

# On the general transformation of the Wirtinger integral

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## §1 Introduction.

The Wirtinger integral is the uniformization to the upper half plane  $H$  of the hypergeometric function defined on the complex projective line  $\mathbf{P}^1$ . In [5] we established the transformation formulas of the Wirtinger integral for the linear fractional transformations  $\tau \rightarrow \tau + 2$  and  $\tau \rightarrow \frac{-1}{\tau}$  with the aide of the theory of theta functions. As a corollary we obtain the transformation formulas of the Wirtinger integral for the linear fractional transformations  $\tau \rightarrow \tau + 2$  and  $\tau \rightarrow \frac{\tau}{-2\tau + 1}$  which are identified with generators of the principal congruence subgroup  $\Gamma(2)$  modulo center. These formulas correspond to the monodromy matrices of the hypergeometric function for generators of the fundamental group of  $\mathbf{P}^1$  minus three points. The purpose of this paper is to generalize this result, namely, to establish the transformation formula of the Wirtinger integral for the general element  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  of  $\Gamma(2)$ , which corresponds to a general monodromy matrix of the hypergeometric function. It seems to be quite difficult to obtain such a formula directly from the hypergeometric function because we have no standard way of expressing a general element of the fundamental group corresponding to the matrix  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(2)$ .

Following the notation of Chandrasekharan [1], we introduce the four theta functions  $\theta(v, \tau), \theta_i(v, \tau)$  ( $i = 1, 2, 3$ ) by

$$\theta(v, \tau) = \frac{1}{i} \sum_{n=-\infty}^{+\infty} (-1)^n e^{(n+\frac{1}{2})^2 \pi i \tau} e^{(2n+1)\pi i v}, \quad \theta_1(v, \tau) = \sum_{n=-\infty}^{+\infty} e^{(n+\frac{1}{2})^2 \pi i \tau} e^{(2n+1)\pi i v}, \quad \theta_2(v, \tau) = \sum_{n=-\infty}^{+\infty} (-1)^n e^{n^2 \pi i \tau} e^{2n\pi i v}, \quad \theta_3(v, \tau) = \sum_{n=-\infty}^{+\infty} e^{n^2 \pi i \tau} e^{2n\pi i v},$$

which are defined for all  $(v, \tau) \in \mathbf{C} \times H$ , where  $\mathbf{C}$  denote the complex plane. Mumford [2] adopts the symbols  $\theta_{00}, \theta_{01}, \theta_{10}, \theta_{11}$  to denote the theta functions above. The relations between the two notations are as follows:  $\theta(v, \tau) = -\theta_{11}(v, \tau)$ ,  $\theta_1(v, \tau) = \theta_{10}(v, \tau)$ ,  $\theta_2(v, \tau) = \theta_{01}(v, \tau)$ ,  $\theta_3(v, \tau) = \theta_{00}(v, \tau)$ . In this paper we also use the following abbreviations:  $\theta_i(v) = \theta_i(v, \tau)$ ,  $\theta_i = \theta_i(0, \tau)$ , etc.

We define two functions  $z_1(\tau)$ ,  $z_2(\tau)$ , which we have been proposing to call *Wirtinger integrals* in our papers [5], [6] (see also [7]), by

$$z_1(\tau) = \theta_3^2 \int_0^{\frac{1}{2}} \theta(v, \tau)^{2\alpha-1} \theta_1(v, \tau)^{2\gamma-2\alpha-1} \theta_2(v, \tau)^{2\beta-2\gamma+3} \theta_3(v, \tau)^{-2\beta-1} dv,$$

$$z_2(\tau) = \theta_3^2 \int_0^{\frac{1}{2}} \theta(v, \tau)^{2\beta-2\gamma+3} \theta_1(v, \tau)^{-2\beta-1} \theta_2(v, \tau)^{2\alpha-1} \theta_3(v, \tau)^{2\gamma-2\alpha-1} dv,$$

where we assume that  $\alpha, \beta, \gamma$  satisfy certain conditions such that these integrals are convergent, and that  $\arg v = 0$  on the interval of integral  $0 < v < 1/2$ . They are the lifts of Gauss' hypergeometric functions of  $SL$  type to the upper half plane, and form a fundamental system of solutions for the lift of Gauss' hypergeometric differential equation of  $SL$  type to the upper half plane. Let  $\Gamma(2)$  be the principal congruence subgroup of level 2, and let  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  be an element

of  $\Gamma(2)$ . Without loss of generality we may assume  $0 < c < d$ . The problem which we will study in this paper is as follows:

**Problem 1.** Determine the constants  $A$  and  $B$  with respect to  $\tau$  such that  $z_1\left(\frac{a\tau+b}{c\tau+d}\right) = Az_1(\tau) + Bz_2(\tau)$ .

One can pose a similar problem for  $z_2\left(\frac{a\tau+b}{c\tau+d}\right)$ , of which we omit the detail in this paper. Since the group  $\Gamma(2)$  modulo center is isomorphic to the fundamental group of the Riemann sphere minus three points (which is the defining region of Gauss' hypergeometric differential equation), the constants  $A, B$  are identified with components of the monodromy matrix for the element of the fundamental group corresponding to the matrix  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ . Substitution  $\tau \rightarrow \frac{a\tau+b}{c\tau+d}$  and  $v \rightarrow \frac{v}{c\tau+d}$  makes the integral representation above for  $z_1(\tau)$  into

$$z_1\left(\frac{a\tau+b}{c\tau+d}\right) = \theta_3\left(0, \frac{a\tau+b}{c\tau+d}\right)^2 \int_0^{(c\tau+d)/2} \theta\left(\frac{v}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right)^{2\alpha-1} \theta_1\left(\frac{v}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right)^{2\gamma-2\alpha-1} \theta_2\left(\frac{v}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right)^{2\beta-2\gamma+3} \theta_3\left(\frac{v}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right)^{-2\beta-1} \frac{dv}{c\tau+d}.$$

The following formulas are well-known (e.g. [3], [4]):

$$\begin{aligned} \theta\left(\frac{v}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) &= \left(\frac{c}{d}\right) e^{\pi i(3d-2+bd-cd)/4} \sqrt{\frac{c\tau+d}{i}} e^{c\pi i v^2/(c\tau+d)} \theta(v, \tau), \\ \theta_1\left(\frac{v}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) &= \left(\frac{c}{d}\right) e^{\pi i(d-4+2c+bd-2cd)/4} \sqrt{\frac{c\tau+d}{i}} e^{c\pi i v^2/(c\tau+d)} \theta_1(v, \tau), \\ \theta_2\left(\frac{v}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) &= \left(\frac{c}{d}\right) e^{\pi i(3d-4+2a-ab+bd-cd)/4} \sqrt{\frac{c\tau+d}{i}} e^{c\pi i v^2/(c\tau+d)} \theta_2(v, \tau), \\ \theta_3\left(\frac{v}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) &= \left(\frac{c}{d}\right) e^{\pi i(3d-3+2a+2c-ab-ad-bc+bd-2cd)/4} \sqrt{\frac{c\tau+d}{i}} e^{c\pi i v^2/(c\tau+d)} \theta_3(v, \tau) \end{aligned}$$

for  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(2)$  with  $c > 0$ , where  $\left(\frac{c}{d}\right)$  denotes Jacobi's symbol. Substituting these formulas into the integral representation for  $z_1\left(\frac{a\tau+b}{c\tau+d}\right)$ , we have

$$z_1\left(\frac{a\tau+b}{c\tau+d}\right) = e^{\pi i d(b+c)/2} e^{\pi i \alpha} e^{\pi i d(\alpha-\gamma)} e^{\pi i c(2-d)(\gamma-\alpha-\beta)/2} e^{\pi i a(b-2)\gamma/2} \theta_3^2 \int_0^{(c\tau+d)/2} \theta(v)^{2\alpha-1} \theta_1(v)^{2\gamma-2\alpha-1} \theta_2(v)^{2\beta-2\gamma+3} \theta_3(v)^{-2\beta-1} dv \quad (1.1)$$

for  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(2)$  with  $c > 0$ . Noticing that the integers  $c, d$  are relatively prime to each other, we define integers  $k_\nu$  by

$$k_0 = 0, \quad k_\nu = \left\lceil \frac{d\nu}{c} \right\rceil + 1 \quad (1 \leq \nu \leq c-1), \quad k_c = d,$$

where for a real number  $x$  the symbol  $\lceil x \rceil$  denotes the maximal integer not exceeding to  $x$ .

The branch cuts  $L_\nu$  ( $\nu = 0, \dots, c$ ) of the integrand  $\theta(v)^{2\alpha-1} \theta_1(v)^{2\gamma-2\alpha-1} \theta_2(v)^{2\beta-2\gamma+3} \theta_3(v)^{-2\beta-1}$  of the integral in (1.1) are given as lines on  $\mathbf{C}$ . Namely,  $L_0$

is defined by the equation  $v = s$  for the real parameter  $s$  such that  $s \leq 0, \frac{1}{2} \leq s$ ; if  $\nu \geq 1$ ,  $L_\nu$  is defined by  $v = s + \frac{\nu}{2}\tau$  for the real parameter  $s$  such that  $s \leq \frac{k_\nu - 1}{2}, \frac{k_\nu}{2} \leq s$ . The integral in (1.1) is decomposed as follows:

$$\int_0^{(c\tau+d)/2} \theta(v)^{2\alpha-1} \theta_1(v)^{2\gamma-2\alpha-1} \theta_2(v)^{2\beta-2\gamma+3} \theta_3(v)^{-2\beta-1} dv = \sum_{\nu=0}^{c-1} (I_\nu + J_\nu), \quad (1.2)$$

where

$$I_\nu = \int_{(k_\nu+\nu\tau)/2}^{(k_{\nu+1}+\nu\tau)/2} \theta(v)^{2\alpha-1} \theta_1(v)^{2\gamma-2\alpha-1} \theta_2(v)^{2\beta-2\gamma+3} \theta_3(v)^{-2\beta-1} dv \quad (1.3)$$

and

$$J_\nu = \int_{(k_{\nu+1}+\nu\tau)/2}^{\{k_{\nu+1}+(\nu+1)\tau\}/2} \theta(v)^{2\alpha-1} \theta_1(v)^{2\gamma-2\alpha-1} \theta_2(v)^{2\beta-2\gamma+3} \theta_3(v)^{-2\beta-1} dv. \quad (1.4)$$

Furthermore, the integral  $I_\nu$  has the following decomposition:

$$I_\nu = \sum_{\mu=0}^{l_{\nu+1}-1} \int_0^{\frac{1}{2}} \theta\left(v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2}\tau\right)^{2\alpha-1} \theta_1\left(v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2}\tau\right)^{2\gamma-2\alpha-1} \theta_2\left(v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2}\tau\right)^{2\beta-2\gamma+3} \theta_3\left(v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2}\tau\right)^{-2\beta-1} dv, \quad (1.5)$$

where  $l_{\nu+1} = k_{\nu+1} - k_\nu$ . Thus, in order to solve Problem 1, it suffices to solve the following:

**Problem 2.** Determine the constants  $P_\nu, Q_\nu, R_\nu, S_\nu$  with respect to  $\tau$  such that  $I_\nu = P_\nu z_1(\tau)/\theta_3^2 + Q_\nu z_2(\tau)/\theta_3^2$  and  $J_\nu = R_\nu z_1(\tau)/\theta_3^2 + S_\nu z_2(\tau)/\theta_3^2$ .

In fact, the constants  $A$  and  $B$  in Problem 1 are written by

$$A = e^{\pi id(b+c)/2} e^{\pi i\alpha} e^{\pi id(\alpha-\gamma)} e^{\pi ic(2-d)(\gamma-\alpha-\beta)/2} e^{\pi ia(b-2)\gamma/2} \sum_{\nu=0}^{c-1} (P_\nu + R_\nu),$$

$$B = e^{\pi id(b+c)/2} e^{\pi i\alpha} e^{\pi id(\alpha-\gamma)} e^{\pi ic(2-d)(\gamma-\alpha-\beta)/2} e^{\pi ia(b-2)\gamma/2} \sum_{\nu=0}^{c-1} (Q_\nu + S_\nu).$$

**Theorem.** The integrals  $I_\nu$ 's and  $J_\nu$ 's in (1.2) are given as follows:  $I_0$  is given by (3.2),  $I_\nu$  for a positive even integer  $\nu$  ( $1 \leq \nu \leq c-1$ ) by (4.11),  $I_\nu$  for a positive odd integer  $\nu$  ( $1 \leq \nu \leq c-1$ ) by (5.10),  $J_0$  by (6.3),  $J_\nu$  for a positive even integer  $\nu$  ( $1 \leq \nu \leq c-1$ ) by (7.8), and  $J_\nu$  for a positive odd integer  $\nu$  ( $1 \leq \nu \leq c-1$ ) by (8.6).

The proof is given in Sections 2–8.

*Remark.* The linear fractional transformation of the general form  $\tau \rightarrow \frac{a\tau + b}{c\tau + d}$  with  $a, d$  odd and  $b, c$  even is composed of a finite number of the linear fractional transformations each of which is of the form  $\tau \rightarrow \tau + m$  with  $m$  even or  $\tau \rightarrow \frac{\tau}{n\tau + 1}$  with  $n$  even. According to the results of [5], the matrices  $M_0(m)$  and  $M_1(n)$

satisfying

$$\begin{pmatrix} z_1(\tau + m) \\ z_2(\tau + m) \end{pmatrix} = M_0(m) \begin{pmatrix} z_1(\tau) \\ z_2(\tau) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} z_1\left(\frac{\tau}{n\tau + 1}\right) \\ z_2\left(\frac{\tau}{n\tau + 1}\right) \end{pmatrix} = M_1(n) \begin{pmatrix} z_1(\tau) \\ z_2(\tau) \end{pmatrix}$$

are given by

$$M_0(m) = \begin{bmatrix} e^{m\pi i(\gamma-1)} & 0 \\ 0 & e^{m\pi i(1-\gamma)} \end{bmatrix}$$

and

$$M_1(n) = \begin{bmatrix} \frac{\sin \pi(\gamma - \alpha) \sin \pi(\gamma - \beta) e^{n\pi i(\alpha + \beta - \gamma + 1)} - \sin \pi \alpha \sin \pi \beta e^{n\pi i(\gamma - \alpha - \beta - 1)}}{\sin \pi \gamma \sin \pi(\gamma - \alpha - \beta)} & \frac{-2\pi i \Gamma(1 - \gamma) \Gamma(2 - \gamma) \sin n\pi(\gamma - \alpha - \beta - 1)}{\Gamma(-\alpha) \Gamma(-\beta) \Gamma(1 + \alpha - \gamma) \Gamma(1 + \beta - \gamma) \sin \pi(\gamma - \alpha - \beta)} \\ \frac{-2\pi i \Gamma(\gamma - 1) \Gamma(\gamma) \sin n\pi(\gamma - \alpha - \beta - 1)}{\Gamma(\alpha + 1) \Gamma(\beta + 1) \Gamma(\gamma - \alpha) \Gamma(\gamma - \beta) \sin \pi(\gamma - \alpha - \beta)} & \frac{\sin \pi(\gamma - \alpha) \sin \pi(\gamma - \beta) e^{n\pi i(\gamma - \alpha - \beta - 1)} - \sin \pi \alpha \sin \pi \beta e^{n\pi i(\alpha + \beta - \gamma + 1)}}{\sin \pi \gamma \sin \pi(\gamma - \alpha - \beta)} \end{bmatrix}.$$

Another way to obtain the results of Theorem is to calculate the components of a product of the matrices  $M_0(m)$  or  $M_1(n)$  which represents the monodromy matrix for the general linear fractional transformation  $\tau \rightarrow \frac{a\tau + b}{c\tau + d}$ . In this paper, however, we do not choose a way to make such a calculation because of its complicatedness.

## §2 Auxiliary formulas.

In this section we prove some auxiliary formulas which are applied to the study of the integrals  $I_\nu$  and  $J_\nu$ .

**Lemma 2.1.** Let  $p, q, r, s$  be complex constants. Then we have

$$\int_0^{\frac{1}{2}} \theta(u)^p \theta_1(u)^q \theta_2(u)^r \theta_3(u)^s du = \int_0^{\frac{1}{2}} \theta(u)^q \theta_1(u)^p \theta_2(u)^s \theta_3(u)^r du.$$

We omit the proof.

**Lemma 2.2.** We have:

(i)

$$\int_0^{\tau/2} \theta(u)^{2\alpha-1} \theta_1(u)^{2\gamma-2\alpha-1} \theta_2(u)^{2\beta-2\gamma+3} \theta_3(u)^{-2\beta-1} du = \frac{1 - e^{2\pi i(\alpha-\gamma)}}{1 - e^{-2\pi i\gamma}} (z_1(\tau)/\theta_3^2) + \frac{e^{\pi i(\alpha-\gamma)}(e^{\pi i\beta} - e^{-\pi i\beta})}{1 - e^{-2\pi i\gamma}} (z_2(\tau)/\theta_3^2).$$

(ii)

$$\int_0^{\tau/2} \theta(u)^{2\gamma-2\alpha-1} \theta_1(u)^{2\alpha-1} \theta_2(u)^{-2\beta-1} \theta_3(u)^{2\beta-2\gamma+3} du = \frac{1 - e^{-2\pi i\alpha}}{1 - e^{-2\pi i\gamma}} (z_1(\tau)/\theta_3^2) + \frac{e^{\pi i(-\alpha-\beta+\gamma)} - e^{\pi i(-\alpha+\beta-\gamma)}}{1 - e^{-2\pi i\gamma}} (z_2(\tau)/\theta_3^2).$$

*Proof.* The first formula follows from the equality

$$0 = \left( \int_{\tau/2}^0 + \int_0^{1/2} + \int_{1/2}^{(1+\tau)/2} + \int_{(1+\tau)/2}^{\tau/2} \right) \theta(u)^{2\alpha-1} \theta_1(u)^{2\gamma-2\alpha-1} \theta_2(u)^{2\beta-2\gamma+3} \theta_3(u)^{-2\beta-1} du,$$

and the second one from

$$0 = \left( \int_{\tau/2}^0 + \int_0^{1/2} + \int_{1/2}^{(1+\tau)/2} + \int_{(1+\tau)/2}^{\tau/2} \right) \theta(u)^{2\gamma-2\alpha-1} \theta_1(u)^{2\alpha-1} \theta_2(u)^{-2\beta-1} \theta_3(u)^{2\beta-2\gamma+3} du.$$

One can prove the following lemma similarly.

**Lemma 2.3.** We have:

(i)

$$\int_0^{\tau/2} \theta(u)^{2\beta-2\gamma+3} \theta_1(u)^{-2\beta-1} \theta_2(u)^{2\alpha-1} \theta_3(u)^{2\gamma-2\alpha-1} du = \frac{e^{\pi i(\alpha+\beta-\gamma)} - e^{\pi i(-\alpha+\beta+\gamma)}}{1 - e^{2\pi i\gamma}} (z_1(\tau)/\theta_3^2) + \frac{1 - e^{2\pi i\beta}}{1 - e^{2\pi i\gamma}} (z_2(\tau)/\theta_3^2).$$

(ii)

$$\int_0^{\tau/2} \theta(u)^{-2\beta-1} \theta_1(u)^{2\beta-2\gamma+3} \theta_2(u)^{2\gamma-2\alpha-1} \theta_3(u)^{2\alpha-1} du = \frac{e^{\pi i(-\alpha-\beta+\gamma)} - e^{\pi i(\alpha-\beta+\gamma)}}{1 - e^{2\pi i\gamma}} (z_1(\tau)/\theta_3^2) + \frac{1 - e^{2\pi i(\gamma-\beta)}}{1 - e^{2\pi i\gamma}} (z_2(\tau)/\theta_3^2).$$

We omit the proof.

