text{if } m \text{ is odd.}\end{cases}.$$

Here we make the convention $F_p^{(m)}(T,p^{k-m-1})=1$ and $L(1+m/2-k,\chi_T^{(m)})=\zeta(1-k)$ if $m=0$. We also define $c_k^{(n)}(T)=0$ if T is not semidefinite. Let $\tilde{E}_k^{(n)}(Z)$ be as in (4.2).

Proposition 5.2. *Let $k\in 2\mathbf{Z}_{>0}$. Assume that $k\geq\frac{3}{4}n+\frac{3}{2}$. Then we have*

$$\tilde{E}_k^{(n)}(Z)=\sum_{T\in A_n}c_k^{(n)}(T)\mathbf{e}(\sigma(TZ)).$$

Let $n,r\in\mathbf{Z}_{>0}$ such that $n\geq r$. Let f be a Hecke eigenform in M_k^r. Then f is expressed as $f=[g]_\nu^r$ with some Hecke eigenform g in S_k^ν $(0\leq\nu\leq r)$. We then define $\tilde{c}^*(f)$ by $\tilde{c}^*(f):=c^*(g)$, and $[f]_r^n:=[g]_\nu^n$. For $T_1\in A_n$ and $T_2\in A_r$, put

$$\epsilon_k^{(n,r)}(T_1,T_2):=\sum_{R\in\mathbf{Z}^{(n,r)}}c_k^{(n+r)}(\begin{pmatrix}T_1 & R/2\\ {}^tR/2 & T_2\end{pmatrix}).$$

Then for any $N\in A_n$ the $g_{k,N}^{(r)}$ defined in (4.3) is expressed as

$$g_{k,N}^{(r)}(W)=\sum_{T\in A_r}\epsilon_k^{(n,r)}(N,T)\mathbf{e}(\sigma(TW)).$$

Now, for a prime number p, let $T(p)$ be the element of $\mathcal{H}^r_{\mathbf{Z}}$ defined by $T(p) = \Gamma_r(1_r \perp p1_r)\Gamma_r$. For each $i \in \mathbf{Z}_{\geq 0}$ and $N \in A_n^+$, write $g_{k,N}^{(r)}|T(p)^i(W)$ as

$$g_{k,N}^{(r)}|T(p)^i(W) = \sum_{T \in A_n^+} \epsilon_{k,p}^{(n,r)}(i,N,T)\mathbf{e}(\sigma(TW)).$$

Let $\{f_j\}$ be a basis of M_k^r consisiting of Hecke eigenforms. Furthermore write

$$f_j|T(p)(z) = \lambda_j f_j(z).$$

Proposition 5.3. *Under the above notation and the assumption, we have*

$$\epsilon_{k,p}^{(n,r)}(i,N,T) = Z(n+r,k)\sum_{j=1}^{d} \lambda_j^i \tilde{c}^*(f_j)a(N,[f_j]_r^n)\overline{a(T,f_j)}$$

for any $N \in A_n^+$, $T \in A_r^+$ and $i \in \mathbf{Z}_{\geq 0}$, where $Z(n,k)$ is defined as in (4.1).

By using Propositions 5.2 and 5.3, we will compute the standard zeta values and the Fourier coefficients in question. We have an explicit formula for $F_p(T,X)$ for any nondegenerate semi-integral matrix T over $\mathbf{Z}_p$ (cf. [15]), but it is rather complicated in general. Thus we use some trick, which enables us to compute $F_p(T,X)$ more easily for some special cases. Let $m,n \in \mathbf{Z}_{>0}$ such that $m \geq n$. For $S \in A_{mp} \cap GL_m(\mathbf{Q})$ and $T \in A_{np} \cap GL_n(\mathbf{Q}_p)$ define the local density $\alpha_p(S,T)$ and the primitive local density $\beta_p(S,T)$ by

$$\alpha_p(S,T) := 2^{\delta_{mn}} \lim_{e \to \infty} p^{(-mn+n(n+1)/2)e} \#\mathcal{A}_e(S,T)$$

and

$$\beta_p(S,T) := 2^{\delta_{mn}} \lim_{e \to \infty} p^{(-mn+n(n+1)/2)e} \#\mathcal{B}_e(S,T),$$

where δ_{mn} is Kronecker's delta,

$$\mathcal{A}_e(S,T) := \{X \in M_{mn}(\mathbf{Z}_p)/p^e M_{mn}(\mathbf{Z}_p) \mid S[X] - T \in p^e A_{np}\},$$

and

$$\mathcal{B}_e(S,T) := \{X \in \mathcal{A}_e(S,T) \mid \mathrm{rank}_{\mathbf{Z}_p/p\mathbf{Z}_p}(X) = n\}.$$

Let $H_k = \overbrace{H \perp ... \perp H}^{k}$ with $H = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$. Now first we remark the following two lemmas (e.g. [18], Lemmas 2.1 and 3.1).

