HOMOTOPY CLASSIFICATION OF GENERALIZED PHRASES IN TURAEV'S THEORY OF WORDS

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ABSTRACT. In 2005 V. Turaev introduced the theory of topology of words and phrases. Turaev defined an equivalence relation on generalized words and phrases which is called homotopy. This is suggested by the Reidemeister moves in the knot theory. Then Turaev gave the homotopy classification of generalized words with less than or equal to five letters. In this paper we give the classification of generalized phrases up to homotopy with less than or equal to three letters. To do this we construct a new homotopy invariant for nanophrases over any α.

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1. INTRODUCTION.

In [8] and [9], V. Turaev introduced the theory of topology of words and phrases. Words are finite sequences of letters in a given alphabet, letters are elements of an alphabet and phrases are finite sequences of words. Turaev defined generalized words which is called étale words as follows : Let α be an alphabet endowed with an involution $\tau : \alpha \to \alpha$. Let $\mathcal{A}$ be an alphabet endowed with a mapping $|\cdot| : \mathcal{A} \to \alpha$ which is called a projection. We call this $\mathcal{A}$ an α-alphabet. Then we call a pair an α-alphabet $\mathcal{A}$ and a word on $\mathcal{A}$ an étale word. If all letters in $\mathcal{A}$ appear exactly twice, then we call this étale word a nanoword.

Turaev introduced an equivalence relation which is called homotopy on nanowords. This equivalence relation is suggested by the Reidemeister moves in the theory of knots. Homotopy of nanowords is generated by isomorphism, and three homotopy moves. The first homotopy move is deformation that changes $xAAy$ into xy. The second homotopy move is deformation that changes $xAByBAz$ into xyz when $|A|$ is equal to $\tau(|B|)$. The third homotopy move is deformation that changes $xAByACzBCt$ into $xBAyCAzCBt$ when $|A|$ and $|B|$ are equal to $|C|$ (Turaev defined more generalized equivalence relation which is called S-homotopy. However, in this paper, we treat only homotopy). Turaev defined homotopy of étale words via desingularization of étale words. Moreover in [9] Turaev defined homotopy of nanophrase in a similar manner.

Theory of words and phrases is applied for studying curves on surfaces. In [3] C. F. Gauss studied planar curves via words. Turaev applied generalized words and phrases for curves and knot diagrams. Turaev showed special cases of the theory of topology of phrases corresponds to the theory of stable equivalent classes of ordered,

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pointed, oriented multi-component curves on surfaces and knot diagrams. Note that the theory stable equivalence classes of ordered, pointed, oriented multi-component curves on surfaces (respectively knot diagrams) is equivalent to the theory of ordered open flat virtual links (respectively ordered open virtual links). In this paper, ordered links means each components of links are numerated. See also [5], [6], [7] and [11] for more details. In this meaning, the theory of topology of words and phrases is combinational extension of the theory of virtual knots and links.

Now the purpose of this paper is classification of generalized phrases (in this paper we call it étale phrases) up to homotopy. Turaev gave the homotopy classification of étale words with less than or equal to five letters in [8]. We will extend this result. More precisely, in this paper, we give the classification of étale phrases with less than or equal to three letters. To do this we use some known invariants which was introduced in [1], [2] and [4]. Moreover we construct a new homotopy invariant.

The rest of this paper is constructed as follows. In the next section we review the theory of topology of nanowords, étale words, nanophrases and étale phrases. In Section 3 we introduce homotopy invariants of nanophrases which was introduced in [1], [2] and [4]. Then we will define a new homotopy invariant for nanophrases. In Section 4 and Section 5 we give the classification of étale phrases with less than or equal to three letters without the condition on length of phrases.

2. Étale Phrases and Nanophrases.

In this section we introduce Turaev's theory of words and phrases (See [8], [9] and [10] for more details).

2.1. Étale words and étale phrases. In this paper an *alphabet* means a finite set and *letters* mean its element. A *word* of length n on an alphabet $\mathcal{A}$ is a mapping $w : \hat{n} \to \mathcal{A}$ where $\hat{n} := \{1, 2, \cdots n\}$ and a *phrase* of length k on $\mathcal{A}$ is a finite sequence of words on $\mathcal{A}$, $(w_1|w_2|\cdots|w_k)$. A multiplicity of a letter $A \in \mathcal{A}$ in a phrase P on $\mathcal{A}$ is a number of A in the phrase P. We denote multiplicity of $A \in \mathcal{A}$ by $m_P(A)$.

