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SINGULARITIES OF HOLOMORPHICALLY EXTENDED SPHERICAL FUNCTIONS

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1. Motivation and background

This is a preliminary account of joint work in progress which serves as an (very) extended abstract for a presentation of one of us (E.M.O.) in the Awajishima conference on Representation Theory, November 16-19, 2004, Japan.

To motivate the questions which we will address we will describe in some detail the beautiful ideas that were initiated by Sarnak [18] and by Bernstein and Reznikov [2], and then further explored by Krötz and Stanton [9].

Inspired by Sarnak [18], Bernstein and Reznikov [2] proposed a new method for estimating the coefficients in the expansion of the square of a Maass form on a compact locally symmetric space $Z = \Gamma\backslash X$ (where $X = G/K$ denotes a noncompact Riemannian symmetric space, and $\Gamma \subset G$ is a co-compact discrete subgroup of G) with respect to an orthonormal basis of $L^2(Z)$ consisting of Maass forms. The method is based on holomorphic extension of irreducible representations of G to a certain G-invariant domain in $X_{\mathbb{C}} := G_{\mathbb{C}}/K_{\mathbb{C}}$ (we assume that $G \subset G_{\mathbb{C}}$).

In [2] the method was applied in the case of $G = SL_2(\mathbb{R})$. The method was carried further by Krötz and Stanton in [9], where the results of [2] were slightly improved for $G = SL_2(\mathbb{R})$, and similar results for other rank 1 Riemannian symmetric spaces G/K were obtained. In addition some higher rank cases were considered in [9]. These considerations gave rise to various interesting issues concerning holomorphic extensions of representations and their matrix coefficients.

A predominant role for these matters is played by the complex crown (or Akhiezer-Gindikin domain) $\Xi \subset X_{\mathbb{C}}$ of X. This is a G-invariant domain in $X_{\mathbb{C}}$ on which G acts properly and which possesses the remarkable universal property that for all irreducible spherical Hilbert representations $(\pi, \mathcal{H})$ of G with spherical vector $v \in \mathcal{H}$ say, the map $X \ni gK = x \to v^x = \pi(g)v$ extends holomorphically to Ξ [9]. Another instance of this universal property of Ξ is the fact that every eigenfunction of the algebra of G-invariant differential operators on X extends holomorphically to Ξ [10]. The domain Ξ was recently studied by several authors from various points of view. It has truly remarkable properties and which show up naturally in other applications as well.

We first discuss some basic facts about Ξ. Then we will describe its distinguished boundary $\partial_d(\Xi)$ (this is the "extremal part" of the subset of the boundary $\partial(\Xi)$ consisting of points with closed $K_{\mathbb{C}}$-orbits). The important rôle of $\partial_d(\Xi)$ for lower estimates of holomorphic functions on Ξ is clear from the fact that a bounded holomorphic function f

which extends continuously to $\overline{\Xi}$ will have the same sup-norm as its restriction to $\partial_d(\Xi)$.

We will then explain the basic computation behind the method of [2] in order to motivate the problem of finding both good upper and lower estimates for the norm of v^x, with v a unit spherical vector of an unitary irreducible spherical representation, when x approaches radially a point of $\partial_d(\Xi)$. These problems will occupy us for the remaining part of this report.

The lower estimates need to be given uniformly in the spectral parameter μ of the spherical vector. For the above application it suffices to do this for the spherical minimal principal series $(\mathcal{H}_{i\lambda}, \pi_{i\lambda})$ with spherical vector $v_{i\lambda}$. We will give an uniform exponential lower bound for the norm of $v_{i\lambda}^x$ when $x \in \Xi$ approaches a point of $\partial_d(\Xi)$. The method is quite direct, using a suitable integral representation for the spherical function $\phi_{i\lambda}$ [9].

We obtain the upper estimates as an application of the theory of hypergeometric functions for root systems (cf. [4], [5], [17]). According to this theory there exists a commutative free $\mathbb{C}[\mathbf{m}_\alpha]$-algebra $\mathcal{R}$ of W-invariant differential operators on A ($A \subset G/K$ a maximal flat subspace) such that the specialization $\mathbf{m}_\alpha \to m_\alpha^X$ of $\mathcal{R}$, with m_α^X the multiplicities of the restricted roots of $\mathfrak{g}$ with respect to $\mathfrak{a} := T_{eK}(A)$, is equal to the algebra $\mathcal{R}_X = \mathrm{Rad}(D(X))$ of radial parts of G-invariant differential operators on X. The key point is the fact that "hypergeometric system of differential equations", which is the holonomic system of eigenfunction equations for this algebra of commuting differential operators, is generically simpler than the original system of eigenfunction equations for the restrictions of the spherical functions of X to A. It is therefore possible to compute the exponents of the hypergeometric system at its singularities, using techniques similar to [14], [15]. We obtain the desired upper estimates for the extended spherical functions by specialization from the appropriate formulae of generic hypergeometric functions.

2. The crown domain of X and holomorphic extensions

Let G be a connected, real semisimple, noncompact algebraic group. Then G is contained in its universal complexification $G_{\mathbb{C}}$. Let K be a maximal compact subgroup of G, and let $X = G/K$. We denote the base point $eK_{\mathbb{C}}$ of $X_{\mathbb{C}}$ by x_0, and we identify X with the totally real submanifold $Gx_0 \subset X_{\mathbb{C}}$ (this can be done by our assumption that $G \subset G_{\mathbb{C}}$).

