The second term of the semi-classical asymptotic expansion for Feynman path integrals with integrand of polynomial growth.

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July 18, 2009

This is a short summary of the joint work with Naoto Kumano-go published in The Journal of Math. Soc. Japan [11].

Feynman path integral was invented by Feynman to quantize the motion of a particle moving in a potential field. It gives the integral kernel of evolution operator, which is the operator describing time evolution of wave function of a particle moving in a potential field. Contrary to Schrödinger equation, it does not use Hamiltonian but uses Lagrangian function.

Let $V(t,x)$ be a time dependent potential on the configuration space $\mathbf{R}^d$. Then the Lagrangian is

$$L(t,\dot{x},x) = \frac{1}{2}|\dot{x}|^2 - V(t,x).$$

A path γ is a continuous or sufficiently smooth map from the time interval $[s,s']$ to $\mathbf{R}^d$. The action $S(\gamma)$ of a path γ is the functional

$$S(\gamma) = \int_s^{s'} L(t, \frac{d}{dt}\gamma(t), \gamma(t))dt. \qquad (2)$$

In [4] Feynman introduced the notion of an integral over the path space Ω, which is called Feynman path integral and is often denoted by

$$\int_\Omega e^{i\nu S(\gamma)} F(\gamma)\mathcal{D}[\gamma], \tag{3}$$

where $\nu = 2\pi h^{-1}$ with Planck's constant h. It was expected that Feynman path integral could have been defined as a measure theoretic integral if a suitable complex measure on the path space had been defined. However, Cameron [2] proved that this is not the case. (cf. also Johnson & Lapidus [13].)

Feynman himself gave the meaning to (3) as the limit of integrals over finite dimensional spaces. We call this method time slicing approximation method. Before we explain it in more detail, we make some preparation.

We assume that $V(t,x)$ is continuous in t and smooth in x and that it satisfies the following estimate: For any non-negative integer m there exists a non-negative constant v_m such that

$$\max_{|\alpha|=m} \sup_{(t,x)\in[0,T]\times\mathbf{R}^d} |\partial_x^\alpha V(t,x)| \le v_m(1+|x|)^{\max\{2-m,0\}}. \tag{4}$$

Our assumption is close to that of Pauli [3].

Let $[s,s']$ be an interval of time. A path γ is called classical if it is a solution to the Euler equation

$$\frac{d^2}{dt^2}\gamma(t) + (\nabla V)(t,\gamma(t)) = 0 \quad \text{for } s < t < s'. \tag{5}$$

Here and hereafter ∇ stands for the nabla operator in the space $\mathbf{R}^d$.

For arbitrary pair of points $x, y \in \mathbf{R}^d$ there exists one and only one classical path γ which satisfies the boundary condition

$$\gamma(s) = y, \quad \gamma(s') = x \tag{6}$$

if $|s'-s| \le \mu$ with sufficiently small μ, say for instance,

$$\frac{\mu^2 d v_2}{8} < 1. \tag{7}$$

In this case the action $S(\gamma)$ of γ is a function of (s', s, x, y) and is denoted by $S(s', s, x, y)$, i.e.,

$$S(s',s,x,y) = \int_s^{s'} L(t, \frac{d}{dt}\gamma(t), \gamma(t))\, dt. \tag{8}$$

Now we explain time slicing approximation method. Let

$$\varDelta : 0 = T_0 < T_1 < \cdots < T_J < T_{J+1} = T \tag{9}$$

be a division of the interval $[0, T]$. Then we set $t_j = T_j - T_{j-1}$ and define the mesh $|\varDelta|$ of the division $\varDelta$ by $|\varDelta| = \max_j\{t_j\}$. We always assume that

$$|\varDelta| \le \mu. \tag{10}$$

Let

$$x_j \in \mathbf{R}^d, \quad j = 0, 1, \ldots, J, J+1, \tag{11}$$

be arbitrary $J+2$ points of the configuration space $\mathbf{R}^d$. The piecewise classical path $\gamma_\varDelta$ with vertices $(x_{J+1}, x_J, \ldots, x_1, x_0) \in \mathbf{R}^{d(J+2)}$ is the broken path that satisfies the Euler equation