Let α be an alphabet endowed with an involution $\tau : \alpha \to \alpha$. An *α-alphabet* is a pair (An alphabet $\mathcal{A}$, mapping $|\cdot| : \mathcal{A} \to \alpha$). We call the mapping $|\cdot|$ *projection.*

In [11], V. Turaev defined generalized words which is called étale words. An *étale word* over α is a pair (An α-alphabet $\mathcal{A}$, A word on $\mathcal{A}$) and A *étale phrase* over α is a pair (An α-alphabet $\mathcal{A}$, A phrase on $\mathcal{A}$).

Remark 2.1. Turaev did not define étale phrases explicitly. However Turaev considered an equivalent object in [9].

A phrase P on an α gives rise to an étale phrase (α, P) where the projection $\alpha \to \alpha$ is the identity mapping. In this meaning étale phrases are generalization of usual phrases.

2.2. Nanowords and nanophrases. A *Gauss word* on an alphabet $\mathcal{A}$ is a word w on $\mathcal{A}$ which all letters in $\mathcal{A}$ appear exactly twice in w. A phrase P on $\mathcal{A}$ is called a *Gauss phrase* if all letters in $\mathcal{A}$ appear exactly twice in P.

In this paper, we consider generalized Gauss words and Gauss phrases. A *nanoword* over α is a pair (An α-alphabet $\mathcal{A}$, A Gauss word on $\mathcal{A}$) and A *nanophrase* over α

is a pair (An α-alphabet $\mathcal{A}$, A Gauss phrase on $\mathcal{A}$). Instead of writing $(\mathcal{A}, P)$ for a nanophrase over α, we often write simply P. The alphabet $\mathcal{A}$ can be uniquely recovered. However the projection $|\cdot| : \mathcal{A} \to \alpha$ should be always specified.

2.3. **Desingularization of étale phrases.** In this section, we introduce a method of associating with any étale phrases over α $(\mathcal{A}, P)$ a nanophrase over α $(\mathcal{A}^d, P^d)$ which is called *desingularization* of étale phrases.

Let $\mathcal{A}^d$ be an α-alphabet $\{A_{i,j} := (A, i, j) | A \in \mathcal{A}, 1 \leq i < j \leq m_P(A)\}$ with the projection $|A_{i,j}| := |A|$ for all $A_{i,j}$. The phrase P^d is obtained from P by first deleting all $A \in \mathcal{A}$ with $m_P(A)$ is less than or equal to one. Then for each $A \in \mathcal{A}$ with $m_P(A)$ is grater than or equal to two and each $i = 1, 2, \dots m_P(A)$, we replace the i-th entry of A in P by

$$A_{1,i}A_{2,i} \dots A_{i-1,i}A_{i,i+1}A_{i,i+2} \dots A_{i,m_P(A)}.$$

The resulting $(\mathcal{A}^d, P^d)$ is a nanophrase with $\sum m_P(A)(m_P(A)-1)$ letters and called a *desingularization of* $(\mathcal{A}, P)$. Note that if $(\mathcal{A}, P)$ is a nanophrase, then desingularization of $(\mathcal{A}, P)$ is a itself.

2.4. **Homotopy of nanophrases and étale phrases.** In [8] and [9], Turaev defined an equivalence relation which is called homotopy on a set of nanophrases and étale words.

To define homotopy, we define isomorphism of étale phrases. A *morphism* of α-alphabets $\mathcal{A}_1$, $\mathcal{A}_2$ is a set-theoric mapping $f : \mathcal{A}_1 \to \mathcal{A}_2$ such that $|A| = |f(A)|$ for all $A \in \mathcal{A}_1$. If f is bijective, then this morphism is an *isomorphism*. Two étale phrases $(\mathcal{A}_1, (w_1| \cdots |w_k)$ and $(\mathcal{A}_2, (v_1| \cdots |v_k))$ over α are *isomorphic* if there is an isomorphism $f : \mathcal{A}_1 \to \mathcal{A}_2$ such that $v_j = f \circ w_j$ for all $j \in \hat{k}$.