Lemma 5.4. *Let T be a nondegenerate semi-integral matrix of size n over $\mathbf{Z}_p$. Then for any $k \in \mathbf{Z}$ with $k \geq n/2$ and a half integral matrix S of size $2k$ over $\mathbf{Z}_p$ such that $2S$ is unimodular, we have*

$$\alpha_p(S,T) = F_p(T, \xi_p(S)p^{-k})\gamma_p(T, \xi_p(S)p^{-k})$$

and, in particular,

$$\alpha_p(H_k,T) = F_p(T, p^{-k})\gamma_p(T, p^{-k}).$$

Lemma 5.5. *Let $n = n_1 + n_2$. Let $T_{11} \in A_{n_1,p} \cap \frac{1}{2}GL_{n_1}(\mathbf{Z}_p)$ and $T_{22} \in A_{n_2,p} \cap GL_{n_2}(\mathbf{Q}_p)$. Then for any $l \geq n$ we have*

$$\alpha_p(H_l, T_{11} \perp T_{22}) = \beta_p(H_l, T_{11})\alpha_p(H_{l-n_1} \perp (-T_{11}), T_{22}).$$

Now the following proposition is due to [18], Proposition 3.2.

Proposition 5.6. *Let $T_{11} \in A_{n_1,p} \cap \frac{1}{2}GL_{n_1}(\mathbf{Z}_p)$ and $T_{22} \in A_{n_2,p}$. Let m be the rank of T_{22}. Then we have*

$$F_p^{(n_1+m)}(T_{11} \perp T_{22}, X) = F_p^{(m)}(T_{22}, \xi_p(T_{11})p^{n_1/2}X).$$

Let $T = (t_{ij}) \in A_3^+$. Let $\tilde{e}_1(T) := \mathrm{GCD}_{1 \leq i,j \leq 3}(t_{ij}), \tilde{e}_2(T) := \mathrm{GCD}_{1 \leq i,j \leq 3}(2^{3-\delta_{ij}}T_{ij})$, and $\tilde{e}_3(T) := 4\det T$, where T_{ij} denotes the (i,j)-th cofactor of T. For a prime number p, let $\eta_p(T) := (-1)^{\delta_{2p}}h_p(T), m_{1p}(T) := \mathrm{ord}_p(\tilde{e}_1(T)), m_{2p}(T) := \mathrm{ord}_p(\tilde{e}_2(T))$, and $m_{3p}(T) := \mathrm{ord}_p(\tilde{e}_3(T))$, where $h_p(T)$ denotes the Hasse invariant defined on $S_3(\mathbf{Q})$, the set of symmetric matrices of size three with entries in $\mathbf{Q}$. Let $p \neq 2$. Then T is $GL_3(\mathbf{Z}_p)$-equivalent to

$$p^{r_1}u_1 \perp p^{r_2}u_2 \perp p^{r_3}u_3$$

with $r_1 \geq r_2 \geq r_3$ and $u_1, u_2, u_3 \in \mathbf{Z}_p^\times$. We note that r_1, r_2, r_3 are uniqely determined by T. Then put $\tilde{\xi}_p(T) := \chi_p(-p^{r_2+r_3}u_2u_3)$ or $(\chi_p(-p^{r_2+r_3}u_2u_3))^2$ according as $r_1 > r_2$ or $r_1 = r_2$. This $\tilde{\xi}_p(T)$ does not depend on the choices of u_1, u_2, u_2. Next let $p = 2$. Then T is $GL_3(\mathbf{Z}_2)$-equivalent to one of the following forms:

$(C1)$ $2^{r_1}u_1 \perp 2^{r_3}K$

with $r_1 \geq r_2, K = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$, and $u_1 \in \mathbf{Z}_2^\times$,

$(C2)$ $2^{r_1}K \perp 2^{r_3}u_3$

with $r_1 \geq r_2 + 2, K = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$, and $u_3 \in \mathbf{Z}_2^\times$,
$(C3)$ $2^{r_1}u_1 \perp 2^{r_2}u_2 \perp 2^{r_3}u_3$
with $r_1 \geq r_2 \geq r_3$ and $u_1, u_2, u_3 \in \mathbf{Z}_2^\times$.
Then define $\tilde{\xi}_2(T)$ by

$$\tilde{\xi}_2(T) := \begin{cases} \chi_2(-\det K) & \text{if } T \text{ is type } C1 \text{ and } r_1 \geq r_3 + 1, \\ \chi_2(-2^{r_2-r_3}u_2u_3) & \text{if } T \text{ is type } C3 \text{ and } r_1 \geq r_2 + 3, \\ \chi_2(-2^{r_2-r_3}u_2u_3)^2 & \text{if } T \text{ is type } C3 \text{ and } r_1 = r_2 + 2, \\ 1 & \text{otherwise.} \end{cases}$$

Furthermore put

$$n_p'(T) := \begin{cases} 1 & \text{if } p \neq 2 \text{ and } m_2 \equiv 0 \bmod 2 \\ & \text{or if } p = 2, m_3 - 2m_2 + m_1 = -4, \text{ and } m_2 \equiv 0 \bmod 2, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have an explicit formula of $F_p(T, X)$ for a nondegenerate semi-integral matrix T of size not greater than four (cf. [14],[15]).