### §3 The integral $I_0$ .

From (1.5) we have

$$I_0 = \sum_{\mu=0}^{k_1-1} \int_0^{1/2} \theta\left(v + \frac{\mu}{2}\right)^{2\alpha-1} \theta_1\left(v + \frac{\mu}{2}\right)^{2\gamma-2\alpha-1} \theta_2\left(v + \frac{\mu}{2}\right)^{2\beta-2\gamma+3} \theta_3\left(v + \frac{\mu}{2}\right)^{-2\beta-1} dv. \quad (3.1)$$

Note that for a non-negative integer  $\mu$

$$\begin{aligned} & \theta\left(v + \frac{\mu}{2}\right)^{2\alpha-1} \theta_1\left(v + \frac{\mu}{2}\right)^{2\gamma-2\alpha-1} \theta_2\left(v + \frac{\mu}{2}\right)^{2\beta-2\gamma+3} \theta_3\left(v + \frac{\mu}{2}\right)^{-2\beta-1} \\ &= \frac{1 + (-1)^\mu}{2} e^{\pi i(1-\gamma)\mu} \theta(v)^{2\alpha-1} \theta_1(v)^{2\gamma-2\alpha-1} \theta_2(v)^{2\beta-2\gamma+3} \theta_3(v)^{-2\beta-1} + \frac{1 + (-1)^{\mu+1}}{2} e^{\pi i(1-\gamma)(\mu-1)} e^{\pi i(2\alpha-2\gamma+1)} \theta(v)^{2\gamma-2\alpha-1} \theta_1(v)^{2\alpha-1} \theta_2(v)^{-2\beta-1} \theta_3(v)^{2\beta-2\gamma+3}. \end{aligned}$$

Substituting this expression into (3.1), we have

$$\begin{aligned} I_0 &= \sum_{\mu=0}^{k_1-1} \left\{ \frac{1 + (-1)^\mu}{2} e^{\pi i(1-\gamma)\mu} + \frac{1 + (-1)^{\mu+1}}{2} e^{\pi i(1-\gamma)\mu} e^{\pi i(2\alpha-\gamma)} \right\} \int_0^{1/2} \theta(v)^{2\alpha-1} \theta_1(v)^{2\gamma-2\alpha-1} \theta_2(v)^{2\beta-2\gamma+3} \theta_3(v)^{-2\beta-1} dv \\ &= \left\{ \frac{1 + e^{\pi i(2\alpha-\gamma)}}{2} \cdot \frac{1 - e^{\pi i(1-\gamma)k_1}}{1 - e^{\pi i(1-\gamma)}} + \frac{1 - e^{\pi i(2\alpha-\gamma)}}{2} \cdot \frac{1 - (-1)^{k_1} e^{\pi i(1-\gamma)k_1}}{1 + e^{\pi i(1-\gamma)}} \right\} \int_0^{1/2} \theta(v)^{2\alpha-1} \theta_1(v)^{2\gamma-2\alpha-1} \theta_2(v)^{2\beta-2\gamma+3} \theta_3(v)^{-2\beta-1} dv, \end{aligned}$$

from which it follows that

$$I_0 = e^{\pi i(2\alpha-\gamma)/2} \left\{ \cos \frac{\pi}{2}(\gamma - 2\alpha) \cdot \frac{1 - e^{\pi i(1-\gamma)k_1}}{1 - e^{\pi i(1-\gamma)}} + i \sin \frac{\pi}{2}(\gamma - 2\alpha) \cdot \frac{1 - (-1)^{k_1} e^{\pi i(1-\gamma)k_1}}{1 + e^{\pi i(1-\gamma)}} \right\} z_1(\tau)/\theta_3^2. \quad (3.2)$$

#### §4 The integrals $I_\nu$ for positive even integers $\nu$ .

In this section let  $\nu$  be a positive even integer. Let  $\left[ \theta \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2}\tau \right)^{2\alpha-1} \theta_1 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2}\tau \right)^{2\gamma-2\alpha-1} \theta_2 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2}\tau \right)^{2\beta-2\gamma+3} \theta_3 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2}\tau \right)^{-2\beta-1} \right]_{\pm}$

be the branches of  $\theta \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2}\tau \right)^{2\alpha-1} \theta_1 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2}\tau \right)^{2\gamma-2\alpha-1} \theta_2 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2}\tau \right)^{2\beta-2\gamma+3} \theta_3 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2}\tau \right)^{-2\beta-1}$  analytically continued on the real interval  $0 < v < \frac{1}{2}$  from the upper side for the plus sign and from the lower side for the minus sign, respectively, such that they coincide with each other on the real interval  $-(\mu+1)/2 < v < -\mu/2$ . Then we have

$$\begin{aligned} & \left[ \theta \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2}\tau \right)^{2\alpha-1} \theta_1 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2}\tau \right)^{2\gamma-2\alpha-1} \theta_2 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2}\tau \right)^{2\beta-2\gamma+3} \theta_3 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2}\tau \right)^{-2\beta-1} \right]_{+} \\ &= \frac{1 + (-1)^\mu}{2} e^{\pi i(1-\gamma)\mu} e^{\pi i(2\alpha-2\gamma+1)\mu} \theta \left( v + \frac{k_\nu - 1}{2} + \frac{\nu}{2}\tau \right)^{2\gamma-2\alpha-1} \theta_1 \left( v + \frac{k_\nu - 1}{2} + \frac{\nu}{2}\tau \right)^{2\alpha-1} \theta_2 \left( v + \frac{k_\nu - 1}{2} + \frac{\nu}{2}\tau \right)^{-2\beta-1} \theta_3 \left( v + \frac{k_\nu - 1}{2} + \frac{\nu}{2}\tau \right)^{2\beta-2\gamma+3} \\ &+ \frac{1 + (-1)^{\mu+1}}{2} e^{\pi i(1-\gamma)(\mu+1)} \theta \left( v + \frac{k_\nu - 1}{2} + \frac{\nu}{2}\tau \right)^{2\alpha-1} \theta_1 \left( v + \frac{k_\nu - 1}{2} + \frac{\nu}{2}\tau \right)^{2\gamma-2\alpha-1} \theta_2 \left( v + \frac{k_\nu - 1}{2} + \frac{\nu}{2}\tau \right)^{2\beta-2\gamma+3} \theta_3 \left( v + \frac{k_\nu - 1}{2} + \frac{\nu}{2}\tau \right)^{-2\beta-1} \end{aligned}$$

and

$$\begin{aligned} & \left[ \theta \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2}\tau \right)^{2\alpha-1} \theta_1 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2}\tau \right)^{2\gamma-2\alpha-1} \theta_2 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2}\tau \right)^{2\beta-2\gamma+3} \theta_3 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2}\tau \right)^{-2\beta-1} \right]_{-} \\ &= \frac{1 + (-1)^\mu}{2} e^{\pi i(\gamma-1)\mu} e^{\pi i(2\gamma-2\alpha-1)\mu} \theta \left( v + \frac{k_\nu - 1}{2} + \frac{\nu}{2}\tau \right)^{2\gamma-2\alpha-1} \theta_1 \left( v + \frac{k_\nu - 1}{2} + \frac{\nu}{2}\tau \right)^{2\alpha-1} \theta_2 \left( v + \frac{k_\nu - 1}{2} + \frac{\nu}{2}\tau \right)^{-2\beta-1} \theta_3 \left( v + \frac{k_\nu - 1}{2} + \frac{\nu}{2}\tau \right)^{2\beta-2\gamma+3} \\ &+ \frac{1 + (-1)^{\mu+1}}{2} e^{\pi i(\gamma-1)(\mu+1)} \theta \left( v + \frac{k_\nu - 1}{2} + \frac{\nu}{2}\tau \right)^{2\alpha-1} \theta_1 \left( v + \frac{k_\nu - 1}{2} + \frac{\nu}{2}\tau \right)^{2\gamma-2\alpha-1} \theta_2 \left( v + \frac{k_\nu - 1}{2} + \frac{\nu}{2}\tau \right)^{2\beta-2\gamma+3} \theta_3 \left( v + \frac{k_\nu - 1}{2} + \frac{\nu}{2}\tau \right)^{-2\beta-1}. \end{aligned}$$

Combining these relations, we have

$$\begin{aligned} & \left[ \theta \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2}\tau \right)^{2\alpha-1} \theta_1 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2}\tau \right)^{2\gamma-2\alpha-1} \theta_2 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2}\tau \right)^{2\beta-2\gamma+3} \theta_3 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2}\tau \right)^{-2\beta-1} \right]_{+} \\ &= \left\{ \frac{1 + (-1)^\mu}{2} e^{2\pi i(1-\gamma)\mu} e^{4\pi i(\alpha-\gamma)} + \frac{1 - (-1)^\mu}{2} e^{2\pi i(1-\gamma)(\mu+1)} \right\} \times \\ & \times \left[ \theta \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2}\tau \right)^{2\alpha-1} \theta_1 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2}\tau \right)^{2\gamma-2\alpha-1} \theta_2 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2}\tau \right)^{2\beta-2\gamma+3} \theta_3 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2}\tau \right)^{-2\beta-1} \right]_{-}. \end{aligned}$$

Now we have

$$\begin{aligned} & \left[ \theta \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2} \tau \right)^{2\alpha-1} \theta_1 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2} \tau \right)^{2\gamma-2\alpha-1} \theta_2 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2} \tau \right)^{2\beta-2\gamma+3} \theta_3 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2} \tau \right)^{-2\beta-1} \right]_- \\ & = e^{\pi i(\alpha+\beta-\gamma+1)} \left[ \theta \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu-1}{2} \tau \right)^{2\beta-2\gamma+3} \theta_1 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu-1}{2} \tau \right)^{-2\beta-1} \theta_2 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu-1}{2} \tau \right)^{2\alpha-1} \theta_3 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu-1}{2} \tau \right)^{2\gamma-2\alpha-1} \right]_+ . \end{aligned}$$

Combining the preceding two relations, we have

$$\begin{aligned} & \left[ \theta \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2} \tau \right)^{2\alpha-1} \theta_1 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2} \tau \right)^{2\gamma-2\alpha-1} \theta_2 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2} \tau \right)^{2\beta-2\gamma+3} \theta_3 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2} \tau \right)^{-2\beta-1} \right]_+ \\ & = \left\{ \frac{1+(-1)^\mu}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1-(-1)^\mu}{2} e^{2\pi i(1-\gamma)} \right\} e^{2\pi i(1-\gamma)\mu} e^{\pi i(\alpha+\beta-\gamma+1)} \times \\ & \quad \times \left[ \theta \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu-1}{2} \tau \right)^{2\beta-2\gamma+3} \theta_1 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu-1}{2} \tau \right)^{-2\beta-1} \theta_2 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu-1}{2} \tau \right)^{2\alpha-1} \theta_3 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu-1}{2} \tau \right)^{2\gamma-2\alpha-1} \right]_+ . \end{aligned} \quad (4.1)$$

On the other hand, we have

$$\begin{aligned} & \left[ \theta \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu-1}{2} \tau \right)^{2\beta-2\gamma+3} \theta_1 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu-1}{2} \tau \right)^{-2\beta-1} \theta_2 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu-1}{2} \tau \right)^{2\alpha-1} \theta_3 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu-1}{2} \tau \right)^{2\gamma-2\alpha-1} \right]_+ \\ & = \frac{1+(-1)^{\mu+l_\nu}}{2} e^{\pi i(\gamma-1)(\mu+l_\nu)} e^{\pi i(2\beta+1)} \theta \left( v + \frac{k_{\nu-1}-1}{2} + \frac{\nu-1}{2} \tau \right)^{-2\beta-1} \theta_1 \left( v + \frac{k_{\nu-1}-1}{2} + \frac{\nu-1}{2} \tau \right)^{2\beta-2\gamma+3} \theta_2 \left( v + \frac{k_{\nu-1}-1}{2} + \frac{\nu-1}{2} \tau \right)^{2\gamma-2\alpha-1} \theta_3 \left( v + \frac{k_{\nu-1}-1}{2} + \frac{\nu-1}{2} \tau \right)^{2\alpha-1} \\ & + \frac{1+(-1)^{\mu+l_\nu+1}}{2} e^{\pi i(\gamma-1)(\mu+l_\nu+1)} \theta \left( v + \frac{k_{\nu-1}-1}{2} + \frac{\nu-1}{2} \tau \right)^{2\beta-2\gamma+3} \theta_1 \left( v + \frac{k_{\nu-1}-1}{2} + \frac{\nu-1}{2} \tau \right)^{-2\beta-1} \theta_2 \left( v + \frac{k_{\nu-1}-1}{2} + \frac{\nu-1}{2} \tau \right)^{2\alpha-1} \theta_3 \left( v + \frac{k_{\nu-1}-1}{2} + \frac{\nu-1}{2} \tau \right)^{2\gamma-2\alpha-1} \end{aligned}$$

and

$$\begin{aligned} & \left[ \theta \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu-1}{2} \tau \right)^{2\beta-2\gamma+3} \theta_1 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu-1}{2} \tau \right)^{-2\beta-1} \theta_2 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu-1}{2} \tau \right)^{2\alpha-1} \theta_3 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu-1}{2} \tau \right)^{2\gamma-2\alpha-1} \right]_- \\ & = \frac{1+(-1)^{\mu+l_\nu}}{2} e^{\pi i(1-\gamma)(\mu+l_\nu)} e^{-\pi i(2\beta+1)} \theta \left( v + \frac{k_{\nu-1}-1}{2} + \frac{\nu-1}{2} \tau \right)^{-2\beta-1} \theta_1 \left( v + \frac{k_{\nu-1}-1}{2} + \frac{\nu-1}{2} \tau \right)^{2\beta-2\gamma+3} \theta_2 \left( v + \frac{k_{\nu-1}-1}{2} + \frac{\nu-1}{2} \tau \right)^{2\gamma-2\alpha-1} \theta_3 \left( v + \frac{k_{\nu-1}-1}{2} + \frac{\nu-1}{2} \tau \right)^{2\alpha-1} \\ & + \frac{1+(-1)^{\mu+l_\nu+1}}{2} e^{\pi i(1-\gamma)(\mu+l_\nu+1)} \theta \left( v + \frac{k_{\nu-1}-1}{2} + \frac{\nu-1}{2} \tau \right)^{2\beta-2\gamma+3} \theta_1 \left( v + \frac{k_{\nu-1}-1}{2} + \frac{\nu-1}{2} \tau \right)^{-2\beta-1} \theta_2 \left( v + \frac{k_{\nu-1}-1}{2} + \frac{\nu-1}{2} \tau \right)^{2\alpha-1} \theta_3 \left( v + \frac{k_{\nu-1}-1}{2} + \frac{\nu-1}{2} \tau \right)^{2\gamma-2\alpha-1} . \end{aligned}$$

Combining these two relations, we have

$$\begin{aligned}
& \left[ \theta \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu - 1}{2} \tau \right)^{2\beta - 2\gamma + 3} \theta_1 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu - 1}{2} \tau \right)^{-2\beta - 1} \theta_2 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu - 1}{2} \tau \right)^{2\alpha - 1} \theta_3 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu - 1}{2} \tau \right)^{2\gamma - 2\alpha - 1} \right]_+ \\
&= \left\{ \frac{1 + (-1)^{\mu + l_\nu}}{2} e^{4\pi i \beta} + \frac{1 - (-1)^{\mu + l_\nu}}{2} e^{2\pi i (\gamma - 1)} \right\} e^{2\pi i (\gamma - 1)(\mu + l_\nu)} \times \\
&\times \left[ \theta \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu - 1}{2} \tau \right)^{2\beta - 2\gamma + 3} \theta_1 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu - 1}{2} \tau \right)^{-2\beta - 1} \theta_2 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu - 1}{2} \tau \right)^{2\alpha - 1} \theta_3 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu - 1}{2} \tau \right)^{2\gamma - 2\alpha - 1} \right]_-.
\end{aligned}$$

Now we have

$$\begin{aligned}
& \left[ \theta \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu - 1}{2} \tau \right)^{2\beta - 2\gamma + 3} \theta_1 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu - 1}{2} \tau \right)^{-2\beta - 1} \theta_2 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu - 1}{2} \tau \right)^{2\alpha - 1} \theta_3 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu - 1}{2} \tau \right)^{2\gamma - 2\alpha - 1} \right]_- \\
&= e^{\pi i (\alpha + \beta - \gamma + 1)} \left[ \theta \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu - 2}{2} \tau \right)^{2\alpha - 1} \theta_1 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu - 2}{2} \tau \right)^{2\gamma - 2\alpha - 1} \theta_2 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu - 2}{2} \tau \right)^{2\beta - 2\gamma + 3} \theta_3 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu - 2}{2} \tau \right)^{-2\beta - 1} \right]_+.
\end{aligned}$$

Combining the preceding two relations, we have

$$\begin{aligned}
& \left[ \theta \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu - 1}{2} \tau \right)^{2\beta - 2\gamma + 3} \theta_1 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu - 1}{2} \tau \right)^{-2\beta - 1} \theta_2 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu - 1}{2} \tau \right)^{2\alpha - 1} \theta_3 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu - 1}{2} \tau \right)^{2\gamma - 2\alpha - 1} \right]_+ \\
&= \left\{ \frac{1 + (-1)^{\mu + l_\nu}}{2} e^{4\pi i \beta} + \frac{1 - (-1)^{\mu + l_\nu}}{2} e^{2\pi i (\gamma - 1)} \right\} e^{2\pi i (\gamma - 1)(\mu + l_\nu)} e^{\pi i (\alpha + \beta - \gamma + 1)} \times \\
&\times \left[ \theta \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu - 2}{2} \tau \right)^{2\alpha - 1} \theta_1 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu - 2}{2} \tau \right)^{2\gamma - 2\alpha - 1} \theta_2 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu - 2}{2} \tau \right)^{2\beta - 2\gamma + 3} \theta_3 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu - 2}{2} \tau \right)^{-2\beta - 1} \right]_+.
\end{aligned} \tag{4.2}$$

From (4.1) and (4.2) we have

$$\begin{aligned}
& \left[ \theta \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2} \tau \right)^{2\alpha - 1} \theta_1 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2} \tau \right)^{2\gamma - 2\alpha - 1} \theta_2 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2} \tau \right)^{2\beta - 2\gamma + 3} \theta_3 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2} \tau \right)^{-2\beta - 1} \right]_+ \\
&= \left\{ \frac{1 + (-1)^\mu}{2} e^{4\pi i (\alpha - \gamma)} + \frac{1 - (-1)^\mu}{2} e^{2\pi i (1 - \gamma)} \right\} \left\{ \frac{1 + (-1)^{\mu + l_\nu}}{2} e^{4\pi i \beta} + \frac{1 - (-1)^{\mu + l_\nu}}{2} e^{2\pi i (\gamma - 1)} \right\} e^{2\pi i (\gamma - 1) l_\nu} e^{2\pi i (\alpha + \beta - \gamma + 1)} \times \\
&\times \left[ \theta \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu - 2}{2} \tau \right)^{2\alpha - 1} \theta_1 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu - 2}{2} \tau \right)^{2\gamma - 2\alpha - 1} \theta_2 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu - 2}{2} \tau \right)^{2\beta - 2\gamma + 3} \theta_3 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu - 2}{2} \tau \right)^{-2\beta - 1} \right]_+.
\end{aligned} \tag{4.3}$$

