The effect of boundary conditions to the dynamics of pulse solutions for reaction-diffusion systems

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Abstract.
We consider pulse-like localized solutions for reaction-diffusion systems on a half line and impose various boundary conditions at one end of the half line. It is shown that the movement of a pulse solution with the homogeneous Neumann boundary condition is completely opposite from that with the Dirichlet boundary condition. As general cases, Robin type bounary conditions are also considered. Introducing one parameter connecting the Neumann and the Dirichlet boundary conditons, we will show the existence of stationary solutions which have been known so far.

key words: Reaction-diffusion systems; Effect of boundary conditions; pulse solutions

Reaction diffusion systems have been widely treated to describe and study spatio-tempral patterns in dissipative systems. Among them, many reaction-diffusion systems which possess various types of localized solutions such as pulse-like localized solutions and front-like ones have been proposed while we omit the detail and merely refer to books ([6], [5]). To understand the dynamics of such solutions, reaction-difusion systems have been studied under various situations such as one or higher dimensional spaces, bounded or unbouded domains, and the Neumann boundary conditions or the Dirichlet ones according to considered problems. In fact, the dynamics solutions drastically change depending on the considered situations. As one example, let us consider the effect of boundary conditions for the Allen-Cahn equation:

$$u_t = \varepsilon^2 u_{xx} + f(u),\ t > 0,\ x \in I \subset \boldsymbol{R} \tag{1}$$

with the Neumann boundary conditions

$$u_x = 0,\ x \in \partial I, \tag{2}$$

or

$$u = \pm 1,\ x \in \partial I, \tag{3}$$

where $f(u) = \frac{1}{2}u(1-u^2)$ and $\varepsilon > 0$ is sufficiently small. For the simplicity, let $I = \boldsymbol{R}_+ := [0, \infty)$ and we impose the boundary conditions only at $x = 0$.

As known well, (1) on the whole line $\boldsymbol{R}$ has a stable standing front solution, say $\Phi(x) := \tanh x/2$ satisfying $\Phi(\pm\infty) = \pm 1$.

If we consider (1) on the half line $\boldsymbol{R}_+$ with the Neumann boundary condition (2) and the initial data is close to $\Phi(x - l_0)$ for $l_0 >> 1$, then it was shown by [1], [3] that the solution $u(t, x)$ remains close to $\Phi(x - l(t))$ and the movement is essentially governed by $\dot{l} = -12\varepsilon e^{-2l}$. That is, a front like localized solution approaches the boundary $x = 0$.

On the other hand, when we impose the Dirichlet boundary condition

$$u = -1, \ x = 0 \tag{4}$$

under the same situations as the previous one except the boundary condition, it is observed as in Fig 0.1 that the solution comes away from the boundary $x = 0$.