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition, and choose $\mathfrak{a}$ a maximal abelian subspace in $\mathfrak{p}$. The complex crown Ξ of X was introduced in [1] and is by definition

$$\Xi := G\exp(i\pi\Omega/2)x_0 \subset X_{\mathbb{C}}, \tag{2.1}$$

where $\Omega \subset \mathfrak{a}$ is the defined in terms of the restricted root system $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ as follows:

$$\Omega := \{X \in \mathfrak{a} \mid |\alpha(X)| < 1\, \forall \alpha \in \Sigma\}. \tag{2.2}$$

The following properties of Ξ are crucially important in all that follows:

(i) Ξ is connected, G-invariant and open in $X_{\mathbb{C}}$ ([1],[9]).
(ii) The G action on Ξ is proper ([1]).
(iii) We have $\Xi \subset N_{\mathbb{C}}A_{\mathbb{C}}x_0$ ([9], [6],[13]).
(iv) In fact we have even $\Xi \subset N_{\mathbb{C}}A\exp(i\pi\Omega/2)x_0$ ([8]).
(v) Ξ is a Stein domain ([10] and see the references therein).

By (i),(iv) one can easily show [9] that the Iwasawa projections $a : G \to A$, $n : G \to N$ and $k : G \to K$ defined by $g = n(g)a(g)k(g)$ all have unique extensions to holomorphic maps $a : \tilde{\Xi} \to A\exp(i\pi\Omega/2)$, $n : \tilde{\Xi} \to N_{\mathbb{C}}$ and $k : \tilde{\Xi} \to K_{\mathbb{C}}$ respectively (where $\tilde{\Xi}$ is the pull-back of Ξ to $G_{\mathbb{C}}$). Since the tube domain $AT_\Omega := A\exp(i\pi\Omega/2) \subset A_{\mathbb{C}}$ is simply connected we obtain that the function

$$\begin{aligned}\Phi : \tilde{\Xi} &\to \mathbb{C}^\times\\ g &\to a(g)^{\mu+\rho} := \exp(\mu+\rho)(\log a(g))\end{aligned}$$

is well defined and holomorphic for all $\mu \in \mathfrak{a}_{\mathbb{C}}$. Using the compact realization of a spherical minimal principal series module $(\pi_\mu, \mathcal{H}_\mu)$ this result allows one to show [9] that the orbit map of a spherical vector $v_\mu \in \mathcal{H}_\mu$

$$\begin{aligned}F : X &\to \mathcal{H}_\mu\\ gK &\to \pi_\mu(g)v_\mu\end{aligned}$$

extends to a holomorphic map

$$\begin{aligned}F : \Xi &\to \mathcal{H}_\mu\\ xK_{\mathbb{C}} &\to \pi_\mu(x)v_\mu.\end{aligned}$$

It follows that every spherical function $\phi_\mu(gK) = \langle v_\mu, \pi(g)v_\mu\rangle$ can be holomorphically continued to Ξ (where we have adopted the physicist's convention that sesquilinear pairings are linear on the right hand side, and anti-linear on the left hand side) to Ξ. For general μ, ϕ_μ can not be continued to a larger G-invariant domain. This can be seen already in the example $G = SL_2(\mathbb{R})$ using the classical integral formula

$$\phi_\mu(g) := \int_K a(kg)^{\mu+\rho} dk \tag{2.3}$$

for the spherical function.

However, the spherical function will continue to a doubly $K_\mathbb{C}$-invariant function on $K_\mathbb{C} A T_\Omega^2 K_\mathbb{C}$ which is holomorphic on the tube AT_Ω^2. This can be easily seen from the following "doubling trick" [9]. If π^* denotes the conjugate contragredient representation, then we have for $at \in AT_\Omega$:

$$\begin{aligned} \phi_\mu(K_\mathbb{C} a^2 t^2 K_\mathbb{C}) &= \langle \pi^*(\overline{a^{-1}t^{-1}}) v_\mu, \pi(at) v_\mu \rangle \\ &= \langle \pi^*(a^{-1}t) v_\mu, \pi(at) v_\mu \rangle \end{aligned} \tag{2.4}$$

In particular we see from this formula in the case of the unitary spherical minimal principal series $\mu = i\lambda \in i\mathfrak{a}^*$ that the function $\phi_{i\lambda}$ is positive on T_Ω^2, and that for all $x = gtK_\mathbb{C} \in GT_\Omega / K_\mathbb{C}$ (recall the notation $v_{i\lambda}^x := \pi(x) v_{i\lambda}$):

$$\begin{aligned} \langle v_{i\lambda}^x, v_{i\lambda}^x \rangle &= \phi_{i\lambda}(t^2) \\ &= \int_K |a(kt)^{2(\lambda+\rho)}| dk. \end{aligned} \tag{2.5}$$

3. The distinguished boundary of $\partial(\Xi)$

The topological boundary $\partial(\Xi)$ of the domain Ξ is a union of G-orbits in $X_\mathbb{C}$. Not all the G-orbits in the boundary contain points of $A_\mathbb{C} x_0$, but for many applications it is in fact enough to consider only those G-orbits in $\partial(\Xi)$ which do meet $A_\mathbb{C} x_0$. This part of the boundary of Ξ is equal to (see [1]) $G\partial(T_\Omega) K_\mathbb{C} / K_\mathbb{C}$. In fact we will restrict further and include only the G-orbits of the extremal points $\partial_e(T_\Omega)$ of $\partial(T_\Omega)$. This set of orbits in the boundary is called the *distinguished boundary* $\partial_d(\Xi)$ of Ξ (see [7]). Its rôle is clearly illustrated by the following elementary result:

Proposition 3.1. *([7]) Let $f \in \mathcal{A}(\Xi)$ be holomorphic function on Ξ which extends to a bounded continuous function on $\overline{\Xi}$. Then*

$$\sup_{x\in\Xi}(|f(x)|) = \sup_{x\in\partial_d(\Xi)}(|f(x)|). \tag{3.1}$$

Essentially the same result holds for the space of bounded continuous plurisubharmonic functions on Ξ ([7]).