Repeating the same procedure, we arrive at the following



**Lemma 4.1.** For a positive even integer  $\nu$  we have

$$\begin{aligned}
& \left[ \theta \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2} \tau \right)^{2\alpha-1} \theta_1 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2} \tau \right)^{2\gamma-2\alpha-1} \theta_2 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2} \tau \right)^{2\beta-2\gamma+3} \theta_3 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2} \tau \right)^{-2\beta-1} \right]_+ \\
&= \left\{ \frac{1 + (-1)^\mu}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1 - (-1)^\mu}{2} e^{2\pi i(1-\gamma)} \right\} \left\{ \frac{1 + (-1)^{\mu+l_\nu}}{2} e^{4\pi i\beta} + \frac{1 - (-1)^{\mu+l_\nu}}{2} e^{2\pi i(\gamma-1)} \right\} \left\{ \frac{1 + (-1)^{\mu+l_\nu+l_{\nu-1}}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1 - (-1)^{\mu+l_\nu+l_{\nu-1}}}{2} e^{2\pi i(1-\gamma)} \right\} \times \\
&\times \left\{ \frac{1 + (-1)^{\mu+l_\nu+l_{\nu-1}+l_{\nu-2}}}{2} e^{4\pi i\beta} + \frac{1 - (-1)^{\mu+l_\nu+l_{\nu-1}+l_{\nu-2}}}{2} e^{2\pi i(\gamma-1)} \right\} \dots \left\{ \frac{1 + (-1)^{\mu+l_\nu+l_{\nu-1}+\dots+l_3}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1 - (-1)^{\mu+l_\nu+l_{\nu-1}+\dots+l_3}}{2} e^{2\pi i(1-\gamma)} \right\} \times \quad (4.4) \\
&\times \left\{ \frac{1 + (-1)^{\mu+l_\nu+l_{\nu-1}+\dots+l_2}}{2} e^{4\pi i\beta} + \frac{1 - (-1)^{\mu+l_\nu+l_{\nu-1}+\dots+l_2}}{2} e^{2\pi i(\gamma-1)} \right\} e^{2\pi i(\gamma-1)(l_\nu+l_{\nu-2}+\dots+l_4+l_2)} e^{\nu\pi i(\alpha+\beta-\gamma+1)} \times \\
&\times \left[ \theta \left( v + \frac{k_\nu + \mu}{2} \right)^{2\alpha-1} \theta_1 \left( v + \frac{k_\nu + \mu}{2} \right)^{2\gamma-2\alpha-1} \theta_2 \left( v + \frac{k_\nu + \mu}{2} \right)^{2\beta-2\gamma+3} \theta_3 \left( v + \frac{k_\nu + \mu}{2} \right)^{-2\beta-1} \right]_+.
\end{aligned}$$

We are now in a position to compute  $I_\nu$  for a positive even integer  $\nu$ . Substituting (4.4) into (1.5), we have

$$\begin{aligned}
I_\nu &= \sum_{\mu=0}^{l_\nu+1-1} \left\{ \frac{1 + (-1)^\mu}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1 - (-1)^\mu}{2} e^{2\pi i(1-\gamma)} \right\} \left\{ \frac{1 + (-1)^{\mu+l_\nu}}{2} e^{4\pi i\beta} + \frac{1 - (-1)^{\mu+l_\nu}}{2} e^{2\pi i(\gamma-1)} \right\} \left\{ \frac{1 + (-1)^{\mu+l_\nu+l_{\nu-1}}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1 - (-1)^{\mu+l_\nu+l_{\nu-1}}}{2} e^{2\pi i(1-\gamma)} \right\} \times \\
&\times \left\{ \frac{1 + (-1)^{\mu+l_\nu+l_{\nu-1}+l_{\nu-2}}}{2} e^{4\pi i\beta} + \frac{1 - (-1)^{\mu+l_\nu+l_{\nu-1}+l_{\nu-2}}}{2} e^{2\pi i(\gamma-1)} \right\} \dots \left\{ \frac{1 + (-1)^{\mu+l_\nu+l_{\nu-1}+\dots+l_3}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1 - (-1)^{\mu+l_\nu+l_{\nu-1}+\dots+l_3}}{2} e^{2\pi i(1-\gamma)} \right\} \times \quad (4.5) \\
&\times \left\{ \frac{1 + (-1)^{\mu+l_\nu+l_{\nu-1}+\dots+l_2}}{2} e^{4\pi i\beta} + \frac{1 - (-1)^{\mu+l_\nu+l_{\nu-1}+\dots+l_2}}{2} e^{2\pi i(\gamma-1)} \right\} e^{2\pi i(\gamma-1)(l_\nu+l_{\nu-2}+\dots+l_4+l_2)} e^{\nu\pi i(\alpha+\beta-\gamma+1)} \times \\
&\times \int_0^{\frac{1}{2}} \theta \left( v + \frac{k_\nu + \mu}{2} \right)^{2\alpha-1} \theta_1 \left( v + \frac{k_\nu + \mu}{2} \right)^{2\gamma-2\alpha-1} \theta_2 \left( v + \frac{k_\nu + \mu}{2} \right)^{2\beta-2\gamma+3} \theta_3 \left( v + \frac{k_\nu + \mu}{2} \right)^{-2\beta-1} dv.
\end{aligned}$$

Since

$$\begin{aligned}
& \theta \left( v + \frac{k_\nu + \mu}{2} \right)^{2\alpha-1} \theta_1 \left( v + \frac{k_\nu + \mu}{2} \right)^{2\gamma-2\alpha-1} \theta_2 \left( v + \frac{k_\nu + \mu}{2} \right)^{2\beta-2\gamma+3} \theta_3 \left( v + \frac{k_\nu + \mu}{2} \right)^{-2\beta-1} \\
&= \frac{1 + (-1)^\mu}{2} e^{\pi i(1-\gamma)\mu} \theta \left( v + \frac{k_\nu}{2} \right)^{2\alpha-1} \theta_1 \left( v + \frac{k_\nu}{2} \right)^{2\gamma-2\alpha-1} \theta_2 \left( v + \frac{k_\nu}{2} \right)^{2\beta-2\gamma+3} \theta_3 \left( v + \frac{k_\nu}{2} \right)^{-2\beta-1} \\
&\quad + \frac{1 - (-1)^\mu}{2} e^{\pi i(1-\gamma)(\mu-1)} e^{\pi i(2\alpha-2\gamma+1)} \theta \left( v + \frac{k_\nu}{2} \right)^{2\gamma-2\alpha-1} \theta_1 \left( v + \frac{k_\nu}{2} \right)^{2\alpha-1} \theta_2 \left( v + \frac{k_\nu}{2} \right)^{-2\beta-1} \theta_3 \left( v + \frac{k_\nu}{2} \right)^{2\beta-2\gamma+3},
\end{aligned}$$

(4.5) is turned to

$$\begin{aligned}
I_\nu = & \sum_{\mu=0}^{l_{\nu+1}-1} \left\{ \frac{1+(-1)^\mu}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1-(-1)^\mu}{2} e^{2\pi i(1-\gamma)} \right\} \left\{ \frac{1+(-1)^{\mu+l_\nu}}{2} e^{4\pi i\beta} + \frac{1-(-1)^{\mu+l_\nu}}{2} e^{2\pi i(\gamma-1)} \right\} \left\{ \frac{1+(-1)^{\mu+l_\nu+l_{\nu-1}}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1-(-1)^{\mu+l_\nu+l_{\nu-1}}}{2} e^{2\pi i(1-\gamma)} \right\} \times \\
& \times \left\{ \frac{1+(-1)^{\mu+l_\nu+l_{\nu-1}+l_{\nu-2}}}{2} e^{4\pi i\beta} + \frac{1-(-1)^{\mu+l_\nu+l_{\nu-1}+l_{\nu-2}}}{2} e^{2\pi i(\gamma-1)} \right\} \dots \left\{ \frac{1+(-1)^{\mu+l_\nu+l_{\nu-1}+\dots+l_3}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1-(-1)^{\mu+l_\nu+l_{\nu-1}+\dots+l_3}}{2} e^{2\pi i(1-\gamma)} \right\} \times \\
& \times \left\{ \frac{1+(-1)^{\mu+l_\nu+l_{\nu-1}+\dots+l_2}}{2} e^{4\pi i\beta} + \frac{1-(-1)^{\mu+l_\nu+l_{\nu-1}+\dots+l_2}}{2} e^{2\pi i(\gamma-1)} \right\} e^{2\pi i(\gamma-1)(l_\nu+l_{\nu-2}+\dots+l_4+l_2)} e^{\nu\pi i(\alpha+\beta-\gamma+1)} \times \\
& \times \left\{ \frac{1+(-1)^\mu}{2} e^{\pi i(1-\gamma)\mu} \int_0^{\frac{1}{2}} \theta \left( v + \frac{k_\nu}{2} \right)^{2\alpha-1} \theta_1 \left( v + \frac{k_\nu}{2} \right)^{2\gamma-2\alpha-1} \theta_2 \left( v + \frac{k_\nu}{2} \right)^{2\beta-2\gamma+3} \theta_3 \left( v + \frac{k_\nu}{2} \right)^{-2\beta-1} dv \right. \\
& \left. + \frac{1-(-1)^\mu}{2} e^{\pi i(1-\gamma)(\mu-1)} e^{\pi i(2\alpha-2\gamma+1)} \int_0^{\frac{1}{2}} \theta \left( v + \frac{k_\nu}{2} \right)^{2\gamma-2\alpha-1} \theta_1 \left( v + \frac{k_\nu}{2} \right)^{2\alpha-1} \theta_2 \left( v + \frac{k_\nu}{2} \right)^{-2\beta-1} \theta_3 \left( v + \frac{k_\nu}{2} \right)^{2\beta-2\gamma+3} dv \right\},
\end{aligned}$$

from which it follows that

$$\begin{aligned}
I_\nu = & \sum_{\mu=0}^{l_{\nu+1}-1} e^{4\pi i(\alpha-\gamma)} \left\{ \frac{1+(-1)^{l_\nu}}{2} e^{4\pi i\beta} + \frac{1-(-1)^{l_\nu}}{2} e^{2\pi i(\gamma-1)} \right\} \left\{ \frac{1+(-1)^{l_\nu+l_{\nu-1}}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1-(-1)^{l_\nu+l_{\nu-1}}}{2} e^{2\pi i(1-\gamma)} \right\} \times \\
& \times \left\{ \frac{1+(-1)^{l_\nu+l_{\nu-1}+l_{\nu-2}}}{2} e^{4\pi i\beta} + \frac{1-(-1)^{l_\nu+l_{\nu-1}+l_{\nu-2}}}{2} e^{2\pi i(\gamma-1)} \right\} \dots \left\{ \frac{1+(-1)^{l_\nu+l_{\nu-1}+\dots+l_3}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1-(-1)^{l_\nu+l_{\nu-1}+\dots+l_3}}{2} e^{2\pi i(1-\gamma)} \right\} \times \\
& \times \left\{ \frac{1+(-1)^{l_\nu+l_{\nu-1}+\dots+l_2}}{2} e^{4\pi i\beta} + \frac{1-(-1)^{l_\nu+l_{\nu-1}+\dots+l_2}}{2} e^{2\pi i(\gamma-1)} \right\} e^{2\pi i(\gamma-1)(l_\nu+l_{\nu-2}+\dots+l_4+l_2)} e^{\nu\pi i(\alpha+\beta-\gamma+1)} \times \\
& \times \frac{1+(-1)^\mu}{2} e^{\pi i(1-\gamma)\mu} \int_0^{\frac{1}{2}} \theta \left( v + \frac{k_\nu}{2} \right)^{2\alpha-1} \theta_1 \left( v + \frac{k_\nu}{2} \right)^{2\gamma-2\alpha-1} \theta_2 \left( v + \frac{k_\nu}{2} \right)^{2\beta-2\gamma+3} \theta_3 \left( v + \frac{k_\nu}{2} \right)^{-2\beta-1} dv \\
& + \sum_{\mu=0}^{l_{\nu+1}-1} e^{2\pi i(1-\gamma)} \left\{ \frac{1-(-1)^{l_\nu}}{2} e^{4\pi i\beta} + \frac{1+(-1)^{l_\nu}}{2} e^{2\pi i(\gamma-1)} \right\} \left\{ \frac{1-(-1)^{l_\nu+l_{\nu-1}}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1+(-1)^{l_\nu+l_{\nu-1}}}{2} e^{2\pi i(1-\gamma)} \right\} \times \\
& \times \left\{ \frac{1-(-1)^{l_\nu+l_{\nu-1}+l_{\nu-2}}}{2} e^{4\pi i\beta} + \frac{1+(-1)^{l_\nu+l_{\nu-1}+l_{\nu-2}}}{2} e^{2\pi i(\gamma-1)} \right\} \dots \left\{ \frac{1-(-1)^{l_\nu+l_{\nu-1}+\dots+l_3}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1+(-1)^{l_\nu+l_{\nu-1}+\dots+l_3}}{2} e^{2\pi i(1-\gamma)} \right\} \times \\
& \times \left\{ \frac{1-(-1)^{l_\nu+l_{\nu-1}+\dots+l_2}}{2} e^{4\pi i\beta} + \frac{1+(-1)^{l_\nu+l_{\nu-1}+\dots+l_2}}{2} e^{2\pi i(\gamma-1)} \right\} e^{2\pi i(\gamma-1)(l_\nu+l_{\nu-2}+\dots+l_4+l_2)} e^{\nu\pi i(\alpha+\beta-\gamma+1)} \times \\
& \times \frac{1-(-1)^\mu}{2} e^{\pi i(1-\gamma)\mu} e^{\pi i(2\alpha-\gamma)} \int_0^{\frac{1}{2}} \theta \left( v + \frac{k_\nu}{2} \right)^{2\gamma-2\alpha-1} \theta_1 \left( v + \frac{k_\nu}{2} \right)^{2\alpha-1} \theta_2 \left( v + \frac{k_\nu}{2} \right)^{-2\beta-1} \theta_3 \left( v + \frac{k_\nu}{2} \right)^{2\beta-2\gamma+3} dv.
\end{aligned} \tag{4.6}$$

Substituting

$$\sum_{\mu=0}^{l_{\nu}+1-1} \frac{1 \pm (-1)^{\mu}}{2} e^{\pi i(1-\gamma)\mu} = \frac{1}{2} \left\{ \frac{1 - e^{\pi i(1-\gamma)l_{\nu}+1}}{1 - e^{\pi i(1-\gamma)}} \pm \frac{1 - (-1)^{l_{\nu}+1} e^{\pi i(1-\gamma)l_{\nu}+1}}{1 + e^{\pi i(1-\gamma)}} \right\}$$

into (4.6), we have

$$\begin{aligned} I_{\nu} = & \left\{ \frac{1 + (-1)^{l_{\nu}}}{2} e^{4\pi i\beta} + \frac{1 - (-1)^{l_{\nu}}}{2} e^{2\pi i(\gamma-1)} \right\} \left\{ \frac{1 + (-1)^{l_{\nu}+l_{\nu}-1}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1 - (-1)^{l_{\nu}+l_{\nu}-1}}{2} e^{2\pi i(1-\gamma)} \right\} \times \\ & \times \left\{ \frac{1 + (-1)^{l_{\nu}+l_{\nu}-1+l_{\nu}-2}}{2} e^{4\pi i\beta} + \frac{1 - (-1)^{l_{\nu}+l_{\nu}-1+l_{\nu}-2}}{2} e^{2\pi i(\gamma-1)} \right\} \dots \left\{ \frac{1 + (-1)^{l_{\nu}+l_{\nu}-1+\dots+l_3}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1 - (-1)^{l_{\nu}+l_{\nu}-1+\dots+l_3}}{2} e^{2\pi i(1-\gamma)} \right\} \times \\ & \times \left\{ \frac{1 + (-1)^{l_{\nu}+l_{\nu}-1+\dots+l_2}}{2} e^{4\pi i\beta} + \frac{1 - (-1)^{l_{\nu}+l_{\nu}-1+\dots+l_2}}{2} e^{2\pi i(\gamma-1)} \right\} e^{4\pi i(\alpha-\gamma)} e^{2\pi i(\gamma-1)(l_{\nu}+l_{\nu}-2+\dots+l_4+l_2)} e^{\nu\pi i(\alpha+\beta-\gamma+1)} \times \\ & \times \frac{1}{2} \left\{ \frac{1 - e^{\pi i(1-\gamma)l_{\nu}+1}}{1 - e^{\pi i(1-\gamma)}} + \frac{1 - (-1)^{l_{\nu}+1} e^{\pi i(1-\gamma)l_{\nu}+1}}{1 + e^{\pi i(1-\gamma)}} \right\} \int_0^{\frac{1}{2}} \theta \left( v + \frac{k_{\nu}}{2} \right)^{2\alpha-1} \theta_1 \left( v + \frac{k_{\nu}}{2} \right)^{2\gamma-2\alpha-1} \theta_2 \left( v + \frac{k_{\nu}}{2} \right)^{2\beta-2\gamma+3} \theta_3 \left( v + \frac{k_{\nu}}{2} \right)^{-2\beta-1} dv \\ & + \left\{ \frac{1 - (-1)^{l_{\nu}}}{2} e^{4\pi i\beta} + \frac{1 + (-1)^{l_{\nu}}}{2} e^{2\pi i(\gamma-1)} \right\} \left\{ \frac{1 - (-1)^{l_{\nu}+l_{\nu}-1}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1 + (-1)^{l_{\nu}+l_{\nu}-1}}{2} e^{2\pi i(1-\gamma)} \right\} \times \\ & \times \left\{ \frac{1 - (-1)^{l_{\nu}+l_{\nu}-1+l_{\nu}-2}}{2} e^{4\pi i\beta} + \frac{1 + (-1)^{l_{\nu}+l_{\nu}-1+l_{\nu}-2}}{2} e^{2\pi i(\gamma-1)} \right\} \dots \left\{ \frac{1 - (-1)^{l_{\nu}+l_{\nu}-1+\dots+l_3}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1 + (-1)^{l_{\nu}+l_{\nu}-1+\dots+l_3}}{2} e^{2\pi i(1-\gamma)} \right\} \times \\ & \times \left\{ \frac{1 - (-1)^{l_{\nu}+l_{\nu}-1+\dots+l_2}}{2} e^{4\pi i\beta} + \frac{1 + (-1)^{l_{\nu}+l_{\nu}-1+\dots+l_2}}{2} e^{2\pi i(\gamma-1)} \right\} e^{2\pi i(1-\gamma)} e^{2\pi i(\gamma-1)(l_{\nu}+l_{\nu}-2+\dots+l_4+l_2)} e^{\nu\pi i(\alpha+\beta-\gamma+1)} e^{\pi i(2\alpha-\gamma)} \times \\ & \times \frac{1}{2} \left\{ \frac{1 - e^{\pi i(1-\gamma)l_{\nu}+1}}{1 - e^{\pi i(1-\gamma)}} - \frac{1 - (-1)^{l_{\nu}+1} e^{\pi i(1-\gamma)l_{\nu}+1}}{1 + e^{\pi i(1-\gamma)}} \right\} \int_0^{\frac{1}{2}} \theta \left( v + \frac{k_{\nu}}{2} \right)^{2\gamma-2\alpha-1} \theta_1 \left( v + \frac{k_{\nu}}{2} \right)^{2\alpha-1} \theta_2 \left( v + \frac{k_{\nu}}{2} \right)^{-2\beta-1} \theta_3 \left( v + \frac{k_{\nu}}{2} \right)^{2\beta-2\gamma+3} dv, \end{aligned}$$

which is turned by some elementary calculation to

$$\begin{aligned} I_{\nu} = & e^{\pi i} \left[ \{ (-1)^{k_{\nu}-k_{\nu}-1} + (-1)^{k_{\nu}-k_{\nu}-3} + \dots + (-1)^{k_{\nu}-k_1} \} (2\beta-\gamma+1) + \{ (-1)^{k_{\nu}-k_{\nu}-2} + (-1)^{k_{\nu}-k_{\nu}-4} + \dots + (-1)^{k_{\nu}-k_2} \} (2\alpha-\gamma-1) \right] e^{2\pi i(\alpha-\gamma)} e^{\pi i(\gamma-1)} e^{2\pi i(\gamma-1)(l_{\nu}+l_{\nu}-2+\dots+l_4+l_2)} e^{2\nu\pi i(\alpha+\beta-\gamma)} \times \\ & \times \frac{1}{2} \left\{ \frac{1 - e^{\pi i(1-\gamma)l_{\nu}+1}}{1 - e^{\pi i(1-\gamma)}} + \frac{1 - (-1)^{l_{\nu}+1} e^{\pi i(1-\gamma)l_{\nu}+1}}{1 + e^{\pi i(1-\gamma)}} \right\} \int_0^{\frac{1}{2}} \theta \left( v + \frac{k_{\nu}}{2} \right)^{2\alpha-1} \theta_1 \left( v + \frac{k_{\nu}}{2} \right)^{2\gamma-2\alpha-1} \theta_2 \left( v + \frac{k_{\nu}}{2} \right)^{2\beta-2\gamma+3} \theta_3 \left( v + \frac{k_{\nu}}{2} \right)^{-2\beta-1} dv \\ & + e^{-\pi i} \left[ \{ (-1)^{k_{\nu}-k_{\nu}-1} + (-1)^{k_{\nu}-k_{\nu}-3} + \dots + (-1)^{k_{\nu}-k_1} \} (2\beta-\gamma+1) + \{ (-1)^{k_{\nu}-k_{\nu}-2} + (-1)^{k_{\nu}-k_{\nu}-4} + \dots + (-1)^{k_{\nu}-k_2} \} (2\alpha-\gamma-1) \right] e^{2\pi i(\gamma-\alpha)} e^{\pi i(1-\gamma)} e^{\pi i(2\alpha-\gamma)} e^{2\pi i(\gamma-1)(l_{\nu}+l_{\nu}-2+\dots+l_4+l_2)} \times \\ & \times e^{2\nu\pi i(\alpha+\beta-\gamma)} \frac{1}{2} \left\{ \frac{1 - e^{\pi i(1-\gamma)l_{\nu}+1}}{1 - e^{\pi i(1-\gamma)}} - \frac{1 - (-1)^{l_{\nu}+1} e^{\pi i(1-\gamma)l_{\nu}+1}}{1 + e^{\pi i(1-\gamma)}} \right\} \int_0^{\frac{1}{2}} \theta \left( v + \frac{k_{\nu}}{2} \right)^{2\gamma-2\alpha-1} \theta_1 \left( v + \frac{k_{\nu}}{2} \right)^{2\alpha-1} \theta_2 \left( v + \frac{k_{\nu}}{2} \right)^{-2\beta-1} \theta_3 \left( v + \frac{k_{\nu}}{2} \right)^{2\beta-2\gamma+3} dv. \end{aligned} \tag{4.7}$$