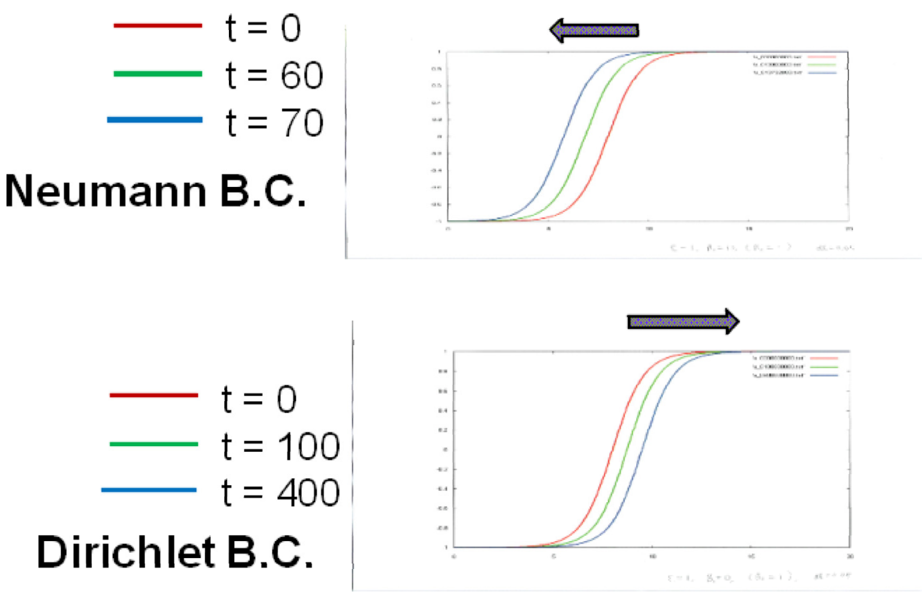

図 0.1: Movements of front solutions of (1) on $\boldsymbol{R}_+$.

In this talk, we consider farely general types of reaction-diffusion systems

$$\boldsymbol{u}_t = D\boldsymbol{u}_{xx} + F(\boldsymbol{u}), \ t > 0, \ x \in \boldsymbol{R}_+, \tag{5}$$

where $\boldsymbol{u} \in \boldsymbol{R}^N$, $D := diag\{d_1, \cdots, d_N)$ and $F : \boldsymbol{R}^N \to \boldsymbol{R}^N$ is a sufficiently smooth function.

First we consider the problem (5) on $\boldsymbol{R}$

$$\boldsymbol{u}_t = D\boldsymbol{u}_{xx} + F(\boldsymbol{u}), \ t > 0, \ x \in \boldsymbol{R}, \tag{6}$$

and assume several conditions for (6) as follows:

A1) There exists a stable symmetric stationary solution, say $S(x)$ satisfying $S(x) \to e^{-\alpha|x|}\boldsymbol{a}$ as $|x| \to \infty$ for $\alpha > 0$ and $\boldsymbol{a} \in \boldsymbol{R}^N$.

Let $L := \partial_{xx} + F'(S(x))$, the linearized operator of (6) with respect to $S(x)$.

A2) The spectral set $\sigma(L)$ of L is given $\sigma(L) = \sigma_0 \cup \sigma_1$, where $\sigma_0 := \{0\}$ and $\sigma_1 \subset \{Re\lambda < -\gamma_0\}$ for $\gamma_0 > 0$. Moreover, 0 is a simple eigenvalue of L.

Then there exists eigenfunction $\phi^*(x)$ of the adjoint operator L^* of L satisfying $L^*\phi^* = 0$ and $\phi^*(x) \to e^{-\alpha x}\boldsymbol{a}^*$ as $x \to +\infty$ for $\boldsymbol{a}^* \in \boldsymbol{R}^N$. Note that we can take $\phi^*(x)$ as an odd function and by the normalization $\langle S_x, \phi^* \rangle_{L^2} = 1$, $\phi^*(x)$ is uniquely determined.

Next coming back the original problem (5) on the half line $\boldsymbol{R}_+$. We impose the boundary condition

$$\boldsymbol{u}_x = \beta \boldsymbol{u}, \ x = 0. \tag{7}$$

Then we have

Theorem 1 *Assume A1) and A2). If the initial data $\boldsymbol{u}(0,x)$ is sufficiently close to $S(x-l_0)$ for $l_0 >> 1$, then the solution $\boldsymbol{u}(t,x)$ of (5) remains close to*

$$\boldsymbol{u}(t,x) = S(x - l(t)) + O(e^{-\alpha l(t)})$$

as long as $l(t) > l^*$ for $l^* >> 1$. $l(t)$ satisfies

$$\frac{dl}{dt} = \frac{2\alpha(\alpha - \beta)}{\alpha + \beta} M_0 e^{-2\alpha l}(1 + O(e^{-\alpha l})),$$

where $M_0 := \langle D\boldsymbol{a}, \boldsymbol{a}^* \rangle$.

In this talk, we will mention more precise analysis about the problems.

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