

# Uniqueness of positive solutions to semilinear elliptic equations with double power nonlinearities

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## Abstract

We consider the problem

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \mathbb{R}^n, \\ \lim_{|x| \rightarrow \infty} u(x) = 0, \end{cases} \quad (1)$$

where

$$f(u) = -\omega u + u^p - u^q, \quad \omega > 0, \quad q > p > 1.$$

It is known that a positive solution to (1) exists if and only if  $F(u) := \int_0^u f(s)ds > 0$  for some  $u > 0$ . Moreover, Ouyang and Shi in 1998 found that the solution is unique if  $f$  satisfies furthermore the condition that  $\tilde{f}(u) := (uf'(u))'f(u) - uf'(u)^2 < 0$  for any  $u > 0$ . In the present paper we remark that this additional condition is unnecessary.

## 1 Introduction

We shall consider a boundary value problem

$$\begin{cases} u_{rr} + \frac{n-1}{r}u_r + f(u) = 0 & \text{for } r > 0, \\ u_r(0) = 0, \\ \lim_{r \rightarrow \infty} u(r) = 0, \end{cases} \quad (2)$$

where  $n \in \mathbb{N}$  and

$$f(u) = -\omega u + u^p - u^q, \quad \omega > 0, \quad q > p > 1.$$

The above problem arises in the study of

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \mathbb{R}^n, \\ \lim_{|x| \rightarrow \infty} u(x) = 0. \end{cases} \quad (3)$$

Indeed, the classical work of Gidas, Ni and Nirenberg [4, 5] tells us that any positive solution to (3) is radially symmetric. On the other hand, for a solution  $u(r)$  of (2),  $v(x) := u(|x|)$  is a solution to (3).

The condition to assure the existence of positive solutions to (3) (and so (2)) was given by Berestycki and Lions [1] and Berestycki, Lions and Peletier [2]:

**Proposition 1.** *A positive solution to (2) exists if and only if*

$$F(u) := \int_0^u f(s)ds > 0, \quad \text{for some } u > 0. \quad (4)$$

Uniqueness of positive solutions to (2) had long remained unknown. Finally in 1998 Ouyang and Shi [9] proved uniqueness for (2) with  $f$  satisfying the additional condition (See also Kwong and Zhang [7]):

**Proposition 2.** *If  $f$  satisfies furthermore the following condition, then the positive solution is unique;*

$$\tilde{f}(u) := (uf'(u))'f(u) - uf'(u)^2 < 0, \quad \text{for any } u > 0. \quad (5)$$

Following is the main result of the present paper:

**Theorem 1.** *If the nonlinearity  $f$  satisfies the existence condition (4), then the uniqueness condition (5) is automatically fulfilled.*

This paper is organized as follows. In section 2 we give a straightforward proof of the theorem. In section 3 we give an alternative proof of the theorem, in which an interesting technical lemma is used. In section 4 we explain the technical lemma.

## 2 Proof of Theorem 1.

**Lemma 1.** *The existence condition (4) is equivalent to*

$$\omega < \omega_{p,q},$$

where

$$\omega_{p,q} = \frac{2(q-p)}{(p+1)(q-1)} \left[ \frac{(p-1)(q+1)}{(p+1)(q-1)} \right]^{\frac{p-1}{q-p}}.$$

(See Ouyang and Shi [9] and the appendix of Fukuizumi [3].)

**Lemma 2.** *The uniqueness condition (5) is equivalent to*

$$\omega < \eta_{p,q},$$

where

$$\eta_{p,q} = \frac{q-p}{q-1} \left[ \frac{p-1}{q-1} \right]^{\frac{p-1}{q-p}}.$$

The proofs of these Lemmas are nothing but straightforward calculation and shall be omitted.

*Proof of Theorem 1.* It is apparent that

$$0 < \omega_{p,q} < \eta_{p,q},$$

which asserts the theorem. □

### 3 Alternative Proof of Theorem 1.

The following is the key lemma. The proof of this lemma is given in the next section as a corollary of more general statement.

**Lemma 3.** *The existence condition (4) is equivalent to the following condition;*

$$\tilde{F}(u) = (uf(u))'F(u) - uf(u)^2 < 0, \quad \text{for any } u > 0. \quad (6)$$

*The uniqueness condition (5) is equivalent to the following condition;*

$$f(u) > 0, \quad \text{for some } u > 0. \quad (7)$$

This Lemma means that the relation between  $f$  and  $\tilde{f}$  is parallel to the relation between  $F$  and  $\tilde{F}$ .

*Alternative Proof of Theorem 1.* From Lemma 3 it is enough to show that if  $F$  has positive parts (i.e. the existence condition (4)) then  $f$  has positive parts (i.e. the condition (7)). The contraposition of this statement is clear by the monotonicity of the integral.  $\square$

### 4 Classification of double power nonlinear functions

To state the main result of this section, from now on we mean a more general function

$$f(u) = -au^p + bu^q - cu^r, \quad \text{for } u > 0, \quad (8)$$

where  $a, b, c > 0$  and  $p < q < r$ , by the same notation  $f$ . We also steal the former notation

$$\tilde{f}(u) = (uf'(u))'f(u) - uf'(u)^2. \quad (9)$$

Following is the statement.

**Theorem 2.** *There only can occur the following three cases;*

$$(a) \quad a < b \frac{r-q}{r-p} \left[ \frac{b(q-p)}{c(r-p)} \right]^{\frac{q-p}{r-q}} \iff f \text{ has positive parts} \iff \tilde{f} \text{ remains negative.}$$

$$(b) \quad a = b \frac{r-q}{r-p} \left[ \frac{b(q-p)}{c(r-p)} \right]^{\frac{q-p}{r-q}} \iff f \text{ has just one zero} \iff \tilde{f} \text{ has just one zero.}$$

$$(c) \quad a > b \frac{r-q}{r-p} \left[ \frac{b(q-p)}{c(r-p)} \right]^{\frac{q-p}{r-q}} \iff f \text{ remains negative} \iff \tilde{f} \text{ has positive parts.}$$

*Proof.* The statement with respect to  $f$  is trivial. We obtain from the definition (9) that

$$\tilde{f} = -ab(q-p)^2u^{q+p-1} + ca(r-p)^2u^{r+p-1} - bc(r-q)^2u^{r+q-1}.$$

This is in the form of (8) and use the result with respect to  $f$ . □

At last we give the proof of Lemma 3, which concludes this paper.

*Proof of Lemma 3.* For the first part, we consider

$$F(u) = -\frac{\omega}{2}u^2 + \frac{u^{p+1}}{p+1} - \frac{u^{q+1}}{q+1}, \quad \omega > 0, \quad q > p > 1,$$

which is in the form of (8) and use the result (a) of Theorem 4.

For the second part, we consider

$$f(u) = -\omega u + u^p - u^q, \quad \omega > 0, \quad q > p > 1,$$

which is in the form of (8) and use the result (a) of Theorem 4. □

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