Now we have

$$\begin{aligned}
& \int_0^{\frac{1}{2}} \theta \left( v + \frac{k_\nu}{2} \right)^{2\alpha-1} \theta_1 \left( v + \frac{k_\nu}{2} \right)^{2\gamma-2\alpha-1} \theta_2 \left( v + \frac{k_\nu}{2} \right)^{2\beta-2\gamma+3} \theta_3 \left( v + \frac{k_\nu}{2} \right)^{-2\beta-1} dv \\
&= \left\{ \frac{1 + (-1)^{k_\nu}}{2} e^{\pi i(1-\gamma)k_\nu} + \frac{1 - (-1)^{k_\nu}}{2} e^{\pi i(1-\gamma)(k_\nu-1)} e^{\pi i(2\alpha-2\gamma+1)} \right\} \int_0^{\frac{1}{2}} \theta(v)^{2\alpha-1} \theta_1(v)^{2\gamma-2\alpha-1} \theta_2(v)^{2\beta-2\gamma+3} \theta_3(v)^{-2\beta-1} dv \\
&= e^{\pi i(1-\gamma)k_\nu} e^{\{1-(-1)^{k_\nu}\}\pi i(2\alpha-\gamma)/2} \int_0^{\frac{1}{2}} \theta(v)^{2\alpha-1} \theta_1(v)^{2\gamma-2\alpha-1} \theta_2(v)^{2\beta-2\gamma+3} \theta_3(v)^{-2\beta-1} dv
\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}
& \int_0^{\frac{1}{2}} \theta \left( v + \frac{k_\nu}{2} \right)^{2\gamma-2\alpha-1} \theta_1 \left( v + \frac{k_\nu}{2} \right)^{2\alpha-1} \theta_2 \left( v + \frac{k_\nu}{2} \right)^{-2\beta-1} \theta_3 \left( v + \frac{k_\nu}{2} \right)^{2\beta-2\gamma+3} dv \\
&= \left\{ \frac{1 + (-1)^{k_\nu}}{2} e^{\pi i(1-\gamma)k_\nu} + \frac{1 - (-1)^{k_\nu}}{2} e^{\pi i(1-\gamma)(k_\nu-1)} e^{\pi i(1-2\alpha)} \right\} \int_0^{\frac{1}{2}} \theta(v)^{2\alpha-1} \theta_1(v)^{2\gamma-2\alpha-1} \theta_2(v)^{2\beta-2\gamma+3} \theta_3(v)^{-2\beta-1} dv \\
&= e^{\pi i(1-\gamma)k_\nu} e^{\{1-(-1)^{k_\nu}\}\pi i(\gamma-2\alpha)/2} \int_0^{\frac{1}{2}} \theta(v)^{2\alpha-1} \theta_1(v)^{2\gamma-2\alpha-1} \theta_2(v)^{2\beta-2\gamma+3} \theta_3(v)^{-2\beta-1} dv.
\end{aligned} \tag{4.9}$$

Substituting (4.8) and (4.9) into (4.7), we have

$$\begin{aligned}
I_\nu &= e^{\pi i} \left[ \{(-1)^{k_\nu-k_\nu-1} + (-1)^{k_\nu-k_\nu-3} + \dots + (-1)^{k_\nu-k_1}\} (2\beta-\gamma+1) + \{(-1)^{k_\nu-k_\nu-2} + (-1)^{k_\nu-k_\nu-4} + \dots + (-1)^{k_\nu-k_2}\} (2\alpha-\gamma-1) \right] \times \\
&\quad \times (-1) e^{\pi i(2\alpha-\gamma)/2} e^{\{2-(-1)^{k_\nu}\}\pi i(2\alpha-\gamma)/2} e^{2\pi i(\gamma-1)(l_\nu+l_\nu-2+\dots+l_4+l_2)} e^{2\nu\pi i(\alpha+\beta-\gamma)} e^{\pi i(1-\gamma)k_\nu} \frac{1}{2} \left\{ \frac{1 - e^{\pi i(1-\gamma)l_\nu+1}}{1 - e^{\pi i(1-\gamma)}} + \frac{1 - (-1)^{l_\nu+1} e^{\pi i(1-\gamma)l_\nu+1}}{1 + e^{\pi i(1-\gamma)}} \right\} z_1(\tau) / \theta_3^2 \\
&+ e^{-\pi i} \left[ \{(-1)^{k_\nu-k_\nu-1} + (-1)^{k_\nu-k_\nu-3} + \dots + (-1)^{k_\nu-k_1}\} (2\beta-\gamma+1) + \{(-1)^{k_\nu-k_\nu-2} + (-1)^{k_\nu-k_\nu-4} + \dots + (-1)^{k_\nu-k_2}\} (2\alpha-\gamma-1) \right] \times \\
&\quad \times (-1) e^{\pi i(2\alpha-\gamma)/2} e^{\{2-(-1)^{k_\nu}\}\pi i(\gamma-2\alpha)/2} e^{2\pi i(\gamma-1)(l_\nu+l_\nu-2+\dots+l_4+l_2)} e^{2\nu\pi i(\alpha+\beta-\gamma)} e^{\pi i(1-\gamma)k_\nu} \frac{1}{2} \left\{ \frac{1 - e^{\pi i(1-\gamma)l_\nu+1}}{1 - e^{\pi i(1-\gamma)}} - \frac{1 - (-1)^{l_\nu+1} e^{\pi i(1-\gamma)l_\nu+1}}{1 + e^{\pi i(1-\gamma)}} \right\} z_1(\tau) / \theta_3^2.
\end{aligned} \tag{4.10}$$

If we introduce  $\Theta_\nu$  by

$$\Theta_\nu = \{(-1)^{k_\nu-k_\nu-1} + (-1)^{k_\nu-k_\nu-3} + \dots + (-1)^{k_\nu-k_1}\} (2\beta-\gamma+1)\pi + \{(-1)^{k_\nu-k_\nu-2} + (-1)^{k_\nu-k_\nu-4} + \dots + (-1)^{k_\nu-k_2}\} (2\alpha-\gamma-1)\pi + \frac{2 - (-1)^{k_\nu}}{2} (2\alpha-\gamma)\pi,$$

(4.10) is turned to

$$I_\nu = -e^{\pi i(2\alpha-\gamma)/2} e^{2\pi i(\gamma-1)(l_\nu+l_\nu-2+\dots+l_4+l_2)} e^{2\nu\pi i(\alpha+\beta-\gamma)} e^{\pi i(1-\gamma)k_\nu} \left\{ \cos \Theta_\nu \cdot \frac{1 - e^{\pi i(1-\gamma)l_\nu+1}}{1 - e^{\pi i(1-\gamma)}} + i \sin \Theta_\nu \cdot \frac{1 - (-1)^{l_\nu+1} e^{\pi i(1-\gamma)l_\nu+1}}{1 + e^{\pi i(1-\gamma)}} \right\} z_1(\tau) / \theta_3^2. \tag{4.11}$$

§5 The integrals  $I_\nu$  for positive odd integers  $\nu$ .

In this section let  $\nu$  be a positive odd integer. The formula (4.1) holds also in this case. In case where  $\nu \geq 3$ , (4.3) holds too. Moreover we have

$$\begin{aligned}
& \left[ \theta \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu - 2}{2} \tau \right)^{2\alpha-1} \theta_1 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu - 2}{2} \tau \right)^{2\gamma-2\alpha-1} \theta_2 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu - 2}{2} \tau \right)^{2\beta-2\gamma+3} \theta_3 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu - 2}{2} \tau \right)^{-2\beta-1} \right]_+ \\
&= \left\{ \frac{1 + (-1)^{\mu+l_\nu+l_{\nu-1}}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1 - (-1)^{\mu+l_\nu+l_{\nu-1}}}{2} e^{2\pi i(1-\gamma)} \right\} e^{2\pi i(1-\gamma)(\mu+l_\nu+l_{\nu-1})} e^{\pi i(\alpha+\beta-\gamma+1)} \times \\
& \times \left[ \theta \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu - 3}{2} \tau \right)^{2\beta-2\gamma+3} \theta_1 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu - 3}{2} \tau \right)^{-2\beta-1} \theta_2 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu - 3}{2} \tau \right)^{2\alpha-1} \theta_3 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu - 3}{2} \tau \right)^{2\gamma-2\alpha-1} \right]_+. \tag{5.1}
\end{aligned}$$

Combining (5.1) with (4.3), we have

$$\begin{aligned}
& \left[ \theta \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2} \tau \right)^{2\alpha-1} \theta_1 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2} \tau \right)^{2\gamma-2\alpha-1} \theta_2 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2} \tau \right)^{2\beta-2\gamma+3} \theta_3 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2} \tau \right)^{-2\beta-1} \right]_+ \\
&= \left\{ \frac{1 + (-1)^\mu}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1 - (-1)^\mu}{2} e^{2\pi i(1-\gamma)} \right\} \left\{ \frac{1 + (-1)^{\mu+l_\nu}}{2} e^{4\pi i\beta} + \frac{1 - (-1)^{\mu+l_\nu}}{2} e^{2\pi i(\gamma-1)} \right\} \times \\
& \times \left\{ \frac{1 + (-1)^{\mu+l_\nu+l_{\nu-1}}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1 - (-1)^{\mu+l_\nu+l_{\nu-1}}}{2} e^{2\pi i(1-\gamma)} \right\} e^{2\pi i(1-\gamma)(\mu+l_{\nu-1})} e^{3\pi i(\alpha+\beta-\gamma+1)} \times \\
& \times \left[ \theta \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu - 3}{2} \tau \right)^{2\beta-2\gamma+3} \theta_1 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu - 3}{2} \tau \right)^{-2\beta-1} \theta_2 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu - 3}{2} \tau \right)^{2\alpha-1} \theta_3 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu - 3}{2} \tau \right)^{2\gamma-2\alpha-1} \right]_+. \tag{5.2}
\end{aligned}$$

Repeating the same procedure, we arrive at the following

**Lemma 5.1.** *For a positive odd integer  $\nu$  we have*

$$\begin{aligned}
& \left[ \theta \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2} \tau \right)^{2\alpha-1} \theta_1 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2} \tau \right)^{2\gamma-2\alpha-1} \theta_2 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2} \tau \right)^{2\beta-2\gamma+3} \theta_3 \left( v + \frac{k_\nu + \mu}{2} + \frac{\nu}{2} \tau \right)^{-2\beta-1} \right]_+ \\
&= \left\{ \frac{1 + (-1)^\mu}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1 - (-1)^\mu}{2} e^{2\pi i(1-\gamma)} \right\} \left\{ \frac{1 + (-1)^{\mu+l_\nu}}{2} e^{4\pi i\beta} + \frac{1 - (-1)^{\mu+l_\nu}}{2} e^{2\pi i(\gamma-1)} \right\} \left\{ \frac{1 + (-1)^{\mu+l_\nu+l_{\nu-1}}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1 - (-1)^{\mu+l_\nu+l_{\nu-1}}}{2} e^{2\pi i(1-\gamma)} \right\} \dots \\
& \dots \left\{ \frac{1 + (-1)^{\mu+l_\nu+l_{\nu-1}+\dots+l_3}}{2} e^{4\pi i\beta} + \frac{1 - (-1)^{\mu+l_\nu+l_{\nu-1}+\dots+l_3}}{2} e^{2\pi i(\gamma-1)} \right\} \left\{ \frac{1 + (-1)^{\mu+l_\nu+l_{\nu-1}+\dots+l_2}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1 - (-1)^{\mu+l_\nu+l_{\nu-1}+\dots+l_2}}{2} e^{2\pi i(1-\gamma)} \right\} \\
& \times e^{2\pi i(1-\gamma)(\mu+l_{\nu-1}+l_{\nu-3}+\dots+l_4+l_2)} e^{\nu\pi i(\alpha+\beta-\gamma+1)} \left[ \theta \left( v + \frac{k_\nu + \mu}{2} \right)^{2\beta-2\gamma+3} \theta_1 \left( v + \frac{k_\nu + \mu}{2} \right)^{-2\beta-1} \theta_2 \left( v + \frac{k_\nu + \mu}{2} \right)^{2\alpha-1} \theta_3 \left( v + \frac{k_\nu + \mu}{2} \right)^{2\gamma-2\alpha-1} \right]_+. \tag{5.3}
\end{aligned}$$

Let us now compute  $I_\nu$  for a positive odd integer  $\nu$ . Substituting (5.3) into (1.5), we have

$$\begin{aligned}
I_\nu &= \sum_{\mu=0}^{l_\nu+1-1} \left\{ \frac{1+(-1)^\mu}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1-(-1)^\mu}{2} e^{2\pi i(1-\gamma)} \right\} \left\{ \frac{1+(-1)^{\mu+l_\nu}}{2} e^{4\pi i\beta} + \frac{1-(-1)^{\mu+l_\nu}}{2} e^{2\pi i(\gamma-1)} \right\} \left\{ \frac{1+(-1)^{\mu+l_\nu+l_{\nu-1}}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1-(-1)^{\mu+l_\nu+l_{\nu-1}}}{2} e^{2\pi i(1-\gamma)} \right\} \dots \\
&\dots \left\{ \frac{1+(-1)^{\mu+l_\nu+l_{\nu-1}+\dots+l_3}}{2} e^{4\pi i\beta} + \frac{1-(-1)^{\mu+l_\nu+l_{\nu-1}+\dots+l_3}}{2} e^{2\pi i(\gamma-1)} \right\} \left\{ \frac{1+(-1)^{\mu+l_\nu+l_{\nu-1}+\dots+l_2}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1-(-1)^{\mu+l_\nu+l_{\nu-1}+\dots+l_2}}{2} e^{2\pi i(1-\gamma)} \right\} \quad (5.4) \\
&\times e^{2\pi i(1-\gamma)(\mu+l_{\nu-1}+l_{\nu-3}+\dots+l_4+l_2)} e^{\nu\pi i(\alpha+\beta-\gamma+1)} \int_0^{\frac{1}{2}} \theta \left( v + \frac{k_\nu + \mu}{2} \right)^{2\beta-2\gamma+3} \theta_1 \left( v + \frac{k_\nu + \mu}{2} \right)^{-2\beta-1} \theta_2 \left( v + \frac{k_\nu + \mu}{2} \right)^{2\alpha-1} \theta_3 \left( v + \frac{k_\nu + \mu}{2} \right)^{2\gamma-2\alpha-1} dv.
\end{aligned}$$

Since

$$\begin{aligned}
&\theta \left( v + \frac{k_\nu + \mu}{2} \right)^{2\beta-2\gamma+3} \theta_1 \left( v + \frac{k_\nu + \mu}{2} \right)^{-2\beta-1} \theta_2 \left( v + \frac{k_\nu + \mu}{2} \right)^{2\alpha-1} \theta_3 \left( v + \frac{k_\nu + \mu}{2} \right)^{2\gamma-2\alpha-1} \\
&= \frac{1+(-1)^\mu}{2} e^{\pi i(\gamma-1)\mu} \theta \left( v + \frac{k_\nu}{2} \right)^{2\beta-2\gamma+3} \theta_1 \left( v + \frac{k_\nu}{2} \right)^{-2\beta-1} \theta_2 \left( v + \frac{k_\nu}{2} \right)^{2\alpha-1} \theta_3 \left( v + \frac{k_\nu}{2} \right)^{2\gamma-2\alpha-1} \\
&\quad + \frac{1-(-1)^\mu}{2} e^{\pi i(\gamma-1)(\mu-1)} e^{\pi i(2\beta+1)} \theta \left( v + \frac{k_\nu}{2} \right)^{-2\beta-1} \theta_1 \left( v + \frac{k_\nu}{2} \right)^{2\beta-2\gamma+3} \theta_2 \left( v + \frac{k_\nu}{2} \right)^{2\gamma-2\alpha-1} \theta_3 \left( v + \frac{k_\nu}{2} \right)^{2\alpha-1},
\end{aligned}$$