Proposition 5.7. (1) *Let $T = (a) \in A_1^+$. Then we have*

$$F_p(T, X) = \sum_{i=0}^{\mathrm{ord}_p(a)} (pX)^i.$$

(2) *Let $T = \begin{pmatrix} a_{11} & a_{12}/2 \\ a_{12}/2 & a_{22} \end{pmatrix} \in A_2^+$. Put $e = e_T = \mathrm{GCD}(a_{11}, a_{12}, a_{22})$. Then we have*

$$\begin{aligned} F_p(T, X) = & \sum_{i=0}^{\mathrm{ord}_p(e_T)} (p^2X)^i \sum_{j=0}^{\mathrm{ord}_p(\mathfrak{f}_T)-i} (p^3X^2)^j \\ & - \chi_T(p)pX \sum_{i=0}^{\mathrm{ord}_p(e_T)} (p^2X)^i \sum_{j=0}^{\mathrm{ord}_p(\mathfrak{f}_T)-i-1} (p^3X^2)^j. \end{aligned}$$

(3) *Let $T \in A_3^+$. Then we have*

$$F_p(T,X) = \sum_{i=0}^{m_1}\Big(\sum_{j=0}^{[(m_2-\delta_{2p}-1)/2]-i} (p^5X^2)^j\Big)(p^3X)^i$$
$$+ \eta_p(T)(p^2X)^{m_3}(p^3X^2)^{-[m_2/2]+\delta_{2p}} \sum_{i=0}^{m_1}\Big(\sum_{j=n'}^{[m_2/2]-\delta_{2p}-i} (p^3X^2)^j\Big)(p^2X)^i$$
$$+ (p^5X^2)^{[m_2/2]}(p^2X)^{-m_1} \sum_{i=0}^{m_3-2m_2+m_1} (p^2X)^i\tilde{\xi}_p(T)^{i+2}\sum_{j=0}^{m_1}(p^2X)^i.$$

Now let p a prime number. For an element $T_1 \in A_{2,p} \cap \frac{1}{2}GL_2(\mathbf{Z}_p)$ and $T_2 \in A_{n,p}$ of rank m, put $G_p^{(m)}(T_1,T_2;k) = F_p^{(m)}(T_2, p^{k-m}\xi_p(T_1))$. Here we make the convention that $F^{(0)}(T_1,T_2,k) = 1$ if $T_2 = O$. Then by Proposition 5.6 we have

Proposition 5.8. *Let $T_1 \in A_n^+$ with $n = 2$ or 3, $T_2 \in A_2^+$ and let $T = \begin{pmatrix} T_1 & R/2 \\ {}^tR/2 & T_2 \end{pmatrix} \in A_{n+2}$ of rank m with $R \in \mathbf{Z}^{(n,2)}$. Let p_0 be a prime number.*

(1) *Let $n=2$. Assume that $2T_1 \in GL_2(\mathbf{Z}_p)$ for any prime number $p \neq p_0$ and $2T_2 \in GL_2(\mathbf{Z}_{p_0})$. Then we have*

$$c_k^{(4)}(T) = 4G_{p_0}^{(m-2)}(T_2, T_1 - \frac{1}{4}T_2^{-1}[{}^tR], k)$$
$$\cdot \prod_{p\neq p_0} G_p^{(m-2)}(T_1, T_2 - \frac{1}{4}T_1^{-1}[R], k) \cdot \begin{cases} L(3-k,\chi_T) & \text{if } m=4, \\ \zeta(5-2k) & \text{if } m=3, \\ \zeta(5-2k)L(2-k) & \text{if } m=2. \end{cases}$$

(2) *Let $n=3$, and write T_1 and R as $T_1 = \begin{pmatrix} \tilde{T}_1 & \mathbf{t}/2 \\ {}^t\mathbf{t} & t_{33} \end{pmatrix}$ and $R = \begin{pmatrix} R_1 \\ \mathbf{r} \end{pmatrix}$ with $\tilde{T}_1 \in A_2^+, \mathbf{t} \in \mathbf{Z}^{(2,1)}, t_{33} \in \mathbf{Z}, R_1 \in \mathbf{Z}^{(2,2)}$, and $\mathbf{r} \in \mathbf{Z}^{(2,1)}$. Assume that $2\tilde{T}_1 \in GL_2(\mathbf{Z}_p)$ for any prime number $p \neq p_0$ and $2T_2 \in GL_2(\mathbf{Z}_{p_0})$. Then we have*

$$c_k^{(5)}(T) = 2^{[(m+1)/2]}G_{p_0}^{(m-2)}(T_2, T_1 - \frac{1}{4}T_2^{-1}[{}^tR], k)$$
$$\cdot \prod_{p\neq p_0} G_p^{(m-2)}(\tilde{T}_1, \begin{pmatrix} t_{33} & \mathbf{r} \\ {}^t\mathbf{r} & T_2 \end{pmatrix} - \frac{1}{4}\tilde{T}_1^{-1}[(\mathbf{t}, R_1)], k) \cdot \begin{cases} \zeta(5-2k) & \text{if } m=3, \\ L(3-k,\chi_T^{(4)}) & \text{if } m=4, \\ 1 & \text{if } m=5. \end{cases}$$