It is therefore an important problem to describe the distinguished boundary of Ξ or, what amounts to the same problem, to describe the set $\partial_e(T_\Omega)$ of extremal points of T_Ω. We can now summarize and extend the results of [7] in the following way:

Theorem 3.2. *Let $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ be the restricted root system and let Σ^l denote the reduced subsystem of roots $\alpha \in \Sigma$ such that $2\alpha \notin \Sigma$. Let D be the dynkin diagram of Σ^l and let D^* denote the corresponding extended diagram (corresponding to the affine root system $\Sigma^a = \Sigma^l \times \mathbb{Z}$). We assume that D is connected. For each vertex $k \in D$ we denote by ω_k the corresponding fundamental coweight of Σ^l. Let $n_k = \theta(\omega_k)$ denote the corresponding coefficient of the highest root $\theta \in \Sigma^l$.*

(i) *([11]) $\partial_e(T_\Omega)$ consists precisely of the W-orbits of the points $\{t_k = \exp(i\pi\omega_k/2n_k)\}$, where k runs over the set of vertices of D such that $D^*\backslash\{k\}$ is connected.*
(ii) *([11],[7]) Thus $\partial_m(T_\Omega) := \{t_k \mid \omega_k \text{ is a minuscule coweight}\} \subset \partial_e(T_\Omega)$. We will refer to $\partial_m T_\Omega$ as the set of minuscule boundary points of T_Ω.*
(iii) *([7]) We have: t_k is minuscule iff the component Gt_k of $\partial_d(\Xi)$ is a non-compactly causal symmetric space.*

4. Holomorphic extensions of Maass forms

This material is from [9], [10]. Let $Y = \Gamma \subset G$ be a cocompact discrete subgroup. A Maass eigenfunction $\psi_\mu \in C^\infty(Z)$ on the compact locally symmetric space $Z = Y/K$ admits a decomposition as a finite linear combination of Maass forms $\psi_\mu^{\eta,v}$ on Z of the form

$$\psi_\mu^{\eta,v}(\Gamma g K) = (\eta, \pi_\mu(g)v) \tag{4.1}$$

where $(\pi_\mu, \mathcal{H}_\mu)$ is a unitary, spherical representation of G with spherical vector v, and where $\eta \in \mathcal{H}_\mu^{-\infty}$ is a Γ-invariant distribution vector. Here we parametrize the spherical unitary dual by a subset of $\mathfrak{a}_\mathbb{C}^*$ as usual. We can and will choose η in such a way that the map

$$\begin{aligned} \mathcal{H}_\mu^\infty &\to C^\infty(Y) \\ v &\to \{\Gamma g \to (\eta, \pi_\mu(g)v)\} \end{aligned} \tag{4.2}$$

extends to an isometry $\mathcal{H}_\mu \to L^2(Y)$. Using the techniques described in the previous sections, it is clear that $\psi_\mu^{\eta,v}$ allows a holomorphic extension to $\Gamma\backslash\Xi$.

Let $\{\psi_i\}_{i\in I}$ denote a orthonormal basis of $L^2(Z)$ consisting of Maass forms $\psi_i = \psi_{\mu_i}^{\eta_i,v_i}$. In [9] a method is explained for obtaining estimates for the coefficients c_i in an expansion of the form

$$\psi_j^2 = \sum_{i \in I} c_i \psi_i. \tag{4.3}$$

These coefficients are thus of the form

$$c_i = \int_Z \psi_j^2 \overline{\psi_i} dz. \tag{4.4}$$

Estimating such integrals of triple products of Maass forms was the original motivation of Sarnak [18].

Let $\epsilon \in (0,1)$ and let $t = \exp(-\pi i H/2) \in \partial(T_\Omega)$. Consider the ray to the boundary point $t \in \partial(T_\Omega)$ given by $t_\epsilon = \exp(-\pi i(1-\epsilon)H/2)$ ($\epsilon \in (0,1)$). For every ϵ we define a function $\psi^\epsilon \in C^\infty(Y)$ by

$$\psi_i^\epsilon(y) := \psi_i(yt_\epsilon) = (\eta_i, \pi_{\mu_i}(y) v_i^{t_\epsilon}). \tag{4.5}$$

The ingredients for obtaining estimates for the c_i are the following:

(i) Uniform (in μ_i) lower bounds for the L^2-norms of ψ_i^ϵ on Y. To obtain the best lower bounds it is clear, in view of Proposition 3.1 and (2.5), that should take t equal to one of the distinguished boundary points and that should take the limit $\epsilon \to 0$.

(ii) Upper estimates for $\|\psi_j^\epsilon\|$ in $L^2(Y)$, where $\epsilon \to 0$ and t is as in (i).

(iii) Upper estimates for $\|\psi_j^\epsilon\|_\infty$ in $L^\infty(Y)$, where $\epsilon \to 0$ and t is as in (i).

The L^∞-estimates are by far the hardest, and this is the battle ground where serious improvements of the estimates have to be conquered. We will restrict ourselves here to improvements of the first two type of estimates in the higher rank cases. These results are basic and are likely to also find applications outside of the present context.

Let us assume for simplicity that $\mu_i = i\lambda \in i\mathfrak{a}^*$ (For the lower estimates this simplifies matters; it is also the most important case for estimating the c_i). Using (4.2), (2.5) we get:

$$\begin{aligned} \|\psi_i^\epsilon\|^2 &= (v_{i\lambda}^{t_\epsilon}, v_{i\lambda}^{t_\epsilon}) \\ &= \phi_{i\lambda}(t_\epsilon^2) \\ &= \int_K |a(kt_\epsilon)^{2(i\lambda+\rho)}| dk. \end{aligned} \tag{4.6}$$

5. Lower L^2 estimates

The lower estimates for $\|\psi_i^\epsilon\|^2$ that we can give are based on the use of the integral in the last equality of 4.6. Since the kernel of the integral is positive we have lower estimates of the form

$$\|\psi_i^\epsilon\|^2 \geq \int_{U_\epsilon} |a(kt_\epsilon)^{2(i\lambda+\rho)}| dk \tag{5.1}$$

where U_ϵ is any neighborhood of $e \in K$. Suppose that $t = t_0$ is a boundary point corresponding to a causal boundary component, i.e. t is a minuscule boundary point of $\partial(T_\Omega)$. Another way of saying this is that $t^4 \in Z(G_\mathbb{C})$, the center of $G_\mathbb{C}$. We have the following useful lemma:

Lemma 5.1. *If t^4 is central in $G_\mathbb{C}$ and $k \in U$, a suitably small neighbourhood of $e \in K$, then $a(kt) \in A_\mathbb{C}$ is well defined and we have $a(kt) = r(kt)t$ with $r(kt) \in A \subset G$.*

The lemma implies that there exists $\epsilon_0 > 0$ and $R > 0$ such that the angular part of $t^{-1}a(kt_\epsilon)$ can be estimated by $R\epsilon$ for all $k \in U$ and for all $\epsilon < \epsilon_0$. Therefore we can now take U_ϵ independent of $\epsilon < \epsilon_0$ in this case, and we obtain:

Lemma 5.2. *Let $t = \exp(-\pi i\omega/2)$ be a boundary point of T_Ω such that t^4 central in $G_\mathbb{C}$. Then there exist constants $\epsilon_0 \in (0,1)$ (independent of $\lambda \in \mathfrak{a}$) and $R > 0$, $C > 0$ (independent of $\lambda \in \mathfrak{a}$ and $\epsilon \in (0, \epsilon_0)$) such that*

$$\|\psi_i^\epsilon\|^2 \geq C \exp(\max_{w \in W} \pi\lambda(w\omega)(1 - R\epsilon)),\ \forall\lambda \in \mathfrak{a}, \forall\epsilon \in (0, \epsilon_0) \tag{5.2}$$

The general case is obtained by considering the above Lemma in the centralizer $G' = C_G(t^4)$, a reductive subgroup of G. Let $K' = K \cap G'$. We obtain

Theorem 5.3. *Let $t = \exp(-\pi i\omega/2)$ be an extremal boundary point of T_Ω. Then there exist constants $\epsilon_0 \in (0,1)$ (independent of λ) and $R > 0$, $C > 0$ (independent of $\lambda \in \mathfrak{a}$, $\epsilon < \epsilon_0$) such that*

$$\|\psi_i^\epsilon\|^2 \geq C\epsilon^{(\dim(K) - \dim(K'))/2} \exp(\max_{w \in W} \pi\lambda(w\omega)(1 - R\epsilon)) \tag{5.3}$$

for all $\lambda \in \mathfrak{a}$, and for all $\epsilon \in (0, \epsilon_0)$.

6. Upper L^2 estimates

In this final section we consider the problem to give upper estimate for the L^2 norm in $L^2(Y)$ of the extended Maass form ψ_i^ϵ. Recall from (4.6) that

$$\begin{aligned}\|\psi_i^\epsilon\|^2 &= (v_{i\lambda}^{t_\epsilon}, v_{i\lambda}^{t_\epsilon}) \\ &= \phi_{i\lambda}(t_\epsilon^2)\end{aligned}$$

Therefore we can concentrate on the question of estimating the holomorphically extended spherical function $\phi_\mu(t_\epsilon^2)$ when $\epsilon \to 0$, where $t = \exp(\pi i\omega/2)$ is a extremal boundary point of T_Ω and where we choose $\omega \in \mathfrak{a}_+$ dominant.

We will deal with this problem using the system of differential eigenfunction equations defining the restriction of the spherical function ϕ_μ to AT_Ω^2. We will establish the estimates by computing the *exponents* of the solutions of these differential equations at the singular point t, when restricted to a small punctured complex disc embedded in $A_{\mathbb{C}}^{\mathrm{reg}}$ and centered at t. We have to take several steps in order to compute these exponents. Before we discuss this in general we remark that in the *complex case* (all root multiplicities are equal to 2 for the elements of Σ_l, and equal to 0 for the other roots) the problem becomes completely elementary because the algebra of differential equations is conjugate to the algebra of W-invariant differential equations by a conjugation with the Weyl denominator. Looking at the explicit form of the eigenfunctions in this case we easily find that:

Proposition 6.1. *In the complex case $\phi_{i\lambda}(t_\epsilon^2) \asymp \epsilon^{n_\omega}$ as $\epsilon \to 0$, where $-n_\omega$ is equal to $|\Sigma_{+,\omega} \backslash \Sigma_{+,\omega}^f|$. Here $\Sigma_{+,\omega}$ is the set of positive affine roots which vanish on ω, and where $\Sigma_{+,\omega}^f := \Sigma_{+,\omega} \cap \Sigma_+^l$. In other words, $-n_\omega$ is the number of roots α in Σ_+^l with $\alpha(\omega) = 1$.*

6.0.1. *Connection form of the equations.* It does not make sense to restrict the system of eigenfunction equations to an embedded complex disc. Therefore we rewrite the system as a flat connection with logarithmic singularities, in such a way that the exponents at t of the restriction of this connection to the disc coincide with the exponents of the restrictions of the solutions of the system of eigenfunction equations for the spherical function.

Using the exponential map $X \to \exp(\pi i X)$ we will work on $i\mathfrak{a} + \Omega \subset \mathfrak{a}_{\mathbb{C}}$ rather than on $AT_\Omega^2 \subset A_{\mathbb{C}}$ (recall that the logarithm is well defined on AT_Ω^2). It is well known [4] that the spherical system of eigenfunction equations can be cast in the form of an integrable connection on $\mathfrak{a}_{\mathbb{C}}$ with singularities along the collection of affine hyperplanes $\alpha(H) \in \mathbb{Z}$ (not the usual $\pi i\mathbb{Z}$, since we have multiplied everything by $(\pi i)^{-1}$). The point $\omega \in \mathfrak{a}$ is in the singular locus of this connection. For the above application we need to adapt the construction in [4] slightly so as to get logarithmic sigularities at (the blow-up of) the point ω.