(5.4) is turned to

$$\begin{aligned}
I_\nu &= \sum_{\mu=0}^{l_\nu+1-1} \left\{ \frac{1+(-1)^\mu}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1-(-1)^\mu}{2} e^{2\pi i(1-\gamma)} \right\} \left\{ \frac{1+(-1)^{\mu+l_\nu}}{2} e^{4\pi i\beta} + \frac{1-(-1)^{\mu+l_\nu}}{2} e^{2\pi i(\gamma-1)} \right\} \left\{ \frac{1+(-1)^{\mu+l_\nu+l_{\nu-1}}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1-(-1)^{\mu+l_\nu+l_{\nu-1}}}{2} e^{2\pi i(1-\gamma)} \right\} \dots \\
&\dots \left\{ \frac{1+(-1)^{\mu+l_\nu+l_{\nu-1}+\dots+l_3}}{2} e^{4\pi i\beta} + \frac{1-(-1)^{\mu+l_\nu+l_{\nu-1}+\dots+l_3}}{2} e^{2\pi i(\gamma-1)} \right\} \left\{ \frac{1+(-1)^{\mu+l_\nu+l_{\nu-1}+\dots+l_2}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1-(-1)^{\mu+l_\nu+l_{\nu-1}+\dots+l_2}}{2} e^{2\pi i(1-\gamma)} \right\} \\
&\times e^{2\pi i(1-\gamma)(\mu+l_{\nu-1}+l_{\nu-3}+\dots+l_4+l_2)} e^{\nu\pi i(\alpha+\beta-\gamma+1)} \left\{ \frac{1+(-1)^\mu}{2} e^{\pi i(\gamma-1)\mu} \int_0^{\frac{1}{2}} \theta \left( v + \frac{k_\nu}{2} \right)^{2\beta-2\gamma+3} \theta_1 \left( v + \frac{k_\nu}{2} \right)^{-2\beta-1} \theta_2 \left( v + \frac{k_\nu}{2} \right)^{2\alpha-1} \theta_3 \left( v + \frac{k_\nu}{2} \right)^{2\gamma-2\alpha-1} dv \right. \\
&\left. + \frac{1-(-1)^\mu}{2} e^{\pi i(\gamma-1)(\mu-1)} e^{\pi i(2\beta+1)} \int_0^{\frac{1}{2}} \theta \left( v + \frac{k_\nu}{2} \right)^{-2\beta-1} \theta_1 \left( v + \frac{k_\nu}{2} \right)^{2\beta-2\gamma+3} \theta_2 \left( v + \frac{k_\nu}{2} \right)^{2\gamma-2\alpha-1} \theta_3 \left( v + \frac{k_\nu}{2} \right)^{2\alpha-1} dv \right\},
\end{aligned}$$

from which it follows that

$$\begin{aligned}
I_\nu = & \sum_{\mu=0}^{l_\nu+1-1} e^{4\pi i(\alpha-\gamma)} \left\{ \frac{1+(-1)^\mu}{2} e^{4\pi i\beta} + \frac{1-(-1)^\mu}{2} e^{2\pi i(\gamma-1)} \right\} \left\{ \frac{1+(-1)^{l_\nu+l_{\nu-1}}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1-(-1)^{l_\nu+l_{\nu-1}}}{2} e^{2\pi i(1-\gamma)} \right\} \dots \\
& \dots \left\{ \frac{1+(-1)^{l_\nu+l_{\nu-1}+\dots+l_3}}{2} e^{4\pi i\beta} + \frac{1-(-1)^{l_\nu+l_{\nu-1}+\dots+l_3}}{2} e^{2\pi i(\gamma-1)} \right\} \left\{ \frac{1+(-1)^{l_\nu+l_{\nu-1}+\dots+l_2}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1-(-1)^{l_\nu+l_{\nu-1}+\dots+l_2}}{2} e^{2\pi i(1-\gamma)} \right\} \\
& \times e^{2\pi i(1-\gamma)(l_{\nu-1}+l_{\nu-3}+\dots+l_4+l_2)} e^{\nu\pi i(\alpha+\beta-\gamma+1)} \frac{1+(-1)^\mu}{2} e^{\pi i(1-\gamma)\mu} \int_0^{\frac{1}{2}} \theta\left(v+\frac{k_\nu}{2}\right)^{2\beta-2\gamma+3} \theta_1\left(v+\frac{k_\nu}{2}\right)^{-2\beta-1} \theta_2\left(v+\frac{k_\nu}{2}\right)^{2\alpha-1} \theta_3\left(v+\frac{k_\nu}{2}\right)^{2\gamma-2\alpha-1} dv \\
& + \sum_{\mu=0}^{l_\nu+1-1} e^{2\pi i(1-\gamma)} \left\{ \frac{1-(-1)^\mu}{2} e^{4\pi i\beta} + \frac{1+(-1)^\mu}{2} e^{2\pi i(\gamma-1)} \right\} \left\{ \frac{1-(-1)^{l_\nu+l_{\nu-1}}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1+(-1)^{l_\nu+l_{\nu-1}}}{2} e^{2\pi i(1-\gamma)} \right\} \dots \\
& \dots \left\{ \frac{1-(-1)^{l_\nu+l_{\nu-1}+\dots+l_3}}{2} e^{4\pi i\beta} + \frac{1+(-1)^{l_\nu+l_{\nu-1}+\dots+l_3}}{2} e^{2\pi i(\gamma-1)} \right\} \left\{ \frac{1-(-1)^{l_\nu+l_{\nu-1}+\dots+l_2}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1+(-1)^{l_\nu+l_{\nu-1}+\dots+l_2}}{2} e^{2\pi i(1-\gamma)} \right\} \\
& \times e^{2\pi i(1-\gamma)(l_{\nu-1}+l_{\nu-3}+\dots+l_4+l_2)} e^{\nu\pi i(\alpha+\beta-\gamma+1)} \frac{1-(-1)^\mu}{2} e^{\pi i(1-\gamma)\mu} e^{\pi i(2\beta-\gamma)} \int_0^{\frac{1}{2}} \theta\left(v+\frac{k_\nu}{2}\right)^{-2\beta-1} \theta_1\left(v+\frac{k_\nu}{2}\right)^{2\beta-2\gamma+3} \theta_2\left(v+\frac{k_\nu}{2}\right)^{2\gamma-2\alpha-1} \theta_3\left(v+\frac{k_\nu}{2}\right)^{2\alpha-1} dv.
\end{aligned} \tag{5.5}$$

Substituting

$$\sum_{\mu=0}^{l_\nu+1-1} \frac{1 \pm (-1)^\mu}{2} e^{\pi i(1-\gamma)\mu} = \frac{1}{2} \left\{ \frac{1 - e^{\pi i(1-\gamma)l_\nu+1}}{1 - e^{\pi i(1-\gamma)}} \pm \frac{1 - (-1)^{l_\nu+1} e^{\pi i(1-\gamma)l_\nu+1}}{1 + e^{\pi i(1-\gamma)}} \right\}$$

into (5.5), we have

$$\begin{aligned}
I_\nu = & \left\{ \frac{1 + (-1)^{l_\nu}}{2} e^{4\pi i \beta} + \frac{1 - (-1)^{l_\nu}}{2} e^{2\pi i (\gamma-1)} \right\} \left\{ \frac{1 + (-1)^{l_\nu + l_{\nu-1}}}{2} e^{4\pi i (\alpha-\gamma)} + \frac{1 - (-1)^{l_\nu + l_{\nu-1}}}{2} e^{2\pi i (1-\gamma)} \right\} \dots \\
& \dots \left\{ \frac{1 + (-1)^{l_\nu + l_{\nu-1} + \dots + l_3}}{2} e^{4\pi i \beta} + \frac{1 - (-1)^{l_\nu + l_{\nu-1} + \dots + l_3}}{2} e^{2\pi i (\gamma-1)} \right\} \left\{ \frac{1 + (-1)^{l_\nu + l_{\nu-1} + \dots + l_2}}{2} e^{4\pi i (\alpha-\gamma)} + \frac{1 - (-1)^{l_\nu + l_{\nu-1} + \dots + l_2}}{2} e^{2\pi i (1-\gamma)} \right\} \\
& \times e^{4\pi i (\alpha-\gamma)} e^{2\pi i (1-\gamma)(l_{\nu-1} + l_{\nu-3} + \dots + l_4 + l_2)} e^{\nu \pi i (\alpha + \beta - \gamma + 1)} \frac{1}{2} \left\{ \frac{1 - e^{\pi i (1-\gamma) l_{\nu+1}}}{1 - e^{\pi i (1-\gamma)}} + \frac{1 - (-1)^{l_{\nu+1}} e^{\pi i (1-\gamma) l_{\nu+1}}}{1 + e^{\pi i (1-\gamma)}} \right\} \\
& \times \int_0^{\frac{1}{2}} \theta \left( v + \frac{k_\nu}{2} \right)^{2\beta - 2\gamma + 3} \theta_1 \left( v + \frac{k_\nu}{2} \right)^{-2\beta - 1} \theta_2 \left( v + \frac{k_\nu}{2} \right)^{2\alpha - 1} \theta_3 \left( v + \frac{k_\nu}{2} \right)^{2\gamma - 2\alpha - 1} dv \\
& + \left\{ \frac{1 - (-1)^{l_\nu}}{2} e^{4\pi i \beta} + \frac{1 + (-1)^{l_\nu}}{2} e^{2\pi i (\gamma-1)} \right\} \left\{ \frac{1 - (-1)^{l_\nu + l_{\nu-1}}}{2} e^{4\pi i (\alpha-\gamma)} + \frac{1 + (-1)^{l_\nu + l_{\nu-1}}}{2} e^{2\pi i (1-\gamma)} \right\} \dots \\
& \dots \left\{ \frac{1 - (-1)^{l_\nu + l_{\nu-1} + \dots + l_3}}{2} e^{4\pi i \beta} + \frac{1 + (-1)^{l_\nu + l_{\nu-1} + \dots + l_3}}{2} e^{2\pi i (\gamma-1)} \right\} \left\{ \frac{1 - (-1)^{l_\nu + l_{\nu-1} + \dots + l_2}}{2} e^{4\pi i (\alpha-\gamma)} + \frac{1 + (-1)^{l_\nu + l_{\nu-1} + \dots + l_2}}{2} e^{2\pi i (1-\gamma)} \right\} \\
& \times e^{2\pi i (1-\gamma)} e^{2\pi i (1-\gamma)(l_{\nu-1} + l_{\nu-3} + \dots + l_4 + l_2)} e^{\nu \pi i (\alpha + \beta - \gamma + 1)} e^{\pi i (2\beta - \gamma)} \frac{1}{2} \left\{ \frac{1 - e^{\pi i (1-\gamma) l_{\nu+1}}}{1 - e^{\pi i (1-\gamma)}} - \frac{1 - (-1)^{l_{\nu+1}} e^{\pi i (1-\gamma) l_{\nu+1}}}{1 + e^{\pi i (1-\gamma)}} \right\} \\
& \times \int_0^{\frac{1}{2}} \theta \left( v + \frac{k_\nu}{2} \right)^{-2\beta - 1} \theta_1 \left( v + \frac{k_\nu}{2} \right)^{2\beta - 2\gamma + 3} \theta_2 \left( v + \frac{k_\nu}{2} \right)^{2\gamma - 2\alpha - 1} \theta_3 \left( v + \frac{k_\nu}{2} \right)^{2\alpha - 1} dv,
\end{aligned}$$

which is turned by some elementary calculation to

$$\begin{aligned}
I_\nu = & e^{\pi i} \left[ \{ (-1)^{k_\nu - k_{\nu-1} + (-1)^{k_\nu - k_{\nu-3} + \dots + (-1)^{k_\nu - k_2}} \} (2\beta - \gamma + 1) + \{ (-1)^{k_\nu - k_{\nu-2} + (-1)^{k_\nu - k_{\nu-4} + \dots + (-1)^{k_\nu - k_1}} \} (2\alpha - \gamma - 1) \right] e^{4\pi i (\alpha-\gamma)} e^{2\pi i (1-\gamma)(l_{\nu-1} + l_{\nu-3} + \dots + l_4 + l_2)} \\
& \times (-1) e^{(2\nu-1)\pi i (\alpha + \beta - \gamma)} \frac{1}{2} \left\{ \frac{1 - e^{\pi i (1-\gamma) l_{\nu+1}}}{1 - e^{\pi i (1-\gamma)}} + \frac{1 - (-1)^{l_{\nu+1}} e^{\pi i (1-\gamma) l_{\nu+1}}}{1 + e^{\pi i (1-\gamma)}} \right\} \\
& \times \int_0^{\frac{1}{2}} \theta \left( v + \frac{k_\nu}{2} \right)^{2\beta - 2\gamma + 3} \theta_1 \left( v + \frac{k_\nu}{2} \right)^{-2\beta - 1} \theta_2 \left( v + \frac{k_\nu}{2} \right)^{2\alpha - 1} \theta_3 \left( v + \frac{k_\nu}{2} \right)^{2\gamma - 2\alpha - 1} dv \\
& + e^{-\pi i} \left[ \{ (-1)^{k_\nu - k_{\nu-1} + (-1)^{k_\nu - k_{\nu-3} + \dots + (-1)^{k_\nu - k_2}} \} (2\beta - \gamma + 1) + \{ (-1)^{k_\nu - k_{\nu-2} + (-1)^{k_\nu - k_{\nu-4} + \dots + (-1)^{k_\nu - k_1}} \} (2\alpha - \gamma - 1) \right] e^{\pi i (2\beta - 3\gamma)} e^{2\pi i (1-\gamma)(l_{\nu-1} + l_{\nu-3} + \dots + l_4 + l_2)} \\
& \times (-1) e^{(2\nu-1)\pi i (\alpha + \beta - \gamma)} \frac{1}{2} \left\{ \frac{1 - e^{\pi i (1-\gamma) l_{\nu+1}}}{1 - e^{\pi i (1-\gamma)}} - \frac{1 - (-1)^{l_{\nu+1}} e^{\pi i (1-\gamma) l_{\nu+1}}}{1 + e^{\pi i (1-\gamma)}} \right\} \\
& \times \int_0^{\frac{1}{2}} \theta \left( v + \frac{k_\nu}{2} \right)^{-2\beta - 1} \theta_1 \left( v + \frac{k_\nu}{2} \right)^{2\beta - 2\gamma + 3} \theta_2 \left( v + \frac{k_\nu}{2} \right)^{2\gamma - 2\alpha - 1} \theta_3 \left( v + \frac{k_\nu}{2} \right)^{2\alpha - 1} dv.
\end{aligned} \tag{5.6}$$



Now we have

$$\begin{aligned}
& \int_0^{\frac{1}{2}} \theta \left( v + \frac{k_\nu}{2} \right)^{2\beta-2\gamma+3} \theta_1 \left( v + \frac{k_\nu}{2} \right)^{-2\beta-1} \theta_2 \left( v + \frac{k_\nu}{2} \right)^{2\alpha-1} \theta_3 \left( v + \frac{k_\nu}{2} \right)^{2\gamma-2\alpha-1} dv \\
&= \left\{ \frac{1 + (-1)^{k_\nu}}{2} e^{\pi i(\gamma-1)k_\nu} + \frac{1 - (-1)^{k_\nu}}{2} e^{\pi i(\gamma-1)(k_\nu-1)} e^{\pi i(2\beta+1)} \right\} \int_0^{\frac{1}{2}} \theta(v)^{2\beta-2\gamma+3} \theta_1(v)^{-2\beta-1} \theta_2(v)^{2\alpha-1} \theta_3(v)^{2\gamma-2\alpha-1} dv \\
&= e^{\pi i(\gamma-1)k_\nu} e^{\{1 - (-1)^{k_\nu}\} \pi i(2\beta-\gamma)/2} \int_0^{\frac{1}{2}} \theta(v)^{2\beta-2\gamma+3} \theta_1(v)^{-2\beta-1} \theta_2(v)^{2\alpha-1} \theta_3(v)^{2\gamma-2\alpha-1} dv
\end{aligned} \tag{5.7}$$

and

$$\begin{aligned}
& \int_0^{\frac{1}{2}} \theta \left( v + \frac{k_\nu}{2} \right)^{-2\beta-1} \theta_1 \left( v + \frac{k_\nu}{2} \right)^{2\beta-2\gamma+3} \theta_2 \left( v + \frac{k_\nu}{2} \right)^{2\gamma-2\alpha-1} \theta_3 \left( v + \frac{k_\nu}{2} \right)^{2\alpha-1} dv \\
&= \left\{ \frac{1 + (-1)^{k_\nu}}{2} e^{\pi i(\gamma-1)k_\nu} + \frac{1 - (-1)^{k_\nu}}{2} e^{\pi i(\gamma-1)(k_\nu-1)} e^{\pi i(2\gamma-2\beta-3)} \right\} \int_0^{\frac{1}{2}} \theta(v)^{2\beta-2\gamma+3} \theta_1(v)^{-2\beta-1} \theta_2(v)^{2\alpha-1} \theta_3(v)^{2\gamma-2\alpha-1} dv \\
&= e^{\pi i(\gamma-1)k_\nu} e^{\{1 - (-1)^{k_\nu}\} \pi i(\gamma-2\beta)/2} \int_0^{\frac{1}{2}} \theta(v)^{2\beta-2\gamma+3} \theta_1(v)^{-2\beta-1} \theta_2(v)^{2\alpha-1} \theta_3(v)^{2\gamma-2\alpha-1} dv.
\end{aligned} \tag{5.8}$$

Substituting (5.7) and (5.8) into (5.6), we have

$$\begin{aligned}
I_\nu &= e^{\pi i} \left[ \{(-1)^{k_\nu-k_{\nu-1}} + (-1)^{k_\nu-k_{\nu-3}} + \dots + (-1)^{k_\nu-k_2}\} (2\beta-\gamma+1) + \{(-1)^{k_\nu-k_{\nu-2}} + (-1)^{k_\nu-k_{\nu-4}} + \dots + (-1)^{k_\nu-k_1}\} (2\alpha-\gamma-1) \right] (-1) e^{\pi i(2\alpha-5\gamma)/2} e^{2\pi i(\alpha-\beta)} \\
&\quad \times e^{\{2 - (-1)^{k_\nu}\} \pi i(2\beta-\gamma)/2} e^{2\pi i(1-\gamma)(l_{\nu-1}+l_{\nu-3}+\dots+l_4+l_2)} e^{2\nu\pi i(\alpha+\beta-\gamma)} e^{\pi i(\gamma-1)k_\nu} \frac{1}{2} \left\{ \frac{1 - e^{\pi i(1-\gamma)l_{\nu+1}}}{1 - e^{\pi i(1-\gamma)}} + \frac{1 - (-1)^{l_{\nu+1}} e^{\pi i(1-\gamma)l_{\nu+1}}}{1 + e^{\pi i(1-\gamma)}} \right\} z_2(\tau)/\theta_3^2 \\
&+ e^{-\pi i} \left[ \{(-1)^{k_\nu-k_{\nu-1}} + (-1)^{k_\nu-k_{\nu-3}} + \dots + (-1)^{k_\nu-k_2}\} (2\beta-\gamma+1) + \{(-1)^{k_\nu-k_{\nu-2}} + (-1)^{k_\nu-k_{\nu-4}} + \dots + (-1)^{k_\nu-k_1}\} (2\alpha-\gamma-1) \right] (-1) e^{\pi i(2\alpha-5\gamma)/2} e^{-2\pi i(\alpha-\beta)} \\
&\quad \times e^{\{2 - (-1)^{k_\nu}\} \pi i(\gamma-2\beta)/2} e^{2\pi i(1-\gamma)(l_{\nu-1}+l_{\nu-3}+\dots+l_4+l_2)} e^{2\nu\pi i(\alpha+\beta-\gamma)} e^{\pi i(\gamma-1)k_\nu} \frac{1}{2} \left\{ \frac{1 - e^{\pi i(1-\gamma)l_{\nu+1}}}{1 - e^{\pi i(1-\gamma)}} - \frac{1 - (-1)^{l_{\nu+1}} e^{\pi i(1-\gamma)l_{\nu+1}}}{1 + e^{\pi i(1-\gamma)}} \right\} z_2(\tau)/\theta_3^2.
\end{aligned} \tag{5.9}$$

If we introduce  $\Theta_\nu$  by

$$\Theta_\nu = \{(-1)^{k_\nu-k_{\nu-1}} + (-1)^{k_\nu-k_{\nu-3}} + \dots + (-1)^{k_\nu-k_2}\} (2\beta-\gamma+1)\pi + \{(-1)^{k_\nu-k_{\nu-2}} + (-1)^{k_\nu-k_{\nu-4}} + \dots + (-1)^{k_\nu-k_1}\} (2\alpha-\gamma-1)\pi + \frac{2 - (-1)^{k_\nu}}{2} (2\beta-\gamma)\pi + 2(\alpha-\beta)\pi,$$

(5.9) is turned to

$$I_\nu = -e^{\pi i(2\alpha-5\gamma)/2} e^{2\pi i(1-\gamma)(l_{\nu-1}+l_{\nu-3}+\dots+l_4+l_2)} e^{2\nu\pi i(\alpha+\beta-\gamma)} e^{\pi i(\gamma-1)k_\nu} \left\{ \cos \Theta_\nu \cdot \frac{1 - e^{\pi i(1-\gamma)l_{\nu+1}}}{1 - e^{\pi i(1-\gamma)}} + i \sin \Theta_\nu \cdot \frac{1 - (-1)^{l_{\nu+1}} e^{\pi i(1-\gamma)l_{\nu+1}}}{1 + e^{\pi i(1-\gamma)}} \right\} z_2(\tau)/\theta_3^2. \tag{5.10}$$

§6 The integral  $J_0$ .