Let $n=2$ or 3, and $T_1 \in A_n^+$. Then by the Hecke theory of Siegel modular forms (cf. [1]), for an element T of A_2^+ and a prime number p, we have the following recursion formula for $\epsilon_{k,p}^{(n,2)}(i,T_1,T)$:

$$\epsilon_{k,p}^{(n,2)}(0,T_1,T)=\epsilon_{k,p}^{(n,2)}(T_1,T),$$

and for $i \geq 1$,

$$\begin{aligned}\epsilon_{k,p}^{(n,2)}(i,T_1,T) =&\epsilon_{k,p}^{(n,2)}(i-1,T_1,pT)+p^{2k-3}\epsilon_{k,p}^{(n,2)}(i-1,T_1,T/p)\\ &+p^{k-2}\sum_{D \in GL_2(\mathbf{Z})U_pGL_2(\mathbf{Z})/GL_2(\mathbf{Z})}\epsilon_{k,p}^{(n,2)}(i-1,T_1,T[D]/p),\end{aligned}$$

where $U_p=\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$. Let $\{f_j\}_{j=1}^d$ be an orthogonal basis of M_k^2 consisting of Hecke eigenforms, and λ_j be the eigenvalue of $T(p)$ on f_j. Recall that by Proposition 5.3 we have

$$\epsilon_{k,p}^{(n,2)}(i,T_1,T)=Z(n+2,k)\sum_{j=1}^d \lambda_j^i \tilde{c}^*(f_j)a(T_1,[f_j]_2^n)\overline{a(T,f_j)}$$

for any $i \in \mathbf{Z}_{\geq 0}$. Thus by using the same argument as in the proof of [18] we have the following.

Proposition 5.9. *Let $T \in A_2^+, N \in A_n^+$ with $n=2$ or 3. Let f be a Hecke eigenform in M_k^2, and put $\lambda=\lambda(T(p),f)$. Furthermore, let $e_i^{(n)}=\epsilon_{k,p}^{(n,2)}(i,N,T)$, and $\Phi(X)=\Phi_{T(p)}(X)=\sum_{i=0}^d b_{d-i}X^i$ the characteristic polynomial of $T(p)$ in M_k^2. Assume that $\Phi'(\lambda) \neq 0$. Then we have*

$$Z(n+2,k)c^*(f)a(N,[f]_2^n)\overline{a(T,f)}=\frac{\sum_{i=0}^{d-1}\sum_{j=i}^{d-1}e_{d-1-j}^{(n)}b_{j-i}\lambda^i}{\Phi'(\lambda)}.$$

Furthermore, let $\tilde{M}_k^2$ be the orthogonal complement of the space spanned by the Siegel-Eisenstein series $\tilde{E}_k^{(2)}$ in M_k^2. Let

$$\tilde{\Phi}(X):=\Phi(X)/(X-(1+p^{k-2})(1+p^{k-1}))$$

and

$$\begin{aligned}\tilde{e}_i^{(n)}&=\tilde{\epsilon}_{k,p}^{(n,2)}(i,N,T)\\ &:=\epsilon_{k,p}^{(n,2)}(i,N,T)-\frac{\zeta(5-2k)}{\zeta(1-k)\zeta(3-2k)}((1+p^{k-2})(1+p^{k-1}))^i c_k^{(n)}(N)c_k^{(2)}(T).\end{aligned}$$

Then $\tilde{\Phi}(X) = \tilde{\Phi}_{T(p)}(X)$ *is a polynomial, and we have*

$$Z(n+2,k)c^*(f)a(N,[f]_2^n)\overline{a(T,f)} = \frac{\sum_{i=0}^{d-2}\sum_{j=i}^{d-2}\tilde{e}_{d-1-j}^{(n)}\tilde{b}_{j-i}\lambda^i}{\tilde{\Phi}'(\lambda)},$$

where we write $\tilde{\Phi}(X) = \sum_{i=0}^{d-1}\tilde{b}_{d-1-i}X^i$.

Proof. The first assertion is proved in the same manner as [16], Theorem 3.6. The second assertion is also proved in the same manner by remarking $\lambda(T(p), E_k^{(2)}) = (1+p^{k-1})(1+p^{k-2})$. □

Now we give some numerical examples. From now on put

$$T_0 = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}, \quad T_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad T_2 = \begin{pmatrix} 1 & 1/2 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

If $\dim S_l^1 = 1$, the normalized Hecke eigenform in S_l^1 is denoted by Δ_l.