We consider a parametrized line $x \to \omega + xV_1$ through this singular point, where V_1 is small and chosen in such a way that this line is not contained in the union of the singular affine hyperplanes. We will choose coordinates $(y_1, y_2, \ldots y_n)$ $(y_i \in (0,1))$ in an open truncated cone $\subset \mathfrak{a}$ which is in the complement of the singular hyperplanes, with top ω and containing a neighborhood of the segment $\omega + (0,1)V_1$ as follows. First we choose $V_2, \dots, V_n$ in $\mathfrak{a}$ such that $\|V_i\|$ is small for all i, and such that $(V_1, V_2, \dots, V_n)$ is a basis of the real vector space $\mathfrak{a}$. Then our coordinate map is given by

$$(y_1, y_2 \dots, y_n) \to \omega + y_1(V_1 + \sum_{i \geq 2} y_i V_i) \in \mathfrak{a}. \tag{6.1}$$

If we lift this coordinate map to the blow-up of $\mathfrak{a}_{\mathbb{C}}$ at ω then the coordinates can be naturally extended to the polydisk $PD := \{(y_1, \dots, y_n) \in \mathbb{C}^n \mid |y_i| < 1\}$, and this is then a coordinate neighborhood of a regular point of the exceptional divisor E. The intersection of this neighbourhood with E is described by the equation $y_1 = 0$.

Let p be a point in the pounctured polydisc PD^* and let $\mathcal{O}_p$ denote the ring of holomorphic germs at p. Following [15], consider the complex vector space U^* spanned by the linear partial differential operators $b_i = y_1^{\deg(q_i)} \partial(q_i)$, where q_i runs over a homogeneous basis of W-harmonic polynomials ordered such that $i \to \deg(q_i)$ is nondecreasing. Let $\mathcal{O}_p \otimes U^*$ be the corresponding free $\mathcal{O}_p$ module. This free $\mathcal{O}_p$-module is a complement, in the localization of the ring of holomorphic partial linear differential operators at p, for the left ideal $\mathcal{I}_\mu$ generated by $D - \gamma(D)(\mu)$, with D running over the commutative ring $\mathcal{R}_X := \mathrm{Rad}(D(X))$ of radial parts of G-invariant operators on $X_{\mathbb{C}}$, and γ the Harish-Chandra homomorphism. We can rewrite the differential equations (with $\mu \in \mathfrak{a}_{\mathbb{C}}^*$)

$$(D - \gamma(D)(\mu))\phi = 0 \ \forall D \in \mathcal{R}_X \tag{6.2}$$

in connection form with respect to the above basis and coordinates. We define matrices $M^i \in \mathrm{End}_{\mathcal{O}_p}(\mathcal{O}_p \otimes U)$ (where U denotes the dual of U^*, with dual basis b_i^*) which are characterized by the requirement that

$$\frac{\partial}{\partial y_i} \circ b_k \in \sum_j (M^i)^{\mathrm{tr}}_{jk} b_j + \mathcal{I}_\mu. \tag{6.3}$$

Then the desired (flat) connection form of (6.2) is defined on the free $\mathcal{O}_p$-module $\mathcal{O}_p \otimes U$ by

$$\frac{\partial \Phi}{\partial y_i} = M^i \Phi \quad (\Phi \in \mathcal{O}_p \otimes U). \tag{6.4}$$

By construction, if ϕ is a solution of (6.2) then $\Phi = \sum_i b_i(\phi) b_i^*$ is a solution vector of (6.4). Conversely, if Φ is a solution vector of (6.4) then the first coordinate $\phi = \langle b_1, \Phi \rangle$ is a solution of (6.2). These are inverse isomorphisms between the solution spaces of these two systems of differential equation.

Remark 6.2. *Since the local solution space of an integrable connection at a regular point p can be identified with the fiber of the underlying vector space at p, the above gives an isomorphism (depending on p) between the local solution space $\mathcal{L}_p(\mu, m^X)$ of (6.2) at p and the complex vector space U.*

By the homogeneity of the basis b_i which we chose, one can easily show (using that the "lowest homogeneous part at ω" of the radial part D contains the highest order term of D) that the matrices M^i $(i = 2, \ldots, n)$ are holomorphic in the coordinates $(y_1, \ldots, y_n)$, and that M^1 has a simple pole at $y_1 = 0$. As in [15] one can in fact show that

Lemma 6.3. *The exponents of the solutions of the system 6.2, pulled back to the punctured coordinate polydisk $PD^* := PD \cap y_1 \neq 0$, are equal to the eigenvalues of the residue R_m of the matrix M^1 at $y_1 = 0$ (it is well known that these eigenvalues are independent of $(y_2, \ldots, y_n)$, cf.* [3]*).*

6.0.2. *Monodromy of the local braid group.* The above analysis "reduces" our problem to the computation of the eigenvalues of the residue matrix R_X (specialized for example at $y_2 = y_3 = \ldots y_n = 0$) of M^1. This seems a daunting task, since we have no explicit information on R_X at all.