From (1.4) we have

$$J_0 = \int_0^{\tau/2} \theta \left( v + \frac{k_1}{2} \right)^{2\alpha-1} \theta_1 \left( v + \frac{k_1}{2} \right)^{2\gamma-2\alpha-1} \theta_2 \left( v + \frac{k_1}{2} \right)^{2\beta-2\gamma+3} \theta_3 \left( v + \frac{k_1}{2} \right)^{-2\beta-1} dv. \quad (6.1)$$

Note that

$$\begin{aligned} & \theta \left( v + \frac{k_1}{2} \right)^{2\alpha-1} \theta_1 \left( v + \frac{k_1}{2} \right)^{2\gamma-2\alpha-1} \theta_2 \left( v + \frac{k_1}{2} \right)^{2\beta-2\gamma+3} \theta_3 \left( v + \frac{k_1}{2} \right)^{-2\beta-1} \\ &= \frac{1 + (-1)^{k_1}}{2} e^{\pi i(1-\gamma)k_1} \theta(v)^{2\alpha-1} \theta_1(v)^{2\gamma-2\alpha-1} \theta_2(v)^{2\beta-2\gamma+3} \theta_3(v)^{-2\beta-1} + \frac{1 - (-1)^{k_1}}{2} e^{\pi i(1-\gamma)(k_1-1)} e^{\pi i(2\alpha-2\gamma+1)} \theta(v)^{2\gamma-2\alpha-1} \theta_1(v)^{2\alpha-1} \theta_2(v)^{-2\beta-1} \theta_3(v)^{2\beta-2\gamma+3}. \end{aligned}$$

Substituting this expression into (6.1), we have

$$\begin{aligned} J_0 &= \frac{1 + (-1)^{k_1}}{2} e^{\pi i(1-\gamma)k_1} \int_0^{\tau/2} \theta(v)^{2\alpha-1} \theta_1(v)^{2\gamma-2\alpha-1} \theta_2(v)^{2\beta-2\gamma+3} \theta_3(v)^{-2\beta-1} dv \\ &\quad + \frac{1 - (-1)^{k_1}}{2} e^{\pi i(1-\gamma)(k_1-1)} e^{\pi i(2\alpha-2\gamma+1)} \int_0^{\tau/2} \theta(v)^{2\gamma-2\alpha-1} \theta_1(v)^{2\alpha-1} \theta_2(v)^{-2\beta-1} \theta_3(v)^{2\beta-2\gamma+3} dv. \end{aligned} \quad (6.2)$$

Applying the formulas of Lemma 2.2 to (6.2), we have

$$\begin{aligned} J_0 &= \left\{ \frac{1 + (-1)^{k_1}}{2} e^{\pi i(1-\gamma)k_1} \frac{1 - e^{2\pi i(\alpha-\gamma)}}{1 - e^{-2\pi i\gamma}} + \frac{1 - (-1)^{k_1}}{2} e^{\pi i(1-\gamma)k_1} e^{\pi i(2\alpha-\gamma)} \frac{1 - e^{-2\pi i\alpha}}{1 - e^{-2\pi i\gamma}} \right\} z_1(\tau)/\theta_3^2 \\ &\quad + \left\{ \frac{1 + (-1)^{k_1}}{2} e^{\pi i(1-\gamma)k_1} \frac{e^{\pi i(\alpha-\gamma)}(e^{\pi i\beta} - e^{-\pi i\beta})}{1 - e^{-2\pi i\gamma}} + \frac{1 - (-1)^{k_1}}{2} e^{\pi i(1-\gamma)k_1} e^{\pi i(2\alpha-\gamma)} \frac{e^{\pi i(\gamma-\alpha-\beta)} - e^{\pi i(-\alpha+\beta-\gamma)}}{1 - e^{-2\pi i\gamma}} \right\} z_2(\tau)/\theta_3^2 \\ &= e^{\pi i(1-\gamma)l_1} e^{\{1+(-1)^{l_1+1}\}\pi i(2\alpha-\gamma)/2} \frac{1 - e^{-\pi i\gamma} e^{(-1)^{l_1}\pi i(2\alpha-\gamma)}}{1 - e^{-2\pi i\gamma}} z_1(\tau)/\theta_3^2 + e^{\pi i(1-\gamma)l_1} e^{\pi i(\alpha-\gamma)} \frac{e^{\pi i\{\gamma+(-1)^{l_1}(2\beta-\gamma)\}/2} - e^{-\pi i\{\gamma+(-1)^{l_1}(2\beta-\gamma)\}/2}}{1 - e^{-2\pi i\gamma}} z_2(\tau)/\theta_3^2 \\ &= e^{\pi i(1-\gamma)l_1} e^{\pi i(\alpha-\gamma)} \frac{e^{\pi i\{\gamma-(-1)^{l_1}(2\alpha-\gamma)\}/2} - e^{-\pi i\{\gamma-(-1)^{l_1}(2\alpha-\gamma)\}/2}}{1 - e^{-2\pi i\gamma}} z_1(\tau)/\theta_3^2 + e^{\pi i(1-\gamma)l_1} e^{\pi i(\alpha-\gamma)} \frac{e^{\pi i\{\gamma+(-1)^{l_1}(2\beta-\gamma)\}/2} - e^{-\pi i\{\gamma+(-1)^{l_1}(2\beta-\gamma)\}/2}}{1 - e^{-2\pi i\gamma}} z_2(\tau)/\theta_3^2, \end{aligned}$$

from which it follows that

$$J_0 = e^{\pi i(1-\gamma)l_1} e^{\pi i\alpha} \frac{\sin[\pi\{\gamma - (-1)^{l_1}(2\alpha - \gamma)\}/2]}{\sin \pi\gamma} z_1(\tau)/\theta_3^2 + e^{\pi i(1-\gamma)l_1} e^{\pi i\alpha} \frac{\sin[\pi\{\gamma + (-1)^{l_1}(2\beta - \gamma)\}/2]}{\sin \pi\gamma} z_2(\tau)/\theta_3^2. \quad (6.3)$$

§7 The integrals  $J_\nu$  for positive even integers  $\nu$ .

In this section let  $\nu$  be a positive even integer. Then we have

$$\begin{aligned} & \left[ \theta \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu}{2}\tau \right)^{2\alpha-1} \theta_1 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu}{2}\tau \right)^{2\gamma-2\alpha-1} \theta_2 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu}{2}\tau \right)^{2\beta-2\gamma+3} \theta_3 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu}{2}\tau \right)^{-2\beta-1} \right]_+ \\ &= \frac{1 + (-1)^{l_{\nu+1}}}{2} e^{\pi i(1-\gamma)l_{\nu+1}} e^{\pi i(2\alpha-2\gamma+1)\theta} \theta \left( v + \frac{k_\nu-1}{2} + \frac{\nu}{2}\tau \right)^{2\gamma-2\alpha-1} \theta_1 \left( v + \frac{k_\nu-1}{2} + \frac{\nu}{2}\tau \right)^{2\alpha-1} \theta_2 \left( v + \frac{k_\nu-1}{2} + \frac{\nu}{2}\tau \right)^{-2\beta-1} \theta_3 \left( v + \frac{k_\nu-1}{2} + \frac{\nu}{2}\tau \right)^{2\beta-2\gamma+3} \\ &+ \frac{1 - (-1)^{l_{\nu+1}}}{2} e^{\pi i(1-\gamma)(l_{\nu+1}+1)\theta} \theta \left( v + \frac{k_\nu-1}{2} + \frac{\nu}{2}\tau \right)^{2\alpha-1} \theta_1 \left( v + \frac{k_\nu-1}{2} + \frac{\nu}{2}\tau \right)^{2\gamma-2\alpha-1} \theta_2 \left( v + \frac{k_\nu-1}{2} + \frac{\nu}{2}\tau \right)^{2\beta-2\gamma+3} \theta_3 \left( v + \frac{k_\nu-1}{2} + \frac{\nu}{2}\tau \right)^{-2\beta-1} \end{aligned}$$

and

$$\begin{aligned} & \left[ \theta \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu}{2}\tau \right)^{2\alpha-1} \theta_1 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu}{2}\tau \right)^{2\gamma-2\alpha-1} \theta_2 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu}{2}\tau \right)^{2\beta-2\gamma+3} \theta_3 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu}{2}\tau \right)^{-2\beta-1} \right]_- \\ &= \frac{1 + (-1)^{l_{\nu+1}}}{2} e^{\pi i(\gamma-1)l_{\nu+1}} e^{\pi i(2\gamma-2\alpha-1)\theta} \theta \left( v + \frac{k_\nu-1}{2} + \frac{\nu}{2}\tau \right)^{2\gamma-2\alpha-1} \theta_1 \left( v + \frac{k_\nu-1}{2} + \frac{\nu}{2}\tau \right)^{2\alpha-1} \theta_2 \left( v + \frac{k_\nu-1}{2} + \frac{\nu}{2}\tau \right)^{-2\beta-1} \theta_3 \left( v + \frac{k_\nu-1}{2} + \frac{\nu}{2}\tau \right)^{2\beta-2\gamma+3} \\ &+ \frac{1 - (-1)^{l_{\nu+1}}}{2} e^{\pi i(\gamma-1)(l_{\nu+1}+1)\theta} \theta \left( v + \frac{k_\nu-1}{2} + \frac{\nu}{2}\tau \right)^{2\alpha-1} \theta_1 \left( v + \frac{k_\nu-1}{2} + \frac{\nu}{2}\tau \right)^{2\gamma-2\alpha-1} \theta_2 \left( v + \frac{k_\nu-1}{2} + \frac{\nu}{2}\tau \right)^{2\beta-2\gamma+3} \theta_3 \left( v + \frac{k_\nu-1}{2} + \frac{\nu}{2}\tau \right)^{-2\beta-1}. \end{aligned}$$

Combining these relations, we have

$$\begin{aligned} & \left[ \theta \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu}{2}\tau \right)^{2\alpha-1} \theta_1 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu}{2}\tau \right)^{2\gamma-2\alpha-1} \theta_2 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu}{2}\tau \right)^{2\beta-2\gamma+3} \theta_3 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu}{2}\tau \right)^{-2\beta-1} \right]_+ \\ &= \left\{ \frac{1 + (-1)^{l_{\nu+1}}}{2} e^{2\pi i(1-\gamma)l_{\nu+1}} e^{4\pi i(\alpha-\gamma)\theta} + \frac{1 - (-1)^{l_{\nu+1}}}{2} e^{2\pi i(1-\gamma)(l_{\nu+1}+1)\theta} \right\} \times \\ & \times \left[ \theta \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu}{2}\tau \right)^{2\alpha-1} \theta_1 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu}{2}\tau \right)^{2\gamma-2\alpha-1} \theta_2 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu}{2}\tau \right)^{2\beta-2\gamma+3} \theta_3 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu}{2}\tau \right)^{-2\beta-1} \right]_-. \end{aligned}$$

Now we have

$$\begin{aligned} & \left[ \theta \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu}{2}\tau \right)^{2\alpha-1} \theta_1 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu}{2}\tau \right)^{2\gamma-2\alpha-1} \theta_2 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu}{2}\tau \right)^{2\beta-2\gamma+3} \theta_3 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu}{2}\tau \right)^{-2\beta-1} \right]_- \\ &= e^{\pi i(\alpha+\beta-\gamma+1)\theta} \left[ \theta \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-1}{2}\tau \right)^{2\beta-2\gamma+3} \theta_1 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-1}{2}\tau \right)^{-2\beta-1} \theta_2 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-1}{2}\tau \right)^{2\alpha-1} \theta_3 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-1}{2}\tau \right)^{2\gamma-2\alpha-1} \right]_+. \end{aligned}$$

Combining the preceding two relations, we have

$$\begin{aligned}
& \left[ \theta \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu}{2} \tau \right)^{2\alpha-1} \theta_1 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu}{2} \tau \right)^{2\gamma-2\alpha-1} \theta_2 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu}{2} \tau \right)^{2\beta-2\gamma+3} \theta_3 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu}{2} \tau \right)^{-2\beta-1} \right]_+ \\
&= \left\{ \frac{1 + (-1)^{l_{\nu+1}}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1 - (-1)^{l_{\nu+1}}}{2} e^{2\pi i(1-\gamma)} \right\} e^{2\pi i(1-\gamma)l_{\nu+1}} e^{\pi i(\alpha+\beta-\gamma+1)} \times \\
& \times \left[ \theta \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-1}{2} \tau \right)^{2\beta-2\gamma+3} \theta_1 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-1}{2} \tau \right)^{-2\beta-1} \theta_2 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-1}{2} \tau \right)^{2\alpha-1} \theta_3 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-1}{2} \tau \right)^{2\gamma-2\alpha-1} \right]_+ .
\end{aligned} \tag{7.1}$$

On the other hand, we have

$$\begin{aligned}
& \left[ \theta \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-1}{2} \tau \right)^{2\beta-2\gamma+3} \theta_1 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-1}{2} \tau \right)^{-2\beta-1} \theta_2 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-1}{2} \tau \right)^{2\alpha-1} \theta_3 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-1}{2} \tau \right)^{2\gamma-2\alpha-1} \right]_+ \\
&= \frac{1 + (-1)^{l_{\nu+1}+l_{\nu}}}{2} e^{\pi i(\gamma-1)(l_{\nu+1}+l_{\nu})} e^{\pi i(2\beta+1)} \theta \left( v + \frac{k_{\nu-1}-1}{2} + \frac{\nu-1}{2} \tau \right)^{-2\beta-1} \theta_1 \left( v + \frac{k_{\nu-1}-1}{2} + \frac{\nu-1}{2} \tau \right)^{2\beta-2\gamma+3} \theta_2 \left( v + \frac{k_{\nu-1}-1}{2} + \frac{\nu-1}{2} \tau \right)^{2\gamma-2\alpha-1} \theta_3 \left( v + \frac{k_{\nu-1}-1}{2} + \frac{\nu-1}{2} \tau \right)^{2\alpha-1} \\
&+ \frac{1 - (-1)^{l_{\nu+1}+l_{\nu}}}{2} e^{\pi i(\gamma-1)(l_{\nu+1}+l_{\nu}+1)} \theta \left( v + \frac{k_{\nu-1}-1}{2} + \frac{\nu-1}{2} \tau \right)^{2\beta-2\gamma+3} \theta_1 \left( v + \frac{k_{\nu-1}-1}{2} + \frac{\nu-1}{2} \tau \right)^{-2\beta-1} \theta_2 \left( v + \frac{k_{\nu-1}-1}{2} + \frac{\nu-1}{2} \tau \right)^{2\alpha-1} \theta_3 \left( v + \frac{k_{\nu-1}-1}{2} + \frac{\nu-1}{2} \tau \right)^{2\gamma-2\alpha-1}
\end{aligned}$$

and

$$\begin{aligned}
& \left[ \theta \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-1}{2} \tau \right)^{2\beta-2\gamma+3} \theta_1 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-1}{2} \tau \right)^{-2\beta-1} \theta_2 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-1}{2} \tau \right)^{2\alpha-1} \theta_3 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-1}{2} \tau \right)^{2\gamma-2\alpha-1} \right]_- \\
&= \frac{1 + (-1)^{l_{\nu+1}+l_{\nu}}}{2} e^{\pi i(1-\gamma)(l_{\nu+1}+l_{\nu})} e^{-\pi i(2\beta+1)} \theta \left( v + \frac{k_{\nu-1}-1}{2} + \frac{\nu-1}{2} \tau \right)^{-2\beta-1} \theta_1 \left( v + \frac{k_{\nu-1}-1}{2} + \frac{\nu-1}{2} \tau \right)^{2\beta-2\gamma+3} \theta_2 \left( v + \frac{k_{\nu-1}-1}{2} + \frac{\nu-1}{2} \tau \right)^{2\gamma-2\alpha-1} \theta_3 \left( v + \frac{k_{\nu-1}-1}{2} + \frac{\nu-1}{2} \tau \right)^{2\alpha-1} \\
&+ \frac{1 - (-1)^{l_{\nu+1}+l_{\nu}}}{2} e^{\pi i(1-\gamma)(l_{\nu+1}+l_{\nu}+1)} \theta \left( v + \frac{k_{\nu-1}-1}{2} + \frac{\nu-1}{2} \tau \right)^{2\beta-2\gamma+3} \theta_1 \left( v + \frac{k_{\nu-1}-1}{2} + \frac{\nu-1}{2} \tau \right)^{-2\beta-1} \theta_2 \left( v + \frac{k_{\nu-1}-1}{2} + \frac{\nu-1}{2} \tau \right)^{2\alpha-1} \theta_3 \left( v + \frac{k_{\nu-1}-1}{2} + \frac{\nu-1}{2} \tau \right)^{2\gamma-2\alpha-1} .
\end{aligned}$$

Combining these two relations, we have

$$\begin{aligned}
& \left[ \theta \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-1}{2} \tau \right)^{2\beta-2\gamma+3} \theta_1 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-1}{2} \tau \right)^{-2\beta-1} \theta_2 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-1}{2} \tau \right)^{2\alpha-1} \theta_3 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-1}{2} \tau \right)^{2\gamma-2\alpha-1} \right]_+ \\
&= \left\{ \frac{1 + (-1)^{l_{\nu+1}+l_{\nu}}}{2} e^{4\pi i\beta} + \frac{1 - (-1)^{l_{\nu+1}+l_{\nu}}}{2} e^{2\pi i(\gamma-1)} \right\} e^{2\pi i(\gamma-1)(l_{\nu+1}+l_{\nu})} \times \\
& \times \left[ \theta \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-1}{2} \tau \right)^{2\beta-2\gamma+3} \theta_1 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-1}{2} \tau \right)^{-2\beta-1} \theta_2 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-1}{2} \tau \right)^{2\alpha-1} \theta_3 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-1}{2} \tau \right)^{2\gamma-2\alpha-1} \right]_- .
\end{aligned}$$

Now we have

$$\begin{aligned} & \left[ \theta \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-1}{2} \tau \right)^{2\beta-2\gamma+3} \theta_1 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-1}{2} \tau \right)^{-2\beta-1} \theta_2 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-1}{2} \tau \right)^{2\alpha-1} \theta_3 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-1}{2} \tau \right)^{2\gamma-2\alpha-1} \right]_- \\ & = e^{\pi i(\alpha+\beta-\gamma+1)} \left[ \theta \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-2}{2} \tau \right)^{2\alpha-1} \theta_1 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-2}{2} \tau \right)^{2\gamma-2\alpha-1} \theta_2 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-2}{2} \tau \right)^{2\beta-2\gamma+3} \theta_3 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-2}{2} \tau \right)^{-2\beta-1} \right]_+ . \end{aligned}$$

Combining the preceding two relations, we have

$$\begin{aligned} & \left[ \theta \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-1}{2} \tau \right)^{2\beta-2\gamma+3} \theta_1 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-1}{2} \tau \right)^{-2\beta-1} \theta_2 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-1}{2} \tau \right)^{2\alpha-1} \theta_3 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-1}{2} \tau \right)^{2\gamma-2\alpha-1} \right]_+ \\ & = \left\{ \frac{1 + (-1)^{l_{\nu+1}+l_{\nu}}}{2} e^{4\pi i\beta} + \frac{1 - (-1)^{l_{\nu+1}+l_{\nu}}}{2} e^{2\pi i(\gamma-1)} \right\} e^{2\pi i(\gamma-1)(l_{\nu+1}+l_{\nu})} e^{\pi i(\alpha+\beta-\gamma+1)} \times \\ & \quad \times \left[ \theta \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-2}{2} \tau \right)^{2\alpha-1} \theta_1 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-2}{2} \tau \right)^{2\gamma-2\alpha-1} \theta_2 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-2}{2} \tau \right)^{2\beta-2\gamma+3} \theta_3 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-2}{2} \tau \right)^{-2\beta-1} \right]_+ . \end{aligned} \quad (7.2)$$