Example 1: The case $n = 3$, $r = 1$, and $k = 12$.
We have $\dim S_{12}^{\nu} = 1$ for $\nu = 1, 2, 3$. By Zagier [48],

$$L^*(11, \Delta_{12}, \mathrm{St}) = \frac{2^{24}}{3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 691}.$$

Let χ_{12} be the Hecke eigenform in S_{12}^2 defined in Kurokawa [22]. We normalize χ_{12} so that its Fourier coefficient at T_0 is 1, and denote it by $\tilde{\chi}_{12}$. By using Kohnen-Skoruppa [21] we obtain the value

$$L^*(10, \tilde{\chi}_{12}, \mathrm{St}) = \frac{2^{41}}{3^{41} \cdot 5^3 \cdot 7^5 \cdot 11^3 \cdot 13^2 \cdot 17^2 \cdot 19 \cdot 131 \cdot 593}.$$

We have $\mathfrak{A}(\Delta_{12}) = (1)$ and $\mathfrak{A}(\tilde{\chi}_{12}) = (1)$. Since the common denominator of the Fourier coefficients of $E_{12}^{(1)}$ (resp. $E_{12}^{(2)}$, $[\Delta_{12}]_1^2$) is 691 (resp. $131 \cdot 593 \cdot 691$, 7), we have $\mu_{12}(1) = 691$ (resp. $\mu_{12}(2) = 7 \cdot 131 \cdot 593 \cdot 691$); for the denominator of Fourier coefficients of $[\Delta_{12}]_1^2$, see Kurokawa [23]. We have

$$\zeta(-19) = \frac{283 \cdot 617}{2^3 \cdot 3 \cdot 5^2 \cdot 11}.$$

Moreover we have

$$\tilde{\epsilon}_{12,2}^{(3,2)}(0, T_2, T_1) = -\frac{1431288859512766464}{53678953},$$

and

$$\tilde{\epsilon}_{12,2}^{(3,2)}(1, T_2, T_1) = \frac{32414222074496738918400}{53678953}.$$

Thus by Proposition 5.9 we have

$$Z(5,12)a(T_1, [\Delta_{12}]_1^2)a(T_2, [\Delta_{12}]_1^3)c^*(\Delta_{12}) = -\frac{2^{15} \cdot 3^7 \cdot 7 \cdot 23 \cdot 1483}{691}.$$

By [36] we have $a(T_1, [\Delta_{12}]_1^2) = 2 \cdot 3^3 \cdot 23/7$. Thus for $p = 283$ or 617 we have

$$\mathrm{ord}_p(\zeta(-19)L^*(11, \Delta_{12}, \mathrm{St})a(T_2, [\Delta_{12}]_1^3)^2) = -1.$$

Thus the conditions of Theorem 3.1 are safisfied with $f = \Delta_{12}$ and $p = 283$ or 617. Let $F_{12} \in S_{12}^3$ be any Hecke eigenform. Then by Theorem 3.1 we have

$$\lambda(T, F_{12}) \equiv \lambda(T, [\Delta_{12}]_1^3) \pmod{283 \cdot 617} \quad \text{for all} \quad T \in \mathcal{H}_{\mathbf{Z}}^3. \tag{5.1}$$

The congruence (5.1) follows also from the first case of Miyawaki's conjecture [29] which was proved by Ikeda [13] telling that

$$L(s, F_{12}, \mathrm{spin}) = L(s-9, \Delta_{12})L(s-10, \Delta_{12})L(s, \Delta_{12} \otimes \Delta_{20}), \tag{5.2}$$

where $L(s, F_{12}, \mathrm{spin})$ is the spinor L-function attached to F_{12} and $L(s, \Delta_{12} \otimes \Delta_{20})$ is the Rankin-Selberg convolution attached to the pair $(\Delta_{12}, \Delta_{20})$; from (5.2) we have in particular

$$\lambda(T(p), F_{12}) - \lambda(T(p), [\Delta_{12}]_1^3) = \lambda(T(p), \Delta_{12})(\lambda(T(p), \Delta_{20}) - \lambda(T(p), E_{20}^{(1)}))$$

for all prime numbers p. Thus the equality (5.2) naturally explains the congruence (5.1) for $T = T(p)$. The other types of $T \in \mathcal{H}_{\mathbf{Z}}^3$ are treated similarly.

Example 2: The case $n = 3$, $r = 2$, and $k = 14$.
We have $\dim M_{14}^2 = 2$ and $\dim S_{14}^2 = 1$. Let χ_{14} be the Hecke eigenform in S_{14}^2 defined in Kurokawa [22]. We normalize χ_{14} so that its Fourier coefficient at T_0 is 1, and denote it by $\tilde{\chi}_{14}$. We note that $a(T_1, \tilde{\chi}_{14}) = -2$. Now we have

$$\tilde{\epsilon}_{14,2}^{(2,2)}(0, T_0, T_1) = \frac{-2^{13} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 23 \cdot 691}{657931}.$$

Hence by Proposition 5.9 we have

$$Z(4,14)c^*(\tilde{\chi}_{14})a(T_1,\tilde{\chi}_{14}) = \frac{-2^{13}\cdot 3^6\cdot 5^3\cdot 7^2\cdot 11\cdot 23\cdot 691}{657931}.$$