Now we will start to make use of the remarkable fact which was already mentioned in the first section, namely that the commutative ring $\mathcal{R}_X$ of radial parts of the ring $D(X)$ of G-invariant operators on $X_{\mathbb{C}}$ is equal to the specialization $\mathbf{m}_\alpha \to m_\alpha$ of a commutative, free $\mathbb{C}[\mathbf{m}_\alpha]$-algebra $\mathcal{R}$ of linear partial differential operators with coefficients in the polynomial ring $\mathbb{C}[\mathbf{m}_\alpha]$. There exists a generalized Harish-Chandra homomorphism $\gamma : \mathcal{R} \to \mathbb{C}[\mathbf{m}_\alpha][\mathfrak{a}_{\mathbb{C}}^*]^W$, and we can define *for all multiplicity functions $m = (m_\alpha)$ on Σ* the *hypergeometric system of differential equations* (cf. [4], [5]) on $A_{\mathbb{C}} x_0 = A_{\mathbb{C}}/F$ (with $F = K_{\mathbb{C}} \cap A_{\mathbb{C}}$, a finite 2-group):

$$(D - \gamma(D)(\mu))\phi = 0 \ \forall D \in \mathcal{R}_m \tag{6.5}$$

The system is equal to (6.2) when we take $m = m^X$, the multiplicity function associated with our symmetric space X. The system is invariant for the action of the (extended) affine Weyl group $W^a = W \ltimes L$ on $\mathfrak{a}_{\mathbb{C}}$, where $L \subset \mathfrak{a}$ such that $\pi i L$ is the lattice of cocharacters of $A_{\mathbb{C}}/F$ (recall that we have multiplied everything by $(\pi i)^{-1}$; observe that the coroot lattice of Σ is always contained in L). Hence the system descends to the space of regular orbits for the action of W^a on $\mathfrak{a}_{\mathbb{C}}$. It is known that it is holonomic of rank $|W|$ and that it is regular singular (see e.g. [4], [17], Remark 6.10).

The whole construction of the connection form of the system and of the residue matrix R_X can now be performed over the ring $\mathbb{C}[\mathbf{m}_\alpha]$, and we obtain a residue matrix R as before, but with coefficients in $\mathbb{C}[\mathbf{m}_\alpha]$, such that $R_X = R_{m^X}$, the specialization of R at m^X.

Still this does not seem to help too much, if anything we have made the situation more complicated-or so it seems. But now something comes to rescue us, namely our explicit knowledge of the action of the fundamental group $\Pi_1(W^a \backslash \mathfrak{a}_{\mathbb{C}}^{\text{reg}}, p)$ (at a regular base point $p \in \mathfrak{a}_{\mathbb{C}}^{\text{reg}}$, chosen near ω) by monodromy on the local solution space $\mathcal{L}_p = \mathcal{L}_p(\mu, m)$ of (6.5). By a well known result of Looijenga and Van der Lek ([12], also see [4], [5], [17]) this fundamental group is isomorphic to the (extended) affine braid group B^a of $W^a = W \ltimes L$. The monodromy action on $\mathcal{L}_p(\mu, m)$ factors through the (extended) affine Hecke algebra $H(W^a, q)$, where q is the label function on the set of simple affine roots $(\alpha_0 = 1 - \theta, \alpha_1, \ldots, \alpha_n)$ of the affine extension Σ^a of Σ^l given by

$$q_i = \exp(-\pi i(m_{\alpha_i} + m_{\alpha_i/2})) \text{ for } i = 1, \ldots, n, \text{ and}$$
$$q_0 = \exp(-\pi i m_\theta).$$

Theorem 6.4. *(cf. [4], [5], [17]) The monodromy action of B^a on $\mathcal{L}(\mu, m)$ factors through the extended affine Heck algebra $H(W^a, q)$ and depends analytically on m and μ.*

Let us assume now that W is irreducible and put $n = \dim_{\mathbb{R}}(\mathfrak{a})$. Let W_ω be the isotropy subgroup of ω in the *non-extended* affine Weyl group $W^{\text{aff}} \subset W^a$, and let Σ_ω be the corresponding root system. We fix simple roots of Σ_ω by taking the complement of $\{k\} \subset D^*$ in the set of affine simple roots, where k is such that $\omega = \omega_k / n_k$ (notations as in Theorem 3.2). According to Theorem 3.2, W_ω is a finite, irreducible reflection group of rank n, isomorphic to a reflection subgroup $\tilde{W}_\omega \subset W$ which is equal to W iff ω is a minuscule extremal point. Put $N_\omega = [W : \tilde{W}_\omega]$ for the index of $\tilde{W}_\omega$ in W.

Let us denote by $B_\omega \subset B^a$ the braid group of W_ω, which we can identify, by Brieskorn's theorem on the fundamental group of the regular orbit space of a finite reflection group, with the fundamental group based at $\bar{p}$ of the regular orbit space of W_ω, acting on the complement of its reflection hyperplanes in a small W_ω-invariant ball centered around ω (and containing p) in $\mathfrak{a}_{\mathbb{C}}$. We will refer to the subgroup $B_\omega \subset B^a$ as the local fundamental group at ω.

Let q_ω be the restriction of q to Σ_ω, and let m_ω be corresponding the corresponding root multiplicity function on Σ_ω. In a dense, open set Q^{reg} of values of m, the finite dimensional Hecke algebra $H_\omega = H(W_\omega, q_\omega)$ (with $q_\omega = q(m_\omega)$) is semisimple. If we assume that $m \in Q^{\text{reg}}$ then, by Tits' deformation lemma, we can index its set of irreducible modules by $\hat{W}_\omega$, the set of irreducible representations of W_ω. Given $\tau \in \hat{W}_\omega$ and $m \in Q^{\text{reg}}$ we will write $\pi_\tau(m)$ for the corresponding irreducible $H_\omega(W_\omega, q(m_\omega))$-module. Upon restriction of the monodromy action on $\mathcal{L}_p(\mu, m)$ to B_ω we get, using the rigidity of semisimple finite dimensional algebras (Tits' deformation lemma):

Corollary 6.5. *Let $q = q(m)$ and $q_\omega = q(m_\omega)$ for $m \in Q^{\text{reg}}$. Under the monodromy action of $H(W_\omega, q_\omega)$, the local solution space $\mathcal{L}_p(\mu, m)$ is isomorphic to*

$$\mathcal{L}_p(\mu, m) \simeq \sum_{\tau \in \hat{W}_\omega} K(\tau, m) \otimes \pi_\tau(m) \tag{6.6}$$

where the multiplicity space $K(\tau, m) = \mathrm{Hom}_{H(W_\omega, q_\omega)}(\pi_\tau(m), \mathcal{L}_p(\mu, m))$ has dimension independent of m. In particular, its dimension is equal to the multiplicity $N_\omega \deg_\tau$ of τ in the restriction of the regular representation of W to $\tilde{W}_\omega$ (this follows by restriction to the case $m = 0$).