From (7.1) and (7.2) we have

$$\begin{aligned} & \left[ \theta \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu}{2} \tau \right)^{2\alpha-1} \theta_1 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu}{2} \tau \right)^{2\gamma-2\alpha-1} \theta_2 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu}{2} \tau \right)^{2\beta-2\gamma+3} \theta_3 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu}{2} \tau \right)^{-2\beta-1} \right]_+ \\ & = \left\{ \frac{1 + (-1)^{l_{\nu+1}}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1 - (-1)^{l_{\nu+1}}}{2} e^{2\pi i(1-\gamma)} \right\} \left\{ \frac{1 + (-1)^{l_{\nu+1}+l_{\nu}}}{2} e^{4\pi i\beta} + \frac{1 - (-1)^{l_{\nu+1}+l_{\nu}}}{2} e^{2\pi i(\gamma-1)} \right\} e^{2\pi i(\gamma-1)l_{\nu}} e^{2\pi i(\alpha+\beta-\gamma+1)} \times \\ & \quad \times \left[ \theta \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-2}{2} \tau \right)^{2\alpha-1} \theta_1 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-2}{2} \tau \right)^{2\gamma-2\alpha-1} \theta_2 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-2}{2} \tau \right)^{2\beta-2\gamma+3} \theta_3 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-2}{2} \tau \right)^{-2\beta-1} \right]_+ . \end{aligned} \quad (7.3)$$

Repeating the same procedure, we arrive at the following

**Lemma 7.1.** For a positive even integer  $\nu$  we have

$$\begin{aligned}
& \left[ \theta \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu}{2} \tau \right)^{2\alpha-1} \theta_1 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu}{2} \tau \right)^{2\gamma-2\alpha-1} \theta_2 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu}{2} \tau \right)^{2\beta-2\gamma+3} \theta_3 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu}{2} \tau \right)^{-2\beta-1} \right]_+ \\
&= \left\{ \frac{1 + (-1)^{l_{\nu+1}}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1 - (-1)^{l_{\nu+1}}}{2} e^{2\pi i(1-\gamma)} \right\} \left\{ \frac{1 + (-1)^{l_{\nu+1}+l_{\nu}}}{2} e^{4\pi i\beta} + \frac{1 - (-1)^{l_{\nu+1}+l_{\nu}}}{2} e^{2\pi i(\gamma-1)} \right\} \times \\
&\times \left\{ \frac{1 + (-1)^{l_{\nu+1}+l_{\nu}+l_{\nu-1}}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1 - (-1)^{l_{\nu+1}+l_{\nu}+l_{\nu-1}}}{2} e^{2\pi i(1-\gamma)} \right\} \left\{ \frac{1 + (-1)^{l_{\nu+1}+l_{\nu}+l_{\nu-1}+l_{\nu-2}}}{2} e^{4\pi i\beta} + \frac{1 - (-1)^{l_{\nu+1}+l_{\nu}+l_{\nu-1}+l_{\nu-2}}}{2} e^{2\pi i(\gamma-1)} \right\} \dots \quad (7.4) \\
&\dots \left\{ \frac{1 + (-1)^{l_{\nu+1}+l_{\nu}+l_{\nu-1}+\dots+l_3}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1 - (-1)^{l_{\nu+1}+l_{\nu}+l_{\nu-1}+\dots+l_3}}{2} e^{2\pi i(1-\gamma)} \right\} \left\{ \frac{1 + (-1)^{l_{\nu+1}+l_{\nu}+l_{\nu-1}+\dots+l_2}}{2} e^{4\pi i\beta} + \frac{1 - (-1)^{l_{\nu+1}+l_{\nu}+l_{\nu-1}+\dots+l_2}}{2} e^{2\pi i(\gamma-1)} \right\} \times \\
&\times e^{2\pi i(\gamma-1)(l_{\nu}+l_{\nu-2}+\dots+l_4+l_2)} e^{\nu\pi i(\alpha+\beta-\gamma+1)} \left[ \theta \left( v + \frac{k_{\nu+1}}{2} \right)^{2\alpha-1} \theta_1 \left( v + \frac{k_{\nu+1}}{2} \right)^{2\gamma-2\alpha-1} \theta_2 \left( v + \frac{k_{\nu+1}}{2} \right)^{2\beta-2\gamma+3} \theta_3 \left( v + \frac{k_{\nu+1}}{2} \right)^{-2\beta-1} \right]_+ .
\end{aligned}$$

Let us now compute  $J_{\nu}$  for a positive even integer  $\nu$ . From (1.4) we have

$$J_{\nu} = \int_0^{\tau/2} \theta \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu}{2} \tau \right)^{2\alpha-1} \theta_1 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu}{2} \tau \right)^{2\gamma-2\alpha-1} \theta_2 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu}{2} \tau \right)^{2\beta-2\gamma+3} \theta_3 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu}{2} \tau \right)^{-2\beta-1} dv. \quad (7.5)$$

Substituting (7.4) into (7.5), we have

$$\begin{aligned}
J_{\nu} &= \left\{ \frac{1 + (-1)^{l_{\nu+1}}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1 - (-1)^{l_{\nu+1}}}{2} e^{2\pi i(1-\gamma)} \right\} \left\{ \frac{1 + (-1)^{l_{\nu+1}+l_{\nu}}}{2} e^{4\pi i\beta} + \frac{1 - (-1)^{l_{\nu+1}+l_{\nu}}}{2} e^{2\pi i(\gamma-1)} \right\} \times \\
&\times \left\{ \frac{1 + (-1)^{l_{\nu+1}+l_{\nu}+l_{\nu-1}}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1 - (-1)^{l_{\nu+1}+l_{\nu}+l_{\nu-1}}}{2} e^{2\pi i(1-\gamma)} \right\} \left\{ \frac{1 + (-1)^{l_{\nu+1}+l_{\nu}+l_{\nu-1}+l_{\nu-2}}}{2} e^{4\pi i\beta} + \frac{1 - (-1)^{l_{\nu+1}+l_{\nu}+l_{\nu-1}+l_{\nu-2}}}{2} e^{2\pi i(\gamma-1)} \right\} \dots \\
&\dots \left\{ \frac{1 + (-1)^{l_{\nu+1}+l_{\nu}+l_{\nu-1}+\dots+l_3}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1 - (-1)^{l_{\nu+1}+l_{\nu}+l_{\nu-1}+\dots+l_3}}{2} e^{2\pi i(1-\gamma)} \right\} \left\{ \frac{1 + (-1)^{l_{\nu+1}+l_{\nu}+l_{\nu-1}+\dots+l_2}}{2} e^{4\pi i\beta} + \frac{1 - (-1)^{l_{\nu+1}+l_{\nu}+l_{\nu-1}+\dots+l_2}}{2} e^{2\pi i(\gamma-1)} \right\} \times \quad (7.6) \\
&\times e^{2\pi i(\gamma-1)(l_{\nu}+l_{\nu-2}+\dots+l_4+l_2)} e^{\nu\pi i(\alpha+\beta-\gamma+1)} \int_0^{\tau/2} \theta \left( v + \frac{k_{\nu+1}}{2} \right)^{2\alpha-1} \theta_1 \left( v + \frac{k_{\nu+1}}{2} \right)^{2\gamma-2\alpha-1} \theta_2 \left( v + \frac{k_{\nu+1}}{2} \right)^{2\beta-2\gamma+3} \theta_3 \left( v + \frac{k_{\nu+1}}{2} \right)^{-2\beta-1} dv.
\end{aligned}$$

Since

$$\begin{aligned}
& \theta \left( v + \frac{k_{\nu+1}}{2} \right)^{2\alpha-1} \theta_1 \left( v + \frac{k_{\nu+1}}{2} \right)^{2\gamma-2\alpha-1} \theta_2 \left( v + \frac{k_{\nu+1}}{2} \right)^{2\beta-2\gamma+3} \theta_3 \left( v + \frac{k_{\nu+1}}{2} \right)^{-2\beta-1} \\
&= \frac{1 + (-1)^{k_{\nu+1}}}{2} e^{\pi i(1-\gamma)k_{\nu+1}} \theta(v)^{2\alpha-1} \theta_1(v)^{2\gamma-2\alpha-1} \theta_2(v)^{2\beta-2\gamma+3} \theta_3(v)^{-2\beta-1} + \frac{1 - (-1)^{k_{\nu+1}}}{2} e^{\pi i(1-\gamma)(k_{\nu+1}-1)} e^{\pi i(2\alpha-2\gamma+1)} \theta(v)^{2\gamma-2\alpha-1} \theta_1(v)^{2\alpha-1} \theta_2(v)^{-2\beta-1} \theta_3(v)^{2\beta-2\gamma+3},
\end{aligned}$$

(7.6) is turned to

$$\begin{aligned}
J_\nu &= \left\{ \frac{1 + (-1)^{l_{\nu+1}}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1 - (-1)^{l_{\nu+1}}}{2} e^{2\pi i(1-\gamma)} \right\} \left\{ \frac{1 + (-1)^{l_{\nu+1}+l_\nu}}{2} e^{4\pi i\beta} + \frac{1 - (-1)^{l_{\nu+1}+l_\nu}}{2} e^{2\pi i(\gamma-1)} \right\} \times \\
&\times \left\{ \frac{1 + (-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1 - (-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}}}{2} e^{2\pi i(1-\gamma)} \right\} \left\{ \frac{1 + (-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}+l_{\nu-2}}}{2} e^{4\pi i\beta} + \frac{1 - (-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}+l_{\nu-2}}}{2} e^{2\pi i(\gamma-1)} \right\} \dots \\
&\dots \left\{ \frac{1 + (-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}+\dots+l_3}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1 - (-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}+\dots+l_3}}{2} e^{2\pi i(1-\gamma)} \right\} \left\{ \frac{1 + (-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}+\dots+l_2}}{2} e^{4\pi i\beta} + \frac{1 - (-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}+\dots+l_2}}{2} e^{2\pi i(\gamma-1)} \right\} \times \\
&\times e^{2\pi i(\gamma-1)(l_\nu+l_{\nu-2}+\dots+l_4+l_2)} e^{\nu\pi i(\alpha+\beta-\gamma+1)} \left\{ \frac{1 + (-1)^{k_{\nu+1}}}{2} e^{\pi i(1-\gamma)k_{\nu+1}} \int_0^{\tau/2} \theta(v)^{2\alpha-1} \theta_1(v)^{2\gamma-2\alpha-1} \theta_2(v)^{2\beta-2\gamma+3} \theta_3(v)^{-2\beta-1} dv \right. \\
&\left. + \frac{1 - (-1)^{k_{\nu+1}}}{2} e^{\pi i(1-\gamma)(k_{\nu+1}-1)} e^{\pi i(2\alpha-2\gamma+1)} \int_0^{\tau/2} \theta(v)^{2\gamma-2\alpha-1} \theta_1(v)^{2\alpha-1} \theta_2(v)^{-2\beta-1} \theta_3(v)^{2\beta-2\gamma+3} dv \right\},
\end{aligned}$$

from which it follows that

$$\begin{aligned}
J_\nu &= e^{\pi i\{(-1)^{k_{\nu+1}-k_\nu} + (-1)^{k_{\nu+1}-k_{\nu-2}} + \dots + (-1)^{k_{\nu+1}-k_2\}}(2\alpha-\gamma-1) + \{(-1)^{k_{\nu+1}-k_{\nu-1}} + (-1)^{k_{\nu+1}-k_{\nu-3}} + \dots + (-1)^{k_{\nu+1}-k_1\}}(2\beta-\gamma+1)} e^{2\pi i(\gamma-1)(l_\nu+l_{\nu-2}+\dots+l_2)} e^{2\nu\pi i(\alpha+\beta-\gamma)} e^{\pi i(1-\gamma)k_{\nu+1}} \times \\
&\times \left\{ \frac{1 + (-1)^{k_{\nu+1}}}{2} \int_0^{\tau/2} \theta(v)^{2\alpha-1} \theta_1(v)^{2\gamma-2\alpha-1} \theta_2(v)^{2\beta-2\gamma+3} \theta_3(v)^{-2\beta-1} dv + \frac{1 - (-1)^{k_{\nu+1}}}{2} e^{\pi i(2\alpha-\gamma)} \int_0^{\tau/2} \theta(v)^{2\gamma-2\alpha-1} \theta_1(v)^{2\alpha-1} \theta_2(v)^{-2\beta-1} \theta_3(v)^{2\beta-2\gamma+3} dv \right\}. \tag{7.7}
\end{aligned}$$

Applying the formulas of Lemma 2.2 to (7.7), we have

$$\begin{aligned}
J_\nu &= e^{\pi i\{(-1)^{k_{\nu+1}-k_\nu} + (-1)^{k_{\nu+1}-k_{\nu-2}} + \dots + (-1)^{k_{\nu+1}-k_2\}}(2\alpha-\gamma-1) + \{(-1)^{k_{\nu+1}-k_{\nu-1}} + (-1)^{k_{\nu+1}-k_{\nu-3}} + \dots + (-1)^{k_{\nu+1}-k_1\}}(2\beta-\gamma+1)} \times \\
&\times e^{2\pi i(\gamma-1)(l_\nu+l_{\nu-2}+\dots+l_2)} e^{2\nu\pi i(\alpha+\beta-\gamma)} e^{\pi i(1-\gamma)k_{\nu+1}} \left\{ \frac{1 + (-1)^{k_{\nu+1}}}{2} \frac{1 - e^{2\pi i(\alpha-\gamma)}}{1 - e^{-2\pi i\gamma}} + \frac{1 - (-1)^{k_{\nu+1}}}{2} e^{\pi i(2\alpha-\gamma)} \frac{1 - e^{-2\pi i\alpha}}{1 - e^{-2\pi i\gamma}} \right\} z_1(\tau)/\theta_3^2 \\
&+ e^{\pi i\{(-1)^{k_{\nu+1}-k_\nu} + (-1)^{k_{\nu+1}-k_{\nu-2}} + \dots + (-1)^{k_{\nu+1}-k_2\}}(2\alpha-\gamma-1) + \{(-1)^{k_{\nu+1}-k_{\nu-1}} + (-1)^{k_{\nu+1}-k_{\nu-3}} + \dots + (-1)^{k_{\nu+1}-k_1\}}(2\beta-\gamma+1)} \times \\
&\times e^{2\pi i(\gamma-1)(l_\nu+l_{\nu-2}+\dots+l_2)} e^{2\nu\pi i(\alpha+\beta-\gamma)} e^{\pi i(1-\gamma)k_{\nu+1}} \left\{ \frac{1 + (-1)^{k_{\nu+1}}}{2} \frac{e^{\pi i(\alpha-\gamma)}(e^{\pi i\beta} - e^{-\pi i\beta})}{1 - e^{-2\pi i\gamma}} + \frac{1 - (-1)^{k_{\nu+1}}}{2} e^{\pi i(2\alpha-\gamma)} \frac{e^{\pi i(\gamma-\alpha-\beta)} - e^{-\pi i(-\alpha+\beta-\gamma)}}{1 - e^{-2\pi i\gamma}} \right\} z_2(\tau)/\theta_3^2 \\
&= e^{\pi i\{(-1)^{k_{\nu+1}-k_\nu} + (-1)^{k_{\nu+1}-k_{\nu-2}} + \dots + (-1)^{k_{\nu+1}-k_2\}}(2\alpha-\gamma-1) + \{(-1)^{k_{\nu+1}-k_{\nu-1}} + (-1)^{k_{\nu+1}-k_{\nu-3}} + \dots + (-1)^{k_{\nu+1}-k_1\}}(2\beta-\gamma+1)} \times \\
&\times e^{2\pi i(\gamma-1)(l_\nu+l_{\nu-2}+\dots+l_2)} e^{2\nu\pi i(\alpha+\beta-\gamma)} e^{\pi i(1-\gamma)k_{\nu+1}} \frac{e^{\pi i(\alpha-\gamma)}}{1 - e^{-2\pi i\gamma}} \left\{ e^{\pi i\gamma/2} e^{(-1)^{k_{\nu+1}}\pi i(\gamma-2\alpha)/2} - e^{-\pi i\gamma/2} e^{(-1)^{k_{\nu+1}}\pi i(2\alpha-\gamma)/2} \right\} z_1(\tau)/\theta_3^2 \\
&+ e^{\pi i\{(-1)^{k_{\nu+1}-k_\nu} + (-1)^{k_{\nu+1}-k_{\nu-2}} + \dots + (-1)^{k_{\nu+1}-k_2\}}(2\alpha-\gamma-1) + \{(-1)^{k_{\nu+1}-k_{\nu-1}} + (-1)^{k_{\nu+1}-k_{\nu-3}} + \dots + (-1)^{k_{\nu+1}-k_1\}}(2\beta-\gamma+1)} \times \\
&\times e^{2\pi i(\gamma-1)(l_\nu+l_{\nu-2}+\dots+l_2)} e^{2\nu\pi i(\alpha+\beta-\gamma)} e^{\pi i(1-\gamma)k_{\nu+1}} \frac{e^{\pi i(\alpha-\gamma)}}{1 - e^{-2\pi i\gamma}} \left\{ e^{\pi i\gamma/2} e^{(-1)^{k_{\nu+1}}\pi i(2\beta-\gamma)/2} - e^{-\pi i\gamma/2} e^{(-1)^{k_{\nu+1}}\pi i(\gamma-2\beta)/2} \right\} z_2(\tau)/\theta_3^2,
\end{aligned}$$

and therefore

$$\begin{aligned}
J_\nu &= e^{\pi i \{(-1)^{k_{\nu+1}-k_\nu} + (-1)^{k_{\nu+1}-k_{\nu-2}} + \dots + (-1)^{k_{\nu+1}-k_2\}} (2\alpha-\gamma-1) + \{(-1)^{k_{\nu+1}-k_{\nu-1}} + (-1)^{k_{\nu+1}-k_{\nu-3}} + \dots + (-1)^{k_{\nu+1}-k_1\}} (2\beta-\gamma+1)} \times \\
&\quad \times e^{2\pi i(\gamma-1)(l_\nu+l_{\nu-2}+\dots+l_2)} e^{2\nu\pi i(\alpha+\beta-\gamma)} e^{\pi i(1-\gamma)k_{\nu+1}} e^{\pi i\alpha} \frac{\sin \pi \{\gamma + (-1)^{k_{\nu+1}}(\gamma-2\alpha)\}/2}{\sin \pi\gamma} z_1(\tau)/\theta_3^2 \\
&\quad + e^{\pi i \{(-1)^{k_{\nu+1}-k_\nu} + (-1)^{k_{\nu+1}-k_{\nu-2}} + \dots + (-1)^{k_{\nu+1}-k_2\}} (2\alpha-\gamma-1) + \{(-1)^{k_{\nu+1}-k_{\nu-1}} + (-1)^{k_{\nu+1}-k_{\nu-3}} + \dots + (-1)^{k_{\nu+1}-k_1\}} (2\beta-\gamma+1)} \times \\
&\quad \times e^{2\pi i(\gamma-1)(l_\nu+l_{\nu-2}+\dots+l_2)} e^{2\nu\pi i(\alpha+\beta-\gamma)} e^{\pi i(1-\gamma)k_{\nu+1}} e^{\pi i\alpha} \frac{\sin \pi \{\gamma + (-1)^{k_{\nu+1}}(2\beta-\gamma)\}/2}{\sin \pi\gamma} z_2(\tau)/\theta_3^2.
\end{aligned} \tag{7.8}$$