Thus we have

$$\mathrm{ord}_{691}(c^*(\tilde{\chi}_{14})) = 1.$$

We note that 691 appears in the numerator of $c^*(\tilde{\chi}_{14})$. This is not so surprising because χ_{14} is the Saito-Kurokawa lift of Δ_{26} and we have

$$L(12,\tilde{\chi}_{14},\mathrm{St}) = \zeta(12)L(24,\Delta_{26})L(25,\Delta_{26}),$$

where $L(s,\Delta_{26})$ is Hecke's L-function attached to Δ_{26}. We also note that it is possible to compute $c^*(\tilde{\chi}_{14})$ exactly by using the result of Kohnen-Skoruppa [21]. But here we have used the method in [16]. Now we have

$$\tilde{\epsilon}^{(3,2)}_{14,2}(0,T_2,T_1) = \frac{2^{15}\cdot 3^8\cdot 5^2\cdot 7^2\cdot 11\cdot 23\cdot 2393}{657931}.$$

Thus we have

$$Z(5,14)a(T_1,\tilde{\chi}_{14})a(T_2,[\tilde{\chi}_{14}]^3_2)c^*(\tilde{\chi}_{14}) = \frac{2^{15}\cdot 3^8\cdot 5^2\cdot 7^2\cdot 11\cdot 23\cdot 2393}{657931}.$$

We note that $Z(4,20)=Z(5,20)$ and it is a 691-unit. Thus we have

$$\mathrm{ord}_{691}(c^*(\tilde{\chi}_{14})a(T_2,[\tilde{\chi}_{14}]^3_2)^2) = -1.$$

We also note that $\zeta(-13)\zeta(-25)\zeta(-23)\zeta(-21)$ is coprime with 691 and that $\mathfrak{A}(\tilde{\chi}_{14}) = (1)$. Since the common denominator of the Fourier coefficients of $E^{(1)}_{14}$ (resp. $E^{(2)}_{14}$) is 1 (resp. 657931), we have $\mu_{14}(1)=1$ (resp. $\mu_{14}(2)=657931$). Let F_{14} be the cusp form in S^3_{14} constructed by Miyawaki [29]. Then S^3_{14} is spanned by F_{14}. Thus by Theorem 3.1 we have

$$\lambda(T,F_{14}) \equiv \lambda(T,[\tilde{\chi}_{14}]^3_2) \pmod{691} \quad \text{for all} \quad T\in\mathcal{H}^3_{\mathbf{Z}}. \tag{5.3}$$

Some Hecke eigenvalues of $[\tilde{\chi}_{14}]^3_2$ and F_{14} have been computed, and we can verify this congruence for some $T(p)\in\mathcal{H}^3_{\mathbf{Z}}$ directly. For example, we have $\lambda(T(2),[\tilde{\chi}_{14}]^3_2) = 12240(1+2^{11})$ and $\lambda(T(2),F_{14}) = -2^7\cdot 2295$. Thus we have

$$\lambda(T(2),F_{14}) - \lambda(T(2),[\tilde{\chi}_{14}]^3_2) = -2^4\cdot 3^3\cdot 5\cdot 17\cdot 691.$$

The congruence (5.3) supports the second case of Miyawaki's conjecture [29] predicting that

$$L(s,F_{14},\mathrm{spin}) = L(s-12,\Delta_{12})L(s-13,\Delta_{12})L(s,\Delta_{12}\otimes\Delta_{26}).$$

Remark 5.10. Similarly we can give examples of congruence between Klingen-Eisenstein series and cusp forms in the case $n = 3$, $r = 2$, and $k = 18$. But we omit the details. We also note that we can compute $c^*(F)$ and $a(T, [F]_2^3)$ for a Hecke eigenform F in S_{16}^2 and some $T \in A_3^+$, but as far as we compute, there is no prime ideal satisfying the conditions in Theorem 3.1.

Example 3: The case $n = 3$, $r = 2$, and $k = 20$.
We have $\dim M_{20}^2 = 5$ and $\dim S_{20}^2 = 3$. We also note that $\dim S_{20}^1 = 1$ and $\dim S_{38}^2 = 2$. Let $\chi_{20}^{(1)}, \chi_{20}^{(2)}$, and $\chi_{20}^{(3)}$ be the Hecke eigenforms in S_{20}^2 defined in Kurokawa [22]. (See also, Skoruppa [43].) We modify them and put $\tilde{\chi}_{20}^{(i)} = \chi_{20}^{(i)}/2$ for $i = 1, 2, 3$. Then these form an orthogonal basis of S_{20}^2, and therefore $[\Delta_{20}]_1^2, \tilde{\chi}_{20}^{(1)}, \tilde{\chi}_{20}^{(2)}$, and $\tilde{\chi}_{20}^{(3)}$ form a basis of $\tilde{M}_{20}^2$. We note that $\tilde{\chi}_{20}^{(1)}$ and $\tilde{\chi}_{20}^{(2)}$ are the Saito-Kurokawa lifts of the normalized Hecke eigenforms in S_{38}^1. Put $\lambda_i = 48(-2025+\sqrt{D})$ and $48(-2025-\sqrt{D})$ with $D = 63737521$, and $K = \mathbf{Q}(\sqrt{D})$. Then λ_1, λ_2 satisfy the equation $X^2+194400X^2-137403408384 = 0$. Then $\lambda(T(2), \tilde{\chi}_{20}^{(i)}) = \lambda_i + 3\cdot 2^{18}$ for $i = 1, 2$, and $\lambda(T(2), \tilde{\chi}_{20}^{(3)}) = -2^8\cdot 3^2\cdot 5\cdot 73$. Furtheremore $\lambda(T(2), [\Delta_{20}]_1^2) = 456(1 + 2^{18})$. Thus