$$\frac{d^2}{dt^2}\gamma_\varDelta(t) + (\nabla V)(t, \gamma_\varDelta(t)) = 0 \tag{12}$$

for $T_{j-1} < t < T_j (j = 1, 2, \ldots, J+1)$ and boundary conditions

$$\gamma_\varDelta(T_j) = x_j, \quad j = 0, 1, \ldots, J, J+1, \tag{13}$$

where $x = x_{J+1}$ and $y = x_0$. When we emphasize the fact that this path $\gamma_\varDelta$ depends on $(x_{J+1}, x_J, \ldots, x_1, x_0)$, we denote it by $\gamma_\varDelta(x_{J+1}, x_J, \ldots, x_1, x_0)$ or $\gamma_\varDelta(t; x_{J+1}, x_J, \ldots, x_1, x_0)$, where t is the time variable.

Let $F(\gamma)$ be a functional defined for paths γ. Its value $F(\gamma_\varDelta)$ at $\gamma_\varDelta$ can be written as a function $F_\varDelta(x_{J+1}, x_J, \ldots, x_1, x_0)$ of $(x_{J+1}, x_J, \ldots, x_1, x_0)$. For example the action functional $S(\gamma_\varDelta)$ of $\gamma_\varDelta$ is given by

$$\begin{aligned} S_\varDelta(x_{J+1}, x_J, \ldots, x_1, x_0) = S(\gamma_\varDelta) &= \int_0^T L(t, \frac{d}{dt}\gamma_\varDelta(t), \gamma_\varDelta(t))dt \\ &= \sum_{j=1}^{J+1} S_j(x_j, x_{j-1}), \end{aligned} \tag{14}$$

where we used the abbreviation

$$S_j(x_j, x_{j-1}) = S(T_j, T_{j-1}, x_j, x_{j-1}) = \int_{T_{j-1}}^{T_j} L(t, \frac{d}{dt}\gamma_\varDelta(t), \gamma_\varDelta(t))\, dt. \tag{15}$$

A piecewise classical time slicing approximation to Feynman path integral (3) with the integrand $F(\gamma)$ is an oscillatory integral

$$
\begin{aligned}
& I[F_\Delta](\Delta; x, y) \qquad\qquad (16)\\
=& \prod_{j=1}^{J+1} \left(\frac{\nu}{2\pi i t_j}\right)^{d/2} \int_{\mathbf{R}^{dJ}} e^{i\nu S(\gamma_\Delta)} F(\gamma_\Delta) \prod_{j=1}^{J} dx_j \\
=& \prod_{j=1}^{J+1} \left(\frac{\nu}{2\pi i t_j}\right)^{d/2} \int_{\mathbf{R}^{dJ}} e^{i\nu S_\Delta(x_{J+1}, x_J, \ldots, x_1, x_0)} F_\Delta(x_{J+1}, x_J, \ldots, x_1, x_0) \prod_{j=1}^{J} dx_j,
\end{aligned}
$$

where $x_{J+1} = x$ and $x_0 = y$.

Feynman's definition of path integral (3) is

$$
\int_\Omega e^{i\nu S(\gamma)} F(\gamma) \mathcal{D}[\gamma] = \lim_{|\Delta| \to 0} I[F_\Delta](\Delta; x, y), \qquad (17)
$$

if the limit on the right hand side exists. See Feynman [4].

One can ask questions:

1. Does this limit exists ?
2. What does this limit looks like if it exists ?

1 Existance of the limit.

In the case $F(\gamma) \equiv 1$ existence of the limit in (17) was proved by [5], [6]. [7],[16] and the Feynman path integral is in fact the fundamental solution of Schrödinger equation as Feynman expected.