Next we define homotopy moves of nanophrases.

Definition 2.1. We define *homotopy moves* (1) - (3) of nanophrases as follows:

(1) $(\mathcal{A}, (xAAy)) \longrightarrow (\mathcal{A} \setminus \{A\}, (xy))$
for all $A \in \mathcal{A}$ and x, y are sequences of letters in $\mathcal{A} \setminus \{A\}$, possibly including the $|$ character.

(2) $(\mathcal{A}, (xAByBAz)) \longrightarrow (\mathcal{A} \setminus \{A, B\}, (xyz))$
if $A, B \in \mathcal{A}$ satisfy $|B| = \tau(|A|)$. x, y, z are sequences of letters in $\mathcal{A} \setminus \{A, B\}$, possibly including $|$ character.

(3) $(\mathcal{A}, (xAByACzBCt)) \longrightarrow (\mathcal{A}, (xBAyCAzCBt))$
if $A, B, C \in \mathcal{A}$ satisfy $|A| = |B| = |C|$. x, y, z, t are sequences of letters in $\mathcal{A}$, possibly including $|$ character.

Now we define homotopy of étale phrases.

Definition 2.2. Two étale phrases $(\mathcal{A}_1, P_1)$ and $(\mathcal{A}_2, P_2)$ over α are *homotopic* (denoted $(\mathcal{A}_1, P_1) \simeq (\mathcal{A}_2, P_2)$) if $((\mathcal{A}_2)^d, (P_2)^d)$ can be obtained from $((\mathcal{A}_1)^d, (P_1)^d)$ by a finite sequence of isomorphism, homotopy moves (1) - (3) and the inverse of moves (1) - (3).

Remark 2.2. By the definition of homotopy of étale phrases, every homotopy invariant I of nanophrases extends to a homotopy invariant I of étale phrases by $I(P) := I(P^d)$.

The gale of this paper is to classify étale phrases of length k over α with less than or equal to three letters up to homotopy for any k and α.

The case of k is equal to one (in other words, the case of étale words) Turaev gave the classification as follows.

Theorem 2.1 (Turaev [8]). *A multiplicity-one-free word of length less than or equal to four in the alphabet α has one of the following forms: aa, aaa, aaaa, aabb, abba, abab with distinct $a, b \in \alpha$ The words aa, aabb, abba are contractible. The words aaa and aaaa are contractible if and only if $\tau(a) = a$. The word abab is contractible if and only if $\tau(a) = b$. Non-contractible words of type aaa, aaaa and abab are homotopic if and only if they are equal.*

3. Homotopy Invariants of Nanophrases.

By the definition of homotopy of étale phrases, we need homotopy invariants of nanophrases. In this section, we introduce homotopy invariants of nanophrases which were defined in [1], [2] and [4]. Moreover we define a new homotopy invariant of nanophrases.

3.1. Simple invariants.

In this subsection, we review homotopy invariants which were defined in [2] and [4].

Let $P = (w_1|w_2|\cdots|w_k)$ be a nanophrase over α. For $l \in \hat{k}$, we define $w(l) \in \mathbb{Z}/2\mathbb{Z}$ by the length of w_l. We call the vector

$$w(P) := (w(1), \cdots, w(k)) \in (\mathbb{Z}/2\mathbb{Z})^k$$

the *component length vector.*

Proposition 3.1 (A. Gibson [4], see also [1]). *The component length vector is a homotopy invariant of nanophrases.*

Next we define another homotopy invariant. Let π be the group which is defined as follows:

$$\pi := (a \in \alpha | a\tau(a) = 1, ab = ba \ for \ all \ a, b \in \alpha \ ).$$

Let $(w_1|w_2|\cdots|w_k)$ be a nanophrase of length k over α. We define $l_P(i, j) \in \pi$ for $i < j$ by

$$l_P(i, j) := \prod_{A \in Im(w_i) \cap Im(w_j)} |A|.$$

We call a vector $lk(P) := (l_P(1, 2), l_P(1, 3), \cdots, l_P(1, k), l_P(2, 3), \cdots, l_P(k-1, k)) \in \pi^{\frac{1}{2}k(k-1)}$ the *linking vector.*

Proposition 3.2 ([2]). *The linking vector of nanophrases is a homotopy invariant of nanophrases.*

3.2. **The invariant T.** In this section we introduce a homotopy invariant T which was defined by the author in [1]. This invariant is defined for nanophrases over α_0 and the one-element set where $\alpha_0 = \{a, b\}$ with the involution $\tau_0 : a \mapsto b$.