6.0.3. *The residual eigenvalues and monodromy.* The following topological observation is crucial for our purpose:

Lemma 6.6. *Let $\beta_\omega \in B_\omega$ denote the braid which corresponds to a reduced expression of the longest element of W_ω. Let σ_ω denote the closed loop based at p going once around the hyperplane $y_1 = 0$, in positive direction, in the punctured polydisc PD^*. In the local fundamental group at ω we have the identity $\beta_\omega^2 = \sigma_\omega$. This element is central in B_ω.*

If $m \in Q^{\text{reg}}$ then the monodromy action of the local fundamental group B_ω on the local solution space $\mathcal{L}_p(\mu, m)$ is semisimple. Hence we have (using standard results on flat connections with logarithmic poles, see e.g. [3]):

Corollary 6.7. *(see [14], [15]) Assume that $m \in Q^{\text{reg}}$. Identify the local solution space $\mathcal{L}_p(\mu, m)$ with the U (see Remark 6.2) and let R_m be the residue of M_m^1 acting on U. Then $\exp(2\pi i R_m)$ is conjugate to the monodromy action of β_ω^2. In particular, R_m is semisimple.*

It is an easy matter to compute the scalar action of the central element β_ω^2 on $\pi_\tau(m)$:

Proposition 6.8. *The scalar value of the central element β_ω^2 on $\pi_\tau(m)$ is equal to $\exp(-\pi i c_\tau(m))$, where $c_\tau(m)$ is the polynomial of degree at* *most* 1 *in the* m_α *given by*

$$c_\tau(m) = \sum_{\alpha\in\Sigma_{\omega,+}} m_\alpha\Big(1 - \frac{\chi_\tau(s_\alpha)}{\deg_\tau}\Big). \tag{6.7}$$

The coefficients of the m_α *in* $c_\tau(m)$ *are positive integral.*

From Corollary 6.7 and Proposition 6.8 we deduce as in [15] that

Corollary 6.9. *The characteristic polynomial of the residue matrix* R *splits completely over the ring* $\mathbb{C}[\mathbf{m}_\alpha]$. *Its roots (counted with multiplicity) are of the form* $n_\tau^i = p_\tau^i - \frac{1}{2}c_\tau(m)$ *for certain integers* p_τ^i, *with* $\tau \in \hat{W}_\omega$ *and where for each* τ, i *runs from* 1 *to* $N_\omega \deg_\tau^2$.

We need to be more precise. Notice that the residue matrix was computed using only the "lowest homogeneous parts at ω" of the operators $D \in \mathcal{R}_m$ (in particular, R_m is independent of μ). Using the theory of Dunkl-Cherednik operators it is easy to see that these lowest homogeneous parts at ω form a subalgbra $\tilde{R}$ of the commutative algebra $\mathcal{R}_{m_\omega}^{\omega,B}$ of "Bessel differential operators" for the root system Σ_ω (with ω as origin of $\mathfrak{a}_\mathbb{C}$), and root multiplicities m_ω. In fact, $\tilde{\mathcal{R}}$ is precisely the inverse image in $\mathcal{R}_{m_\omega}^{\omega,B}$ under the "Harish-Chandra isomorphism" $\gamma^{\omega,B} : \mathcal{R}_{m_\omega}^{\omega,B} \tilde{\to} \mathbb{C}[\mathfrak{a}^*]^{W_\omega}$ of $\mathbb{C}[\mathfrak{a}^*]^W \subset \mathbb{C}[\mathfrak{a}^*]^{W_\omega}$. Let $\mathcal{L}_p^{\omega,B}(m)$ be the local solution space of the homogeneous system of equations

$$D\phi = 0, \ \forall D \in (\gamma^{\omega,B})^{-1}(\mathbb{C}[\mathfrak{a}^*]^W). \tag{6.8}$$

From [15] it is easy to see that, provided that we are in the semisimple situation $m \in Q^{\mathrm{reg}}$, the monodromy representations on $\mathcal{L}_p^{\omega,B}(m)$ and $\mathcal{L}_p(\mu, m)$ of B_ω are isomorphic (for all $\mu \in \mathfrak{a}_\mathbb{C}^*$, as the monodromy type of $\mathcal{L}_p(\mu, m)$ is independent of μ in the semisimple case).

Using general theory of regular singular connections [3] we can show that there exists (for all m an μ) a natural map $\mathrm{gr}^\omega : \mathcal{L}_p(\mu, m) \to \mathcal{L}_p^{\omega,B}(m)$ such that $\mathrm{gr}^\omega(\phi)$ is the lowest homogeneous term of ϕ with respect to the coordinate y_1 the logarithmic expansion of ϕ in the coordinate y_1 on the punctured polydisc PD^* (in the coordinate system $(y_1, \ldots, y_n)$ as we used before). Using the flatness of the residue matrix R_m with respect to the residual connection on the singular hyperplane E (the exceptional divisor of the blow-up of $\mathfrak{a}_\mathbb{C}$ at ω) we deduce

Proposition 6.10. *The map* $\mathrm{gr}^\omega : \mathcal{L}_p(\mu, m) \to \mathcal{L}_p^{\omega,B}(m)$ *commutes with the local monodromy action of* B_ω. *The residue* R_m, *considered as an endomorphism* U, *commutes with the local monodromy action on* $\mathcal{L}_p^{\omega,B}(m)$.