In the case $F(\gamma) \not\equiv$ *constant* we give here a sufficient condition for the limit in (17) to converge. To explain our assumptions we make some preparation. The set $\Gamma(\Delta)$ of all piecewise classical paths associated with the division Δ forms a differentiable manifold of dimension $d(J+2)$. For a pair of divisions Δ' and Δ we use symbol $\Delta \prec \Delta'$ if Δ' is a refinement of Δ. The set Γ of all piecewise classical paths is the inductive limit of $\{\Gamma(\Delta), \prec\}$,i.e., $\Gamma = \lim_{\to} \Gamma(\Delta)$. Γ is a dense subset of the Sobolev space $H^1([0,T]; \mathbf{R}^d)$ of order 1 with values in $\mathbf{R}^d$ and hence it is also dense in the space $C([0,T]; \mathbf{R}^d)$ of all continuous paths. Let $\gamma_\Delta \in \Gamma(\Delta)$. Then the tangent space $T_{\gamma_\Delta}\Gamma$ to Γ at γ_Δ is the inductive limit $\lim_{\to} T_{\gamma_\Delta}\Gamma(\Delta')$, which is a dense linear subspace of the Sobolev space $H^1([0,T]; \mathbf{R}^d)$.

Let $F(\gamma)$ be a functional defined on Γ. We denote its Fréchet differential at $\gamma \in \Gamma$ by DF_γ if it exists. And $DF_\gamma[\zeta]$ stands for its value at the tangent vector $\zeta \in T_\gamma\Gamma$. For any integer $n > 0$ and for $\zeta_j \in T_\gamma\Gamma \quad (j = 1, 2, \dots, n)$, we denote by $D^n F_\gamma[\zeta_1 \otimes \zeta_2 \otimes \cdots \otimes \zeta_n]$, the symmetric n-linear form on the tangent space arising from the n-th jet modulo $n-1$-th jet of F at γ.

Our assumptions are the followings:.

Assumption 1 *Let $m \geq 0$. For any non-negative integer K there exist positive constants A_K and X_K such that for any division Δ of the form (9) and any integer $n_j \quad (0 \leq j \leq J+1)$ with $0 \leq n_j \leq K$*

$$\left| D^{\sum_{j=0}^{J+1} n_j} F_{\gamma_\Delta}[\otimes_{j=0}^{J+1} \otimes_{k=1}^{n_j} \zeta_{j,k}] \right| \leq A_K X_K^{J+2} (1 + \|\gamma_\Delta\| + \||\gamma_\Delta\||)^m \prod_{j=0}^{J+1} \prod_{k=1}^{n_j} \|\zeta_{j,k}\|, \tag{18}$$

as far as $\zeta_{j,k} \in T_{\gamma_\Delta}\Gamma$ satisfies

$$\mathrm{supp}\zeta_{j,k} \subset \begin{cases} [0, T_1] & \text{if} \quad j = 0 \\ [T_{j-1}, T_{j+1}] & \text{if} \quad 1 \leq j \leq J \\ [T_J, T_{J+1}] & \text{if} \quad j = J+1, \end{cases} \tag{19}$$

where $\|\zeta\| = \max_{0 \leq t \leq T} |\zeta(t)|$ and $\||\gamma_\Delta\|| =$ total variation of γ_Δ.

Assumption 2 *[15] [9]. There exists a positive bounded Borel measure ρ on $[0, T]$ such that with the same A_K, X_K as above*

$$\left| D^{1+\sum_{j=0}^{J+1} n_j} F_{\gamma_\Delta}[\eta \otimes \otimes_{j=0}^{J+1} \otimes_{k=1}^{n_j} \zeta_{j,k}] \right| \leq A_K X_K^{J+2} (1 + \|\gamma_\Delta\| + \||\gamma_\Delta\||)^m \int_{[0,T]} |\eta(t)| \rho(dt) \prod_{j=0}^{J+1} \prod_{k=1}^{n_j} \|\zeta_{j,k}\|, \tag{20}$$

for any division Δ, integer $n_j \leq K$ and $\zeta_{j,k}$ which are the same as in Assumption 1. And η is an arbitrary element of $T_{\gamma_\Delta}\Gamma$.

Example:

Let $m \geq 0$, $f(t, x)$ be a smooth function and ρ be a function of bounded variation. Assume that for any multi-index α there exists a constant a constant C_α such that

$$|\partial_x^\alpha f(t, x)| \leq C_\alpha (1 + |x|)^m. \tag{21}$$

Then the Stieltjes integral

$$F(\gamma) = \int_0^T f(t, \gamma(t))\, d\rho. \tag{22}$$

is an example satisfying our assumptions.

