Reachability and Observability of Reactor Diffusion Equations in Chemical Engineering – An Application of Analytic Semigroup

Shin-ichi NAKAGIRI
Graduate School of Engineering, Kobe University, Japan

In this lecture, as an application of analytic semigroup, we explain about the reachability and observability of a system of reactor diffusion equations appearing in chemical engineering. There are many types of chemical reactor equations. However, as a specific type of equations, we investigate a plug-flow reactor diffusion equation, which is fundamental in the theory of chemical reaction. Let C_1 and C_2 be a reactor and a product chemicals, respectively, and let b be a chemical reaction coefficient. Then the dynamical model of chemical reaction $C_1 \to b\,C_2$ is described by the following system of transport diffusion equations (cf. Winkin et al. [3]):

$$(S)\begin{cases} \dfrac{\partial z_1}{\partial t}(t,x) = D\dfrac{\partial^2 z_1}{\partial x^2}(t,x) - \alpha\dfrac{\partial z_1}{\partial x}(t,x) - k_0 z_1(t,x), \quad (t,x) \in (0,\infty)\times[0,1], \\ \dfrac{\partial z_2}{\partial t}(t,x) = D\dfrac{\partial^2 z_2}{\partial x^2}(t,x) - \alpha\dfrac{\partial z_2}{\partial x}(t,x) + bk_0 z_1(t,x), \quad (t,x) \in (0,\infty)\times[0,1], \\ D\dfrac{\partial z_1}{\partial x}(t,0) - \alpha z_1(t,0) = -\alpha u_1(t), \quad D\dfrac{\partial z_2}{\partial x}(t,0) - \alpha z_2(t,0) = -\alpha u_2(t), \\ \dfrac{\partial z_1}{\partial x}(t,1) = 0, \quad \dfrac{\partial z_2}{\partial x}(t,1) = 0, \quad t \in (0,\infty), \\ z_1(0,x) = \varphi_1(x), \quad z_2(0,x) = \varphi_2(x), \quad x \in [0,1], \end{cases}$$

where $z_1(t,x)$, $z_2(t,x) \in \mathbf{R}$ denote the densities of chemicals C_1, C_2 at time t at point $x \in [0,1]$, respectively, and $\varphi_1(x)$, $\varphi_2(x)$ are initial values, $u_1(t)$, $u_2(t) \in \mathbf{R}$ are the control inputs, $\alpha > 0$ is the speed of flud, $D > 0$ is the diffusion constant of chemicals, and k_0, b are positive constants of chemical reaction process. The observation of boundary control system (S) is given by a state observation at $x = 1$:

$$(O) \qquad y(t) = [y_1(t),\, y_2(t)]^T = [z_1(t,1),\, z_2(t,1)]^T, \quad t \in (0,\infty),$$

where $y_1(t)$, $y_2(t) \in \mathbf{R}$ denote the observation outputs.

Define the unbounded operator $A_1 : D(A_1) \subset L^2(0,1) \to L^2(0,1)$ by

$$(A_1\varphi)(x) = -D\frac{d^2\varphi(x)}{dx^2} + \alpha\frac{d\varphi(x)}{dx} + k_0\varphi(x),$$
$$D(A_1) = \{\, \varphi \in H^2(0,1)\,;\, D\frac{d\varphi}{dx}(0) - \alpha\varphi(0) = 0,\ \frac{d\varphi}{dx}(1) = 0 \,\}.$$

Then $-A_1$ is a Riesz-spectral operator in $L^2(0,1)$. The spectrum of $-A_1$ consists only of isolated eigenvalues $\{-\lambda_n : \; n \geq 1\}$, and the corresponding set of eigenfunctions $\{\phi_n : \; n \geq 1\}$ of multiplicity 1 exists. The set of eigenfunctions of the adoint operator $-A_1^*$ is denoted by $\{\psi_n : \; n \geq 1\}$. By the pertubation theorem for C_0-semigroups (cf. Tanabe [2]), the operator A associated with system (S) in the product Hilbert space $X := [L^2(0,1)]^2$ generates an exponentially stable analytic semigroup $e^{tA} =: T(t)$ on $X = [L^2(0,1)]^2$ represented by

$$T(t)[\varphi_1, \varphi_2]^T = [z_1(t), z_2(t)]^T,$$

$$z_1(t) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \langle \varphi_1, \psi_n \rangle \phi_n \in L^2(0,1),$$

$$z_2(t) = \sum_{n=1}^{\infty} e^{-(\lambda_n - k_0)t} \langle b\varphi_1 + \varphi_2, \psi_n \rangle \phi_n - \sum_{n=1}^{\infty} e^{-\lambda_n t} \langle b\varphi_1, \psi_n \rangle \phi_n \in L^2(0,1).$$

In Winkin et al. [3] we have seen:

- The *approximate systems* for system (S) with $u_2 \equiv 0$ are reachable in some proper finite dimensional spaces.
- The necessary and sufficient conditions of observability for the *approximate* systems for system (S) are obtained.
- The obtained results are insufficient for the original system (S).

In this lecture, differently from [3], we discuss the reachabilty and observability of (S) and (O) *without approximating*. Our obtained results are complete and new as stated as below:

- The system (S) is always reachable. Further, the system with $u_2 \equiv 0$ is also reachable except for the *special values* of D, α, b, and k_0. For the *exceptional cases*, the unreachable subspaces are determined.
- For the observed system $(S)+(O)$, we can prove the observabilty results similar to those for the reachability.

(cf. Sano and Nakagiri [1]). We shall state the results for other types of reactor diffusion equations.

References

[1] H. Sano and S. Nakagiri: On reachability and observability of a plug-flow reactor diffusion equation; Transactions of the Institute of Systems, Control and Information Engineers, Vol. 22, No. 5, (2009), to appear

[2] H. Tanabe: Equations of Evolution, Pitman, London, (1979)

[3] J.J. Winkin, D. Dochain and P. Ligarius: Dynamical analysis of distributed parameter tubular reactors; Automatica, Vol. 36, pp. 349–361 (2000)