Definition 3.1. Let $P = (\mathcal{A}, (w_1|\cdots|w_k))$ be a nanophrase over α_0 and A,$B \in \mathcal{A}$. Then we define $\sigma_P(A, B)$ as follows: If A and B form $\cdots A \cdots B \cdots A \cdots B \cdots$ in P and $|B| = a$, or $\cdots B \cdots A \cdots B \cdots A \cdots$ in P and $|B| = b$, then $\sigma_P(A, B) := 1$. If $\cdots A \cdots B \cdots A \cdots B \cdots$ in P and $|B| = b$, or $\cdots B \cdots A \cdots B \cdots A \cdots$ in P and $|B| = a$, then $\sigma_P(A, B) := -1$. Otherwise $\sigma_P(A, B) := 0$.

Definition 3.2. For $A \in \mathcal{A}$ we define $\varepsilon(A) \in \{\pm 1\}$ by

$$\varepsilon(A) := \begin{cases} 1 \ (\ if \ |A| = a \ ), \\ -1 \ (\ if \ |A| = b \ ). \end{cases}$$

Definition 3.3. Let $P = (\mathcal{A}, (w_1|w_2|\cdots|w_k))$ be a nanophrase of length k over α_0. For $A \in \mathcal{A}$ such that there exist $i \in \{1, 2, \cdots, k\}$ such that $Card(w_i^{-1}(A)) = 2$, we define $T_P(A) \in \mathbb{Z}$ by

$$T_P(A) := \varepsilon(A) \sum_{B \in \mathcal{A}} \sigma_P(A, B),$$

and we define $T_P(w_i) \in \mathbb{Z}$ by

$$T_P(w_i) := \sum_{A \in \mathcal{A}, \ Card(w_i^{-1}(A))=2} T_P(A).$$

Then we define $T(P) \in \mathbb{Z}^k$ by

$$T(P) := (T_P(w_1), T_P(w_2), \cdots, T_P(w_k)).$$

Theorem 3.1 ([1])**.** *T is a homotopy invariant of nanophrases over α_0.*

Next we define an invariant T for nanophrases over the one-element set (we use the same notation "T" because of the Remark 3.1).

Definition 3.4. Let $P := (\mathcal{A}, (w_1|\cdots|w_k))$ be a nanophrase over the one-element set $\alpha := \{a\}$. Let $A, B \in \mathcal{A}$ be letters. Then we define $\tilde{\sigma}_P(A, B) \in \mathbb{Z}/2\mathbb{Z}$ as follows: If A and B forms $\cdots A \cdots B \cdots A \cdots B \cdots$ or $\cdots B \cdots A \cdots B \cdots A \cdots$ in P, then $\tilde{\sigma}_P(A, B) := 1$. Otherwise $\tilde{\sigma}_P(A, B) := 0$.

Definition 3.5. Let $P := (\mathcal{A}, (w_1|\cdots|w_k))$ be a nanophrase over $\alpha := \{a\}$. For $A \in \mathcal{A}$ such that there exist an $i \in \{1, 2, \cdots, k\}$ such that $Card(w_i^{-1}(A)) = 2$, we define $T_P(A) \in \mathbb{Z}/2\mathbb{Z}$ by

$$T_P(A) := \sum_{B \in \mathcal{A}} \tilde{\sigma}_P(A, B) \in \mathbb{Z}/2\mathbb{Z},$$

and $T_P(w_i) \in \mathbb{Z}/2\mathbb{Z}$ by

$$T_P(w_i) := \sum_{A \in \mathcal{A}, \ Card(w_i^{-1}(A))=2} T_P(A).$$

Then we define $T(P) \in (\mathbb{Z}/2\mathbb{Z})^k$ by

$$T(P) := (T_P(w_1), T_P(w_2), \cdots, T_P(w_k)).$$

Then the next theorem follows.