We have the following

Theorem 1 *Assume that the integrand $F(\gamma)$ satisfies Assumption 1 and Assumption 2 above and T is so small that $|T| \le \mu$, Then the limit of the right hand side of (17) converges compact-uniformly with respect to $(x, y) \in \mathbf{R}^{2d}$.*

We remark that Feynman [4] used also piecewise linear paths in place of piecewise classical paths. N. Kumano-go [15] proved the limit in (17) exists in the case of more general class of functional F using piecewise linear paths in place of piecewise classical paths.

We shall make more precise statement. In what follows we always assume that $0 < T \le \mu$. For any fixed $(x, y) \in \mathbf{R}^{2d}$ the action $S(\gamma)$ has a unique critical point γ^*, which is the unique classical path starting y at time 0 and reaching x at time T. The critical point is non-degenerate. Similarly, if $0 < T \le \mu$, the function $S_\Delta(x_{J+1}, x_J, \dots, x_1, x_0)$ of $(x_J, \dots, x_1)$ has only one critical point, which is non-degenerate. Regarding ν as a parameter satisfying $\nu \ge 1$, we can apply stationary phase method to (16). Stationary phase method says that $I[F_\Delta](\Delta; x, y)$ is of the following form:

$$\begin{aligned} & I[F_\Delta](\Delta; x, y) \\ &= \left(\frac{\nu}{2\pi i T}\right)^{d/2} D(\Delta; x, y)^{-1/2} e^{i\nu S(\gamma^*)} \Big(F(\gamma^*) + \nu^{-1} R_\Delta[F_\Delta](\nu, x, y) \Big). \end{aligned} \tag{23}$$

Here we used the following symbol

$$D(\Delta; x, y) = \left(\frac{t_{J+1} t_J \dots t_1}{T}\right)^d \det \mathrm{Hess} S(\gamma_\Delta), \tag{24}$$

where $\mathrm{Hess} S(\gamma_\Delta)$ denotes the Hessian of $S(\gamma_\Delta)$ with respect to $(x_J, x_{J-1}, \dots, x_1)$ evaluated at the critical point.

We know (cf. [7]) that $D(T, x, y) = \lim_{|\Delta| \to 0} D(\Delta; x, y)$ exists.

The function $T^{-d} D(T, x, y)$ coincides with the famous Morette-VanVleck determinant (cf. [7]).

Theorem 2 *Under the same assumption as in Theorem 1 we can write the limit* $\lim_{|\Delta|\to 0} I[F_\Delta](\Delta; x, y)$ *in the following way:*

$$\int_\Omega e^{i\nu S(\gamma)} F(\gamma)\mathcal{D}[\gamma] = \lim_{|\Delta|\to 0} I[F_\Delta](\Delta; x, y) \tag{25}$$
$$= \Big(\frac{\nu}{2\pi i T}\Big)^{d/2} D(T, x, y)^{-1/2} e^{i\nu S(\gamma^*)}\Big(F(\gamma^*) + \nu^{-1} R[F](\nu, x, y)\Big).$$

For any non-negative integer K there exist a positive constant C_K and a non-negative integer $M(K)$ independent of ν such that

$$|\partial_x^\alpha \partial_y^\beta R[F](\nu, x, y)| \le C_K A_{M(K)} T(T + \rho([0, T]))(1 + |x| + |y|)^m. \tag{26}$$

2 The second term of semi-classical asymptotic expansion

Although ν is a constant in Physics, it is offten treated as a large positive parameter. It is expected that the Newton's classical mechanics is the limit of $\nu \to \infty$ (semi-classical limit) of quantum mechanics. Feynman discussed the asymptotic behaviour of Feynman path integral (3) as $\nu \to \infty$. i.e., the semiclassical asymptotic behaviour of Feynman path integrals. And he explained that the asymptotic behaviour of (3) is a result of "stationary phase method on path space". This is very interesting and challenging idea. **Can one make it mathematically rigorous?** Here is our partial answer.

It is expected that the following semi-classical asymptotic expansion holds;

$$\int_\Omega e^{i\nu S(\gamma)} F(\gamma)\mathcal{D}[\gamma] \tag{27}$$
$$= \Big(\frac{\nu}{2\pi i T}\Big)^{d/2} D(T, x, y)^{-1/2} e^{i\nu S(\gamma^*)}\Big(A_0 + \nu^{-1} A_1 + O(\nu^{-2})\Big)$$

as $\nu \to \infty$.

