Rotationally symmetric 1-harmonic flows from $D^2$ to $S^2$.

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Given a domain $\Omega$ in $\mathbb{R}^N$, a real constant $p \geq 1$, a smoothly embedded compact submanifold (without boundary) $M$ of $\mathbb{R}^{N+1}$ and a mapping $u : \Omega \to M$, let

$$E_p(u) := \frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx \quad (1)$$

(if $p = 1$, $E_1$ is the total variation of $u$). The so-called $p$-harmonic flow for $E_p$ (i.e. the $L^2$-gradient flow for $E_p$ with constraint of values in $M$) is given by

$$u_t = -\pi_u(-\text{div} \left( |\nabla u|^p \nabla u \right)), \quad (2)$$

where $\pi_u$ denotes the orthogonal projection of $\mathbb{R}^{N+1}$ onto the tangent space $T_u M$ of $M$ at $u \in M$. We consider the case in which $N = 2$, $\Omega$ is the unit disk ($\Omega = D^2 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$) and $M$ is the unit sphere ($M = S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$). In this case (2) may be explicitly written as

$$u_t = \text{div} \left( |\nabla u|^{p-2} \nabla u \right) + u |\nabla u|^p. \quad (3)$$

Equation (3) arises in various contexts – multi-grain problems [10], theory of liquid crystals [9], ferromagnetism [3] and image processing [11] – as a prototype of often quite complicated problems containing reaction-diffusion systems for the evolution of director fields. The Dirichlet problem for (3) amounts to impose the initial-boundary condition

$$u = u_0 \quad \text{on} \quad \partial \left( (0, \infty) \times \Omega \right). \quad (4)$$

For $p = 1$, (3) may be seen as a constrained gradient system of total variation:

$$u_t = \text{div} \left( \frac{Du}{|Du|} \right) + u |Du|. \quad (5)$$

Equation (5) was proposed in [12, 13] as a tool to denoise color images by smoothing the chromaticity data while the contrast is being preserved. The scalar and unconstrained version of (5) – the so-called total variation flow –

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corresponds to the gray image denoising and is by now well understood (see the monograph [1] and the references therein). However, very few is known for the vectorial case (see [5, 6, 7, 8]).

In this talk I will make an overview of what is known for the problem (3)-(4) (specially for the $p = 2$ case) and I will present some new results obtained in collaboration with R. Dal Passo and L. Giacomelli ([2, 4]) for the special case of rotationally symmetric solutions to (5)-(4).

References