$$\begin{aligned}&\tilde{\Phi}_{T(2)}(X) = (X - 456(1 + 2^{18}))\\ &\cdot ((X - 3 \cdot 2^{18})^2 + 194400(X - 3 \cdot 2^{18}) - 137403408384)(X + 2^8 \cdot 3^2 \cdot 5 \cdot 73).\end{aligned}$$

We note that $\mathbf{Q}(\tilde{\chi}_{20}^{(i)}) = K$ for $i = 1, 2$, and $\mathbf{Q}(\tilde{\chi}_{20}^{(3)}) = \mathbf{Q}$. As for Fourier coefficients of these Hecke eigenforms, we have

$$a(T_0, \tilde{\chi}_{20}^{(i)}) = -5092 - \lambda_i/96 \ (i = 1, 2), \quad a(T_0, \tilde{\chi}_{20}^{(3)}) = 1,$$

and

$$a(T_1, \tilde{\chi}_{20}^{(i)}) = -10(4816 + \lambda_i/96) \ (i = 1, 2), \quad a(T_1, \tilde{\chi}_{20}^{(3)}) = 4.$$

Thus we have

$$N_{K/\mathbf{Q}}(a(T_0, \tilde{\chi}_{20}^{(i)})) = 2^2 \cdot 3^4 \cdot 5 \cdot 19 \cdot 23,$$

and

$$N_{K/\mathbf{Q}}(a(T_1, \tilde{\chi}_{20}^{(i)})) = -2^5 \cdot 3 \cdot 5^2 \cdot 23 \cdot 2659$$

for $i = 1, 2$. Then by using Mathematica, we compute

$$\tilde{\epsilon}_{20,2}^{(2,2)}(0, T_0, T_1) = \frac{-71291349782988999612052416421134370210790 40}{996291536301166998227},$$

$$\tilde{\epsilon}_{20,2}^{(2,2)}(1, T_0, T_1) = \frac{-1656012331833988565149518023826738738188002007040 0}{26926798278409918871},$$

$$\tilde{\epsilon}_{20,2}^{(2,2)}(2,T_0,T_1) = \frac{-1958314136249483542383671248105980953272860946852098048000}{26926798278409918871},$$

and

$$\tilde{\epsilon}_{20,2}^{(2,2)}(3,T_0,T_1) = \frac{-2333537090930830834209851678491260588936668700762500806867353600000}{26926798278409918871}.$$

Thus by Proposition 5.9 we can show that

$$N_{K/\mathbf{Q}}(Z(4,20)c^*(\tilde{\chi}_{20}^{(i)})a(T_0,\tilde{\chi}_{20}^{(i)})a(T_1,\tilde{\chi}_{20}^{(i)})) = \frac{-2^{36}\cdot 3^{20}\cdot 5^{15}\cdot 7^{8}\cdot 11^{3}\cdot 13^{3}\cdot 17^{3}\cdot 29\cdot 31\cdot 37\cdot 67\cdot 83\cdot 2659\cdot 43867^{2}\cdot 635893\cdot 701159}{181\cdot 349\cdot 1009\cdot 14581603\cdot 154210205991661} \tag{5.4}$$

for $i=1,2$, and

$$Z(4,20)c^*(\tilde{\chi}_{20}^{(3)})a(T_0,\tilde{\chi}_{20}^{(3)})a(T_1,\tilde{\chi}_{20}^{(3)}) = -\frac{2^{21}\cdot 3^{11}\cdot 5^{5}\cdot 7^{6}\cdot 11^{2}\cdot 13\cdot 17\cdot 199\cdot 691}{35059}.$$

Now we consider the primes 43867 and 691, which appear in the numerator of the above standard zeta values. We note that 43867 remains prime in K and 691 splits in K. Now we have

$$\tilde{\epsilon}_{20,2}^{(3,2)}(0,T_2,T_1) = \frac{-4227691337764919223557889580047123097190 40}{26926798278409918871},$$

$$\tilde{\epsilon}_{20,2}^{(3,2)}(1,T_2,T_1) = \frac{-30654706595736959052315755340992945935371337728 00}{26926798278409918871},$$

$$\tilde{\epsilon}_{20,2}^{(3,2)}(2,T_2,T_1) = \frac{-3449228179457029157086994569813071133141176119674798080 00}{26926798278409918871},$$

and

$$\tilde{\epsilon}_{20,2}^{(3,2)}(3,T_2,T_1) = \frac{-4111781665181543154455466944016069854288180326650225439329484 8000}{26926798278409918871}.$$