Theorem 2 implies $A_0 = F(\gamma^*)$. What is the next term A_1 ?

In the case $F(\gamma) \equiv 1$ assuming the existence of expansion, Birkhoff gave the answer [1]. In fact, he gave even higher order terms of asymptotic expansion. However, if $F(\gamma) \neq$ *constant*, then his method does not apply.

We write down the second term A_1 of (27) for general $F(\gamma)$ and prove that the asymptotic expression (27) actually holds. For this purpose we introduce a new piece-wise classical path. Let ϵ be an arbitrary small positive number. And $\Delta(t,\epsilon)$ be the division

$$\Delta(t,\epsilon) : 0 = T_0 < t < t+\epsilon < T. \tag{28}$$

Let z be an arbitrary point in $\mathbf{R}^d$. We abbreviate the piecewise classical path $\gamma_{\Delta(t,\epsilon)}(s; x, \gamma^*(t+\epsilon), z, y)$ associated with the division $\Delta(t,\epsilon)$ by $\gamma_{\{t,\epsilon\}}(s,z)$, i.e., $\gamma_{\{t,\epsilon\}}(s,z)$ is the piecewise classical path which satisfies conditions:

$$\gamma_{\{t,\epsilon\}}(0,z) = y, \quad \gamma_{\{t,\epsilon\}}(t,z) = z, \quad \gamma_{\{t,\epsilon\}}(t+\epsilon,z) = \gamma^*(t+\epsilon), \quad \gamma_{\{t,\epsilon\}}(T,z) = x. \tag{29}$$

It is clear that $\gamma_{\{t,\epsilon\}}(s,z)$ coincides with $\gamma^*(s)$ for $t+\epsilon \le s \le T$ independently of z. Therefore, $\partial_z \gamma_{\{t,\epsilon\}}(s,z) = 0$ for $t+\epsilon \le s \le T$.

Lemma 1 *Under Assumption 1 and Assumption 2 we have bounded convergence of the limit*

$$q(t) = \lim_{\epsilon \to +0} \Big[\Delta_z (D(t,z,y)^{-1/2} F(\gamma_{\{t,\epsilon\}}(*,z)))\Big|_{z=\gamma^*(t)} \Big], \tag{30}$$

where Δ_z stands for the Laplacian with respect to z.

Theorem 3 *In addition to Assumptions 1 and 2 we further assume that the function $q(t)$ of Lemma 1 is Riemannian integrable over $[0,T]$. Set*

$$A_1 = \frac{i}{2} \int_0^T D(t, \gamma^*(t), y)^{1/2} q(t) dt. \tag{31}$$

Then, there holds the asymptotic formula, as $\nu \to \infty$,

$$\int_\Omega e^{i\nu S(\gamma)} F(\gamma) \mathcal{D}[\gamma] \tag{32}$$

$$= \Big(\frac{\nu}{2\pi i T}\Big)^{d/2} D(T,x,y)^{-1/2} e^{i\nu S(\gamma^*)} \Big(A_0 + \nu^{-1} A_1 + \nu^{-2} r(\nu,x,y)\Big),$$

where for any α, β the remainder term $r(\nu,x,y)$ satisfies estimate

$$|\partial^\alpha \partial^\beta r(\nu,x,y)| \le C_{\alpha,\beta} T^2 (1+|x|+|y|)^m. \tag{33}$$

We can calculate $q(t)$ in more detail for simple functionals $F(\gamma)$ of example (22).

Since our method is based on "stationary phase method of oscillatory integrals over a space of large dimension [8] and [10] ", it is completely different from Birkhoff's method, it may be interesting to see that this formula coincides with Birkhoff's result in the case of $F(\gamma) \equiv 1$.

Remark 1 *In this note the Lagrangian has no vector potential. Kitada-Kumano-go[14], Yajima [17] and Tshuchida-Fujiwara [12] discussed the case of Lagrangian with non zero vector potential. They proved that the limit (17) exists and the limit is the fundamental solution of Schrödinger equation if* $F(\gamma) \equiv 1$*. However we do not know whether the limit (17) exists or not if* $F(\gamma) \neq$ *constant and Lagrangian has non-zero vector potential.*

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