Theorem 3.2 ([1])**.** *T is a homotopy invariant of nanophrases over the one-element set.*

Remark 3.1. In a preprint [2], the author extended the invariant T to nanophrases over any α. However in this paper we only use this invariant for nanophrases over α_0 and nanophrases over the one element set.

3.3. **The invariant S_o.** In [4] A.Gibson defined a homotopy invariant of nanophrases over the one-element set which is stronger than the invariant T for nanophrases over the one-element set. In this subsection we introduce Gibson's S_o invariant.

First we define some notations. Let $(\mathcal{A}, P = (w_1|\cdots|w_k))$ be a nanophrase over the one-element set. For a letter $A \in \mathcal{A}_i := \{A \in \mathcal{A}|Card(w_i^{-1}(A)) = 2\}$, we define $l_j(A) \in \mathbb{Z}/2\mathbb{Z}$ as follows : When we write P as $xAyAz$ where x, y and z are words in $\mathcal{A}$ possibly including "$|$" character, $l_j(A)$ is modulo 2 of the number of letters which appear exactly once in y and once in the j-th component of the phrase P. Then we define $l(A) \in (\mathbb{Z}/2\mathbb{Z})^k$ by

$$l(A) := (l_1(A), l_2(A), \cdots, l_k(A)).$$

Let v be a vector in $(\mathbb{Z}/2\mathbb{Z})^k$. Then we define $d_j(v) \in \mathbb{Z}$ by

$$d_j(v) := Card(\{A \in \mathcal{A}_j|l(A) = v\}),$$

and we define $B_j(P) \in 2^{(\mathbb{Z}/2\mathbb{Z})^k}$ by

$$B_j(P) := \{v \in (\mathbb{Z}/2\mathbb{Z})^k \setminus \{0\}|d_j(v) = 1 \ mod \ 2\}.$$

Then we define the $S_o(P) \in (2^{(\mathbb{Z}/2\mathbb{Z})^k})^k$ by

$$S_o(P) := (B_1(P), B_2(P), \cdots, B_k(P)).$$

Theorem 3.3 (Gibson [4])**.** *S_o is a homotopy invariant of nanophrases over the one-element set.*

3.4. **The invariant U_L.** In this section we introduce a new invariant of nanophrases.

First we prepare some notations. Since the set α is a finite set, we obtain following orbit decomposition of the τ : $\alpha/\tau = \{\widehat{a_{i_1}}, \widehat{a_{i_2}}, \cdots, \widehat{a_{i_l}}, \widehat{a_{i_{l+1}}}, \cdots, \widehat{a_{i_{l+m}}}\}$, where $\widehat{a_{i_j}} := \{a_{i_j}, \tau(a_{i_j})\}$ such that $Card(\widehat{a_{i_j}}) = 2$ for all $j \in \{1, \cdots, l\}$ and $Card(\widehat{a_{i_j}}) = 1$ for all $j \in \{l+1, \cdots, l+m\}$ (we fix a complete representative system $crs(\alpha/\tau) := \{a_{i_1}, a_{i_2}, \cdots, a_{i_l}, a_{i_{l+1}}, \cdots, a_{i_{l+m}}\}$ which satisfy the above condition). Let L be a subset of $crs(\alpha/\tau)$. For a nanophrase $(\mathcal{A}, P)$ over α, we define a nanophrase $U_L((\mathcal{A}, P))$ over $L \cup \tau(L)$ as follows: deleting all letters $A \in \mathcal{A}$ such that $|A| \notin L \cup \tau(L)$ from both $\mathcal{A}$ and P.

Proposition 3.3. *U_L is a homotopy invariant of nanophrases.*

Proof. First, isomorphism does not change $U_L(P)$ up to isomorphic is clear.

Consider the first homotopy move

$$P_1 := (\mathcal{A}, (xAAy)) \longrightarrow P_2 := (\mathcal{A} \setminus \{A\}, (xy))$$

where x and y are words on $\mathcal{A}$, possibly including "|" character. Suppose $|A| \in L \cup \tau(L)$. Then

$$U_L(P_1) = x_L AA y_L \simeq x_L y_L = U_L(P_2)$$

where x_L and y_L are words which obtained by deleting all letters $X \in \mathcal{A}$ such that $X \notin L \cup \tau(L)$ from x and y respectively.

Suppose $|A| \notin L \cup \tau(L)$. Then

$$U_L(P_1) = x_L y_L = U_L(P_2).$$

So the first homotopy move does not change the homotopy class of $U_L(P)$.

Consider the second homotopy move

$$P_1 := (\mathcal{A}, (xAByBAz)) \longrightarrow P_2 := (\mathcal{A} \setminus \{A, B\}, (xyz))$$

where $|A| = \tau(|B|)$, and x, y and z are words on $\mathcal{A}$ possibly including "|" character. Suppose $|A| \in L \cup \tau(L)$. Then $|B| \in L \cup \tau(L)$ since $|A| = \tau(|B|)$. So

$$U_L(P_1) = x_L AB y_L BA z_L \simeq x_L y_L z_L = U_L(P_2).$$

Suppose $|A| \notin L \cup \tau(L)$. Then $|B| \notin L \cup \tau(L)$ since $|A| = \tau(|B|)$. So

$$U_L(P_1) = x_L y_L z_L = U_L(P_2).$$

By the above, the second homotopy move does not change the homotopy class of $U_L(P)$.

Consider the third homotopy move

$$P_1 := (\mathcal{A}, (xAByACzBCt)) \to P_2 := (\mathcal{A}, (xBAyCAzCBt))$$

where $|A| = |B| = |C|$, and x, y, z and t are words on $\mathcal{A}$ possibly including "|" character. Suppose $|A| \in L \cup \tau(L)$. Then $|B|, |C| \in L \cup \tau(L)$ since $|A| = |B| = |C|$. So we obtain

$$U_L(P_1) = x_L AB y_L AC z_L AC t_L \simeq x_L BA y_L CA z_L CB t_L = U_L(P_2).$$

Suppose $|A| \notin L \cup \tau(L)$. Then $|B|, |C| \notin L \cup \tau(L)$ since $|A| = |B| = |C|$. So we obtain

$$U_L(P_1) = x_L y_L z_L t_L = U_L(P_2).$$

So the third homotopy move does not change the homotopy class of $U_L(P)$.

By the above, U_L is a homotopy invariant of nanophrases. □

4. Homotopy Classification of Étale Phrases.

In this section we classify étale phrases with less than or equal to three letters up to homotopy. First we recall lemmas in [1].

Lemma 4.1. *Let β be τ-invariant subset of α. If two nanophrases over β are homotopic in the class of nanophrases over α, then they are homotopic in the class of nanophrases over β.*

Lemma 4.2. *Let $P_1 = (w_1|w_2|\cdots|w_k)$ and $P_2 = (v_1|v_2|\cdots|v_k)$ be nanophrases of length k over α. If P_1 and P_2 are homotopic as nanophrases, then w_i and v_i are homotopic as étale words for all $i \in \{1, 2, , \cdots, k\}$.*

Next we prepare some notations. Let α be an alphabet endowed with an involution $\tau : \alpha \to \alpha$. Then we set
$P_a^{1,1;l_1,l_2} := (\emptyset|\cdots|\emptyset|\overset{l_1}{\check{a}}|\emptyset|\cdots|\emptyset|\overset{l_2}{\check{a}}|\emptyset|\cdots|\emptyset)$,
$P_a^{3;l} := (\emptyset|\cdots|\emptyset|\overset{l}{\check{a^3}}|\emptyset|\cdots|\emptyset)$,
$P_a^{2,1;l_1,l_2} := (\emptyset|\cdots|\emptyset|\overset{l_1}{\check{a^2}}|\emptyset|\cdots|\emptyset|\overset{l_2}{\check{a}}|\emptyset|\cdots|\emptyset)$,
$P_a^{1,2;l_1,l_2} := (\emptyset|\cdots|\emptyset|\overset{l_1}{\check{a}}|\emptyset|\cdots|\emptyset|\overset{l_2}{\check{a^2}}|\emptyset|\cdots|\emptyset)$,
$P_a^{1,1,1;l_1,l_2,l_3} := (\emptyset|\cdots|\emptyset|\overset{l_1}{\check{a}}|\emptyset|\cdots|\overset{l_2}{\check{a}}|\emptyset|\cdots|\emptyset|\overset{l_3}{\check{a}}|\emptyset|\cdots|\emptyset)$,
where $a \in \alpha$ and l, l_1,l_2,$l_3 \in \hat{k}$ with $l_1 < l_2 < l_3$. Note that if $a = \tau(a)$, then $P_a^{3;l}$ is homotopic to the nanophrase $(\emptyset)_k := (\emptyset|\cdots|\emptyset)$. So when we use the notation $P_a^{3;l}$, we always assume that $a \neq \tau(a)$.

Remark 4.1. For two different integers k_1 and k_2, an étale phrase of length k_1 and an étale phrase of length k_2 are not homotopic each other. So we do not write length of phrases in above notations.

Now we describe the main results of this paper.

Theorem 4.1. *Let P be a multiplicity-one-free étale phrase over α with less than or equal to three letters. Then P is either homotopic to $(\emptyset)_k$ or homotopic to one of the following étale phrases: $P_a^{1,1;l_1,l_2}$, $P_a^{3;l}$, $P_a^{2,1;l_1,l_2}$, $P_a^{1,2;l_1,l_2}$, $P_a^{1,1,1;l_1,l_2,l_3}$ for some l_1, l_2, $l_3 \in \hat{k}$ and $a \in \alpha$. Moreover $P_a^{1,1;l_1,l_2}$, $P_a^{3;l}$, $P_a^{2,1;l_1,l_2}$, $P_a^{1,2;l_1,l_2}$, $P_a^{1,1,1;l_1,l_2,l_3}$ are homotopic if and only if they are equal.*

To prove this theorem, we prepare following lemmas.

Lemma 4.3. *Étale phrases $P_a^{1,1;l_1,l_2}$, $P_a^{3;l}$, $P_a^{2,1;l_1,l_2}$, $P_a^{1,2;l_1,l_2}$ and $P_a^{1,1,1;l_1,l_2,l_3}$ are not homotopic to $(\emptyset)_k$.*

Lemma 4.4. *If a is not equal to b, then $P_a^{X_1;Y_1}$ and $P_b^{X_2;Y_2}$ are not homotopic for all $(X_1;Y_1),(X_2;Y_2) \in \{(1,1;l_1,l_2),(3;l),(2,1;l_1,l_2),(1,2;l_1,l_2),(1,1,1;l_1,l_2,l_3)\}$.*

Lemma 4.5. *Two étale phrases $P_a^{X_1;Y_1}$ and $P_a^{X_2;Y_2}$ are homotopic if and only if $(X_1;Y_1)$ is equal to $(X_2;Y_2)$.*

If we show above lemmas, then we obtain the main theorem. We prove these lemmas in the next section.

5. Proof of Lemmas.

In this section, we prove Lemma 4.3, Lemma 4.4 and Lemma4.5.

5.1. **Proof of the Lemma 4.3.** The first claim of this lemma is easily checked. We show the second part of the lemma.

- The case $P_a^{1,1;l_1,l_2} \not\simeq (\emptyset)_k$.

In this case, component length vector

$$w(P_a^{1,1;l_1,l_2}) = \mathbf{e}_{l_1} + \mathbf{e}_{l_2}$$

where $\mathbf{e}_j = (0, \cdots, 0, \overset{j}{\check{1}}, 0, \cdots, 0)$. On the other hand

$$w((\emptyset)_k) = 0.$$

So we obtain $P_a^{1,1;l_1,l_2} \not\simeq (\emptyset)_k$.

- The case $P_a^{3;l} \not\simeq (\emptyset)_k$.

By the Theorem 2.1, $a^3 = aaa$ with $a \neq \tau(a)$ is not homotopic to empty nanoword $\emptyset$. Combining this fact and Lemma 4.2, we obtain $P_a^{3;l} \not\simeq (\emptyset)_k$.

- The case $P_a^{2,1;l_1,l_2} \not\simeq (\emptyset)_k$.

By Lemma 4.1, we can assume $\alpha = \{a, \tau(a)\}$. In this case

$$T(P_a^{2,1;l_1,l_2}) = \mathbf{e}_{l_1} \neq 0$$

(both the case $a = \tau(a)$ and the case $a \neq \tau(a)$). So we obtain $P_a^{2,1;l_1,l_2} \not\simeq (\emptyset)_k$.

- The case $P_a^{1,2;l_1,l_2} \not\simeq (\emptyset)_k$.

In this case

$$T(P_a^{1,2;l_1,l_2}) = \mathbf{e}_{l_2} \neq 0$$

(both the case $a = \tau(a)$ and the case $a \neq \tau(a)$). So we obtain $P_a^{1,2;l_1,l_2} \not\simeq (\emptyset)_k$.

- The case $P_a^{1,1,1;l_1,l_2,l_3} \not\simeq (\emptyset)_k$.

In this case

$$l_{P_a^{1,1,1;l_1,l_2,l_3}}(l_1, l_2) = a \in \pi.$$

Note that a is not equal to the unit element 1 in π. So we obtain $P_a^{1,1,1;l_1,l_2,l_3} \not\simeq (\emptyset)_k$.

Now we finished prove the lemma.

5.2. **Proof of the Lemma 4.4.** By Lemma 4.1, we can assume $\alpha = \{a, \tau(a), b, \tau(b)\}$.

Suppose $\hat{a} \neq \hat{b}$. Let $crs(\alpha/\tau) = \{a, b\}$ and $L = \{a\}$. Then

$$U_L(P_a^{X_1;Y_1}) = P_a^{X_1;Y_1}.$$

On the other hand,

$$U_L(P_b^{X_2;Y_2}) = (\emptyset)_k.$$

So by Lemma 4.3, we obtain $P_a^{X_1;Y_1}$ is not homotopic to $P_b^{X_2;Y_2}$ for all $(X_1; Y_1), (X_2; Y_2) \in \{(1, 1; l_1, l_2), (3; l), (2, 1; l_1, l_2), (1, 2; l_1, l_2), (1, 1, 1; l_1, l_2, l_3)\}$.

Suppose $\hat{a} = \hat{b}$. By the assumption a is not equal to b, $Card(\hat{a}) = 2$ and $b = \tau(a)$.

- On $P_a^{1,1;l_1,l_2} \not\simeq P_{\tau(a)}^{1,1;m_1,m_2}$.

In this case

$$l_{P_a^{1,1;l_1,l_2}}(l_1, l_2) = a.$$

On the other hand,

$$l_{P_{\tau(a)}^{1,1;m_1,m_2}}(l_1, l_2) = \begin{cases} \tau(a) = a^{-1} \ (\ if \ (l_1, l_2) = (m_1.m_2)\ ), \\ 1 \ \ (otherwise). \end{cases}$$

So $l_{P_a^{1,1;l_1,l_2}}(l_1, l_2)$ is not equal to $l_{P_{\tau(a)}^{1,1;m_1,m_2}}(l_1, l_2)$. So we obtain $P_a^{1,1;l_1,l_2} \not\simeq P_{\tau(a)}^{1,1;m_1,m_2}$.

- On $P_a^{1,1;l_1,l_2} \not\simeq P_{\tau(a)}^{X;Y}$ for all $(X; Y) \neq (1, 1; m_1, m_2)$.

In this case,

$$w(P_a^{1,1;l_1,l_2}) = \mathbf{e}_{l_1} + \mathbf{e}_{l_2},$$

and

$$w(P^{X;Y}_{\tau(a)}) = 0.$$

So we obtain $P^{1,1;l_1,l_2}_a \not\simeq P^{X;Y}_{\tau(a)}$.

• On $P^{3;l}_a \not\simeq P^{X;Y}_{\tau(a)}$ for all $(X;Y) \neq (1,1;m_1,m_2)$.
  This is obtained from Theorem 2.1 and Lemma 4.2.

• On $P^{2,1;l_1,l_2}_a \not\simeq P^{X;Y}_{\tau(a)}$ for all $(X;Y) \neq (1,1;m_1,m_2),(3;m)$.
  In this case,

$$l_{P^{2,1;l_1,l_2}_a}(l_1,l_2) = a^2.$$

On the other hand

$$l_{P^{2,1;m_1,m_2}_{\tau(a)}}(l_1,l_2) = \begin{cases} \tau(a)^2 = a^{-2} \ (\ if \ (l_1,l_2)=(m_1.m_2)\ ), \\ 1 \ \ (otherwise), \end{cases