Thus we have

$$N_{K/\mathbf{Q}}(Z(5,20)a(T_1,\tilde{\chi}_{20}^{(i)})a(T_2,[\tilde{\chi}_{20}^{(i)}]_2^3)c^*(\tilde{\chi}_{20}^{(i)})) = \frac{-2^{41}\cdot 3^{22}\cdot 5^{12}\cdot 7^{7}\cdot 11^{2}\cdot 13\cdot 17^{2}\cdot 29\cdot 31\cdot 67\cdot 83\cdot 2659\cdot 635893\cdot 701159\cdot 93044315702749}{181\cdot 349\cdot 1009\cdot 14581603\cdot 154210205991661}$$

for $i = 1, 2$, and

$$Z(5,20)a(T_1, \tilde{\chi}_{20}^{(3)})a(T_2, [\tilde{\chi}_{20}^{(3)}]_2^3)c^*(\tilde{\chi}_{20}^{(3)}) = -\frac{2^{24} \cdot 3^{11}\dot{5}^6 \cdot 7^4 \cdot 11 \cdot 13 \cdot 17 \cdot 199 \cdot 1301}{35059}.$$

We note that $Z(4,20) = Z(5,20)$ and $\mathrm{ord}_{691}(Z(4,20)) = \mathrm{ord}_{43867}(Z(4,20)) = 0$. This implies

$$\mathrm{ord}_{43867}(c^*(\tilde{\chi}_{20}^{(i)})a(T_2, [\tilde{\chi}_{20}^{(i)}]_2^3)^2) = -1$$

for $i = 1, 2$ and

$$\mathrm{ord}_{691}(c^*(\tilde{\chi}_{20}^{(3)})a(T_2, [\tilde{\chi}_{20}^{(3)}]_2^3)^2) = -1.$$

The prime 43867 does not satisfy the condition (v) for $f = \tilde{\chi}_{20}^{(i)}$ with $i = 1$ or 2 since both $c^*(\tilde{\chi}_{20}^{(1)})$ and $c^*(\tilde{\chi}_{20}^{(2)})$ are divisible by 43867. Hence we consider only the prime 691. We check the conditions of Theorem 3.1 for $f = \chi_{20}^{(3)}$. As above, the conditions (i)-(iii) are satisfied. We easily see that $\zeta(-19)\zeta(-37)\zeta(-35)\zeta(-33)$ is coprime with 691. By DUMMIGAN [7],

$$\mathrm{ord}_{691}(c^*(\Delta_{20})) = 0.$$

By [30], p. 124, we have

$$\mathfrak{A}(\tilde{\chi}_{20}^{(i)}) = 2^{14} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11(\sqrt{D}) \quad \text{in} \quad \mathcal{O}_K \quad \text{for} \quad i = 1, 2 \qquad (5.5)$$

with $D = 181 \cdot 349 \cdot 1009$ and $\mathfrak{A}(\tilde{\chi}_{20}^{(3)}) = (\kappa(\tilde{\chi}_{20}^{(3)})) = (2^9 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11)$ in $\mathbf{Z}$. For any prime ideal $\mathfrak{q}$ in K lying over 691, we have $\mathrm{ord}_{\mathfrak{q}}(c^*(\tilde{\chi}_{20}^{(i)})) \geq 0$ by Lemma 4.4 and (5.5). Hence from (5.4) we have $\mathrm{ord}_{\mathfrak{q}}(c^*(\tilde{\chi}_{20}^{(i)})) = 0$ for $i = 1, 2$. Since the common denominator of the Fourier coefficients of $E_{20}^{(1)}$ (resp. $E_{20}^{(2)}$, $[\Delta_{20}]_1^2$) is $283 \cdot 617$ (resp. $283 \cdot 617 \cdot 154210205991661$, $11 \cdot 71^2$), we have

$$\mu_{20}(1) = 283 \cdot 617, \quad \mu_{20}(2) = 11 \cdot 71^2 \cdot 283 \cdot 617 \cdot 154210205991661.$$

Hence by Theorem 3.1 there exists a Hecke eigenform $G \in S_{20}^3$ such that

$$N_{\mathbf{Q}(G)/\mathbf{Q}}(\lambda(T,G) - \lambda(T, [\tilde{\chi}_{20}^{(3)}]_2^3)) \equiv 0 \pmod{691} \quad \text{for all} \quad T \in \mathcal{H}_{\mathbf{Z}}^3.$$

It seems difficult to construct this G concretely since $\dim S_{20}^3 = 6$ by TSUYUMINE [45] and RUNGE [39].

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