From this result we obtain:

Theorem 6.11. *(extends results of* [15]*) Identify $\mathcal{L}_p^{\omega,B}(m)$ with U as before.*

(i) *The R-eigenvalues are of the form*

$$n_\tau^i(\mathbf{m}) = p_\tau^i - \frac{1}{2}c_\tau(\mathbf{m}) \tag{6.9}$$

where $i = 1, \ldots, N_\omega \deg_\tau$ and where p_τ^i denotes the embedding degrees of $\tau \in \hat{W}_\omega$ in the graded vector space of W-harmonic polynomials. Each eigenvalue n_τ^i has multiplicity $\deg_\tau$.

(ii) *If we assume $m \in Q^{\mathrm{reg}}$ then we can decompose $\mathcal{L}_p^{\omega,B}(m)$ into R_m-eigenspaces $E_\tau^i(m)$ with eigenvalue $n_\tau^i(m)$ which are invariant and irreducible of type $\pi_\tau(m)$ for the local monodromy action of B_ω. Thus R_m is semisimple in this case.*

Remark 6.12. *We choose the harmonic embedding degrees so that $i \longrightarrow p_\tau^i$ is a non-decreasing sequence. In particular, p_τ^1 is the "harmonic birthday" of τ in the W-harmonic polynomials.*

6.0.4. *Computation of the exponents.* Let $\phi_{i\lambda,m}$ denote the hypergeometric function, the unique normalized solution of the hypergeometric equations (6.5) which extends holomorphically to a neighborhood of $0 \in \mathfrak{a}_{\mathbb{C}}$. It is holomorphic in m, and it is easy to see that it extends holomorphically to $i\mathfrak{a} + \Omega$. At $m = m^X$ it is identical with our holomorphically extended spherical function. In particular, these functions have the property that they are fixed vectors in $\mathcal{L}_p(i\lambda, m)$ for the action of the subgroup $B_\omega^f \subset B_\omega$ of the local braid group corresponding to the walls of Σ_ω^f (we may and will assume ω is dominant).

Using Theorem 6.11 we can now compute the exponents of ϵ which show up when we take the limit $\epsilon \longrightarrow 0$ in $\phi_{i\lambda,m}(t_{\epsilon,V}^2)$, where $t_{\epsilon,V} = \exp(\pi i(\omega - \epsilon V)/2)$ with ω an extremal point of Ω, and V a regular direction. Before we explain this we warn the reader that there are some minor subtleties which deserve our attention at this point.

First of all, in the original problem which we set out we took $V = \omega$, and this is not a regular direction (unless we are in the rank 1 case). One can show however (using that the monodromy along the walls of Σ_ω^f commutes with R_m) that the logarithmic expansions with respect to y_1 extend over the walls of Σ_ω^f. Therefore the result of sending V to the singular direction ω will possibly be a positive integral jump in the exponent for generic regular directions V (but no worse things can happen).

Next we should underline that m^X is usually NOT in Q^{reg}. In some sense, the split real case corresponds to the worst possible non-semisimple degeneration of the Hecke algebra $H(W_\omega, q(m_\omega))$, since all second order relations are now of the form $(T_i - 1)^2 = 0$. Therefore logarithmic terms can show up in the leading ϵ-expansion for the limit $\epsilon \to 0$. The above method also gives good estimates for the power of $\log(\epsilon)$ which may show up (in fact, $\log(\epsilon)$ will appear only in a limited number of cases), but we will ignore this for now.

Finally we may have a positive integral jump in the leading exponent if we specialize m at m^X, simply because the leading asymptotic term vanishes for the substitution $m = m^X$ in the family $m \to \phi_{i\lambda,m}$.

For these reasons we will now only formulate the exact leading exponent for a generic $m \in Q^{\mathrm{reg}}$ and a ray $t_{\epsilon,V}$ in a generic regular direction V. By the above remarks we get for the spherical function itself possibly additional powers of $log(\epsilon)$ (but this can be handled with some care), and the order ϵ^n-estimate that we get by substitution of m^X instead of the generic m may not be optimal by the phenomena described above. *We believe however that the results are in fact optimal for generic $i\lambda$.*

With all these reservations we now state the main result:

Theorem 6.13. *Assume that $m \in Q^{\mathrm{reg}}$. The leading exponent $\epsilon^{n_\omega(m)}$ of $\phi_{i\lambda,m}(t^2_{\epsilon,V})$ for $\epsilon \to 0$ is equal to*

$$n_\omega(m) = \min_\tau n^1_\tau(m) \tag{6.10}$$

(see Theorem 6.11 and Remark 6.12), where τ runs over the set of irreducible constituents of the induced representation $\mathrm{Ind}_{W^f_\omega}^{W_\omega}(1)$ (where $W^f_\omega := W(\Sigma^f_\omega)$ (see Proposition 6.1).

This value can be easily determined by hand for all classical root systems using the Littlewood-Richardson rule. For the exceptional cases they can also be determined, with some help of the Chevie package of GAP. We remark that the specialization of this value for the complex case matches with Proposition 6.1, and this is a main point in the proof that the computed value is optimal generically in m.

Example 6.14. *For $\Sigma = A_{n-1}$ all the nodes of the Dynkin diagram are minuscule and thus extremal according to Theorem 3.2. We order the nodes of the diagram linearly. Let $k \leq n/2$, and let ω_k be the k-th node of the Dynkin diagram. Then*

$$n_{\omega_k}(m) = k(1 - (n + 1 - k)m/2)). \tag{6.11}$$

This value corresponds to the representation τ_k of S_n labeled by the partition $(n - k, k)$ occurring in the induction $\mathrm{Ind}^{S_n}_{S_{n-k} \times S_k}(1)$. If $m = 2$ (complex case, see Proposition 6.1) we get: $n_{\omega_k}(2) = -k(n - k)$, which is indeed in accordance with Proposition 6.1 (check!). Notice also the *case of* $SL_2(\mathbb{R})$ *(i.e.* $n = 2$ *and* $m = 1$*). Here we get* $n_\omega(1) = 0$*, but there will occur a* $\log(\epsilon)$ *term in the* ϵ*-expansion of the spherical function. This is the only case (in type A) where a logarithmic term occurs.*

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