### §8 The integrals $J_\nu$ for positive odd integers $\nu$ .

In this section let  $\nu$  be a positive odd integer. The formula (7.1) holds also in this case. In case where  $\nu \geq 3$ , (7.3) holds too. Moreover we have

$$\begin{aligned}
&\left[ \theta \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-2}{2}\tau \right)^{2\alpha-1} \theta_1 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-2}{2}\tau \right)^{2\gamma-2\alpha-1} \theta_2 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-2}{2}\tau \right)^{2\beta-2\gamma+3} \theta_3 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-2}{2}\tau \right)^{-2\beta-1} \right]_+ \\
&= \left\{ \frac{1 + (-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1 - (-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}}}{2} e^{2\pi i(1-\gamma)} \right\} e^{2\pi i(1-\gamma)(l_{\nu+1}+l_\nu+l_{\nu-1})} e^{\pi i(\alpha+\beta-\gamma+1)} \times \\
&\quad \times \left[ \theta \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-3}{2}\tau \right)^{2\beta-2\gamma+3} \theta_1 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-3}{2}\tau \right)^{-2\beta-1} \theta_2 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-3}{2}\tau \right)^{2\alpha-1} \theta_3 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-3}{2}\tau \right)^{2\gamma-2\alpha-1} \right]_+.
\end{aligned} \tag{8.1}$$

Combining (8.1) with (7.3), we have

$$\begin{aligned}
&\left[ \theta \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu}{2}\tau \right)^{2\alpha-1} \theta_1 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu}{2}\tau \right)^{2\gamma-2\alpha-1} \theta_2 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu}{2}\tau \right)^{2\beta-2\gamma+3} \theta_3 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu}{2}\tau \right)^{-2\beta-1} \right]_+ \\
&= \left\{ \frac{1 + (-1)^{l_{\nu+1}}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1 - (-1)^{l_{\nu+1}}}{2} e^{2\pi i(1-\gamma)} \right\} \left\{ \frac{1 + (-1)^{l_{\nu+1}+l_\nu}}{2} e^{4\pi i\beta} + \frac{1 - (-1)^{l_{\nu+1}+l_\nu}}{2} e^{2\pi i(\gamma-1)} \right\} \times \\
&\quad \times \left\{ \frac{1 + (-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1 - (-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}}}{2} e^{2\pi i(1-\gamma)} \right\} e^{2\pi i(1-\gamma)(l_{\nu+1}+l_\nu+l_{\nu-1})} e^{3\pi i(\alpha+\beta-\gamma+1)} \times \\
&\quad \times \left[ \theta \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-3}{2}\tau \right)^{2\beta-2\gamma+3} \theta_1 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-3}{2}\tau \right)^{-2\beta-1} \theta_2 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-3}{2}\tau \right)^{2\alpha-1} \theta_3 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu-3}{2}\tau \right)^{2\gamma-2\alpha-1} \right]_+.
\end{aligned} \tag{8.2}$$

Repeating the same procedure, we arrive at the following



**Lemma 8.1.** For a positive odd integer  $\nu$  we have

$$\begin{aligned}
& \left[ \theta \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu}{2} \tau \right)^{2\alpha-1} \theta_1 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu}{2} \tau \right)^{2\gamma-2\alpha-1} \theta_2 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu}{2} \tau \right)^{2\beta-2\gamma+3} \theta_3 \left( v + \frac{k_{\nu+1}}{2} + \frac{\nu}{2} \tau \right)^{-2\beta-1} \right]_+ \\
&= \left\{ \frac{1 + (-1)^{l_{\nu+1}}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1 - (-1)^{l_{\nu+1}}}{2} e^{2\pi i(1-\gamma)} \right\} \left\{ \frac{1 + (-1)^{l_{\nu+1}+l_{\nu}}}{2} e^{4\pi i\beta} + \frac{1 - (-1)^{l_{\nu+1}+l_{\nu}}}{2} e^{2\pi i(\gamma-1)} \right\} \\
&\times \left\{ \frac{1 + (-1)^{l_{\nu+1}+l_{\nu}+l_{\nu-1}}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1 - (-1)^{l_{\nu+1}+l_{\nu}+l_{\nu-1}}}{2} e^{2\pi i(1-\gamma)} \right\} \dots \left\{ \frac{1 + (-1)^{l_{\nu+1}+l_{\nu}+\dots+l_3}}{2} e^{4\pi i\beta} + \frac{1 - (-1)^{l_{\nu+1}+l_{\nu}+\dots+l_3}}{2} e^{2\pi i(\gamma-1)} \right\} \\
&\times \left\{ \frac{1 + (-1)^{l_{\nu+1}+l_{\nu}+\dots+l_2}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1 - (-1)^{l_{\nu+1}+l_{\nu}+\dots+l_2}}{2} e^{2\pi i(1-\gamma)} \right\} e^{2\pi i(1-\gamma)(l_{\nu+1}+l_{\nu-1}+l_{\nu-3}+\dots+l_4+l_2)} e^{\nu\pi i(\alpha+\beta-\gamma+1)} \\
&\times \left[ \theta \left( v + \frac{k_{\nu+1}}{2} \right)^{2\beta-2\gamma+3} \theta_1 \left( v + \frac{k_{\nu+1}}{2} \right)^{-2\beta-1} \theta_2 \left( v + \frac{k_{\nu+1}}{2} \right)^{2\alpha-1} \theta_3 \left( v + \frac{k_{\nu+1}}{2} \right)^{2\gamma-2\alpha-1} \right]_+. \tag{8.3}
\end{aligned}$$

Let us now compute  $J_{\nu}$  for a positive odd integer  $\nu$ . Substituting (8.3) into (7.5), we have

$$\begin{aligned}
J_{\nu} &= \left\{ \frac{1 + (-1)^{l_{\nu+1}}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1 - (-1)^{l_{\nu+1}}}{2} e^{2\pi i(1-\gamma)} \right\} \left\{ \frac{1 + (-1)^{l_{\nu+1}+l_{\nu}}}{2} e^{4\pi i\beta} + \frac{1 - (-1)^{l_{\nu+1}+l_{\nu}}}{2} e^{2\pi i(\gamma-1)} \right\} \times \\
&\times \left\{ \frac{1 + (-1)^{l_{\nu+1}+l_{\nu}+l_{\nu-1}}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1 - (-1)^{l_{\nu+1}+l_{\nu}+l_{\nu-1}}}{2} e^{2\pi i(1-\gamma)} \right\} \dots \left\{ \frac{1 + (-1)^{l_{\nu+1}+l_{\nu}+\dots+l_3}}{2} e^{4\pi i\beta} + \frac{1 - (-1)^{l_{\nu+1}+l_{\nu}+\dots+l_3}}{2} e^{2\pi i(\gamma-1)} \right\} \times \\
&\times \left\{ \frac{1 + (-1)^{l_{\nu+1}+l_{\nu}+\dots+l_2}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1 - (-1)^{l_{\nu+1}+l_{\nu}+\dots+l_2}}{2} e^{2\pi i(1-\gamma)} \right\} e^{2\pi i(1-\gamma)(l_{\nu+1}+l_{\nu-1}+l_{\nu-3}+\dots+l_4+l_2)} e^{\nu\pi i(\alpha+\beta-\gamma+1)} \\
&\times \int_0^{\tau/2} \theta \left( v + \frac{k_{\nu+1}}{2} \right)^{2\beta-2\gamma+3} \theta_1 \left( v + \frac{k_{\nu+1}}{2} \right)^{-2\beta-1} \theta_2 \left( v + \frac{k_{\nu+1}}{2} \right)^{2\alpha-1} \theta_3 \left( v + \frac{k_{\nu+1}}{2} \right)^{2\gamma-2\alpha-1} dv. \tag{8.4}
\end{aligned}$$

Since

$$\begin{aligned}
& \theta \left( v + \frac{k_{\nu+1}}{2} \right)^{2\beta-2\gamma+3} \theta_1 \left( v + \frac{k_{\nu+1}}{2} \right)^{-2\beta-1} \theta_2 \left( v + \frac{k_{\nu+1}}{2} \right)^{2\alpha-1} \theta_3 \left( v + \frac{k_{\nu+1}}{2} \right)^{2\gamma-2\alpha-1} \\
&= \frac{1 + (-1)^{k_{\nu+1}}}{2} e^{\pi i(\gamma-1)k_{\nu+1}} \theta(v)^{2\beta-2\gamma+3} \theta_1(v)^{-2\beta-1} \theta_2(v)^{2\alpha-1} \theta_3(v)^{2\gamma-2\alpha-1} + \frac{1 - (-1)^{k_{\nu+1}}}{2} e^{\pi i(\gamma-1)(k_{\nu+1}-1)} e^{\pi i(2\beta+1)} \theta(v)^{-2\beta-1} \theta_1(v)^{2\beta-2\gamma+3} \theta_2(v)^{2\gamma-2\alpha-1} \theta_3(v)^{2\alpha-1},
\end{aligned}$$

(8.4) is turned to

$$\begin{aligned}
J_\nu = & \left\{ \frac{1 + (-1)^{l_{\nu+1}}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1 - (-1)^{l_{\nu+1}}}{2} e^{2\pi i(1-\gamma)} \right\} \left\{ \frac{1 + (-1)^{l_{\nu+1}+l_\nu}}{2} e^{4\pi i\beta} + \frac{1 - (-1)^{l_{\nu+1}+l_\nu}}{2} e^{2\pi i(\gamma-1)} \right\} \times \\
& \times \left\{ \frac{1 + (-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1 - (-1)^{l_{\nu+1}+l_\nu+l_{\nu-1}}}{2} e^{2\pi i(1-\gamma)} \right\} \dots \left\{ \frac{1 + (-1)^{l_{\nu+1}+l_\nu+\dots+l_3}}{2} e^{4\pi i\beta} + \frac{1 - (-1)^{l_{\nu+1}+l_\nu+\dots+l_3}}{2} e^{2\pi i(\gamma-1)} \right\} \times \\
& \times \left\{ \frac{1 + (-1)^{l_{\nu+1}+l_\nu+\dots+l_2}}{2} e^{4\pi i(\alpha-\gamma)} + \frac{1 - (-1)^{l_{\nu+1}+l_\nu+\dots+l_2}}{2} e^{2\pi i(1-\gamma)} \right\} e^{2\pi i(1-\gamma)(l_{\nu+1}+l_{\nu-1}+l_{\nu-3}+\dots+l_4+l_2)} e^{\nu\pi i(\alpha+\beta-\gamma+1)} \\
& \times \left\{ \frac{1 + (-1)^{k_{\nu+1}}}{2} e^{\pi i(\gamma-1)k_{\nu+1}} \int_0^{\tau/2} \theta(v)^{2\beta-2\gamma+3} \theta_1(v)^{-2\beta-1} \theta_2(v)^{2\alpha-1} \theta_3(v)^{2\gamma-2\alpha-1} dv \right. \\
& \left. + \frac{1 - (-1)^{k_{\nu+1}}}{2} e^{\pi i(\gamma-1)k_{\nu+1}} e^{\pi i(2\beta-\gamma)} \int_0^{\tau/2} \theta(v)^{-2\beta-1} \theta_1(v)^{2\beta-2\gamma+3} \theta_2(v)^{2\gamma-2\alpha-1} \theta_3(v)^{2\alpha-1} dv \right\},
\end{aligned}$$

from which it follows that

$$\begin{aligned}
J_\nu = & e^{\pi i\{(-1)^{k_{\nu+1}-k_\nu} + (-1)^{k_{\nu+1}-k_{\nu-2}} + \dots + (-1)^{k_{\nu+1}-k_1\}}(2\alpha-\gamma-1) + \{(-1)^{k_{\nu+1}-k_{\nu-1}} + (-1)^{k_{\nu+1}-k_{\nu-3}} + \dots + (-1)^{k_{\nu+1}-k_2\}}(2\beta-\gamma+1)} e^{\pi i(\alpha-\beta-2\gamma)} \\
& \times e^{2\pi i(1-\gamma)(l_{\nu+1}+l_{\nu-1}+\dots+l_4+l_2)} e^{2\nu\pi i(\alpha+\beta-\gamma)} e^{\pi i(\gamma-1)k_{\nu+1}} \left\{ \frac{1 + (-1)^{k_{\nu+1}}}{2} \int_0^{\tau/2} \theta(v)^{2\beta-2\gamma+3} \theta_1(v)^{-2\beta-1} \theta_2(v)^{2\alpha-1} \theta_3(v)^{2\gamma-2\alpha-1} dv \right. \\
& \left. + \frac{1 - (-1)^{k_{\nu+1}}}{2} e^{\pi i(2\beta-\gamma)} \int_0^{\tau/2} \theta(v)^{-2\beta-1} \theta_1(v)^{2\beta-2\gamma+3} \theta_2(v)^{2\gamma-2\alpha-1} \theta_3(v)^{2\alpha-1} dv \right\}. \tag{8.5}
\end{aligned}$$

Applying the formulas of Lemma 2.3 to (8.5), we have

$$\begin{aligned}
J_\nu = & e^{\pi i\{(-1)^{k_{\nu+1}-k_\nu} + (-1)^{k_{\nu+1}-k_{\nu-2}} + \dots + (-1)^{k_{\nu+1}-k_1\}}(2\alpha-\gamma-1) + \{(-1)^{k_{\nu+1}-k_{\nu-1}} + (-1)^{k_{\nu+1}-k_{\nu-3}} + \dots + (-1)^{k_{\nu+1}-k_2\}}(2\beta-\gamma+1)} e^{\pi i(\alpha-\beta-2\gamma)} \\
& \times e^{2\pi i(1-\gamma)(l_{\nu+1}+l_{\nu-1}+\dots+l_4+l_2)} e^{2\nu\pi i(\alpha+\beta-\gamma)} e^{\pi i(\gamma-1)k_{\nu+1}} \left\{ \frac{1 + (-1)^{k_{\nu+1}}}{2} \frac{e^{\pi i(\alpha+\beta-\gamma)} - e^{\pi i(\gamma-\alpha+\beta)}}{1 - e^{2\pi i\gamma}} + \frac{1 - (-1)^{k_{\nu+1}}}{2} \frac{e^{\pi i(-\alpha+\beta)} - e^{\pi i(\alpha+\beta)}}{1 - e^{2\pi i\gamma}} \right\} z_1(\tau)/\theta_3^2 \\
& + e^{\pi i\{(-1)^{k_{\nu+1}-k_\nu} + (-1)^{k_{\nu+1}-k_{\nu-2}} + \dots + (-1)^{k_{\nu+1}-k_1\}}(2\alpha-\gamma-1) + \{(-1)^{k_{\nu+1}-k_{\nu-1}} + (-1)^{k_{\nu+1}-k_{\nu-3}} + \dots + (-1)^{k_{\nu+1}-k_2\}}(2\beta-\gamma+1)} e^{\pi i(\alpha-\beta-2\gamma)} \\
& \times e^{2\pi i(1-\gamma)(l_{\nu+1}+l_{\nu-1}+\dots+l_4+l_2)} e^{2\nu\pi i(\alpha+\beta-\gamma)} e^{\pi i(\gamma-1)k_{\nu+1}} \left\{ \frac{1 + (-1)^{k_{\nu+1}}}{2} \frac{1 - e^{2\pi i\beta}}{1 - e^{2\pi i\gamma}} + \frac{1 - (-1)^{k_{\nu+1}}}{2} \frac{e^{\pi i(2\beta-\gamma)} - e^{\pi i\gamma}}{1 - e^{2\pi i\gamma}} \right\} z_2(\tau)/\theta_3^2 \\
= & e^{\pi i\{(-1)^{k_{\nu+1}-k_\nu} + (-1)^{k_{\nu+1}-k_{\nu-2}} + \dots + (-1)^{k_{\nu+1}-k_1\}}(2\alpha-\gamma-1) + \{(-1)^{k_{\nu+1}-k_{\nu-1}} + (-1)^{k_{\nu+1}-k_{\nu-3}} + \dots + (-1)^{k_{\nu+1}-k_2\}}(2\beta-\gamma+1)} e^{\pi i(\alpha-3\gamma)} \\
& \times e^{2\pi i(1-\gamma)(l_{\nu+1}+l_{\nu-1}+\dots+l_4+l_2)} e^{2\nu\pi i(\alpha+\beta-\gamma)} \frac{e^{\pi i(\gamma-1)k_{\nu+1}}}{e^{\pi i\gamma} - e^{-\pi i\gamma}} \left\{ e^{\pi i\gamma/2} e^{(-1)^{k_{\nu+1}}\pi i(\gamma-2\alpha)/2} - e^{-\pi i\gamma/2} e^{(-1)^{k_{\nu+1}}\pi i(2\alpha-\gamma)/2} \right\} z_1(\tau)/\theta_3^2 \\
& + e^{\pi i\{(-1)^{k_{\nu+1}-k_\nu} + (-1)^{k_{\nu+1}-k_{\nu-2}} + \dots + (-1)^{k_{\nu+1}-k_1\}}(2\alpha-\gamma-1) + \{(-1)^{k_{\nu+1}-k_{\nu-1}} + (-1)^{k_{\nu+1}-k_{\nu-3}} + \dots + (-1)^{k_{\nu+1}-k_2\}}(2\beta-\gamma+1)} e^{\pi i(\alpha-3\gamma)} \\
& \times e^{2\pi i(1-\gamma)(l_{\nu+1}+l_{\nu-1}+\dots+l_4+l_2)} e^{2\nu\pi i(\alpha+\beta-\gamma)} \frac{e^{\pi i(\gamma-1)k_{\nu+1}}}{e^{\pi i\gamma} - e^{-\pi i\gamma}} \left\{ e^{\pi i\gamma/2} e^{(-1)^{k_{\nu+1}}\pi i(2\beta-\gamma)/2} - e^{-\pi i\gamma/2} e^{(-1)^{k_{\nu+1}}\pi i(\gamma-2\beta)/2} \right\} z_2(\tau)/\theta_3^2,
\end{aligned}$$

and therefore

$$\begin{aligned}
J_\nu = & e^{\pi i \{ (-1)^{k_{\nu+1}-k_\nu} + (-1)^{k_{\nu+1}-k_{\nu-2}} + \dots + (-1)^{k_{\nu+1}-k_1} \} (2\alpha-\gamma-1) + \{ (-1)^{k_{\nu+1}-k_{\nu-1}} + (-1)^{k_{\nu+1}-k_{\nu-3}} + \dots + (-1)^{k_{\nu+1}-k_2} \} (2\beta-\gamma+1)} e^{2\pi i (1-\gamma)(l_{\nu+1}+l_{\nu-1}+\dots+l_4+l_2)} \\
& \times e^{2\nu\pi i(\alpha+\beta-\gamma)} e^{\pi i(\gamma-1)k_{\nu+1}} e^{\pi i(\alpha-3\gamma)} \frac{\sin \pi \{ \gamma + (-1)^{k_{\nu+1}}(\gamma-2\alpha) \} / 2}{\sin \pi \gamma} z_1(\tau) / \theta_3^2 \\
& + e^{\pi i \{ (-1)^{k_{\nu+1}-k_\nu} + (-1)^{k_{\nu+1}-k_{\nu-2}} + \dots + (-1)^{k_{\nu+1}-k_1} \} (2\alpha-\gamma-1) + \{ (-1)^{k_{\nu+1}-k_{\nu-1}} + (-1)^{k_{\nu+1}-k_{\nu-3}} + \dots + (-1)^{k_{\nu+1}-k_2} \} (2\beta-\gamma+1)} e^{2\pi i (1-\gamma)(l_{\nu+1}+l_{\nu-1}+\dots+l_4+l_2)} \\
& \times e^{2\nu\pi i(\alpha+\beta-\gamma)} e^{\pi i(\gamma-1)k_{\nu+1}} e^{\pi i(\alpha-3\gamma)} \frac{\sin \pi \{ \gamma + (-1)^{k_{\nu+1}}(2\beta-\gamma) \} / 2}{\sin \pi \gamma} z_2(\tau) / \theta_3^2.
\end{aligned} \tag{8.6}$$

*Acknowledgement.* This work was supported by JSPS. KAKENHI (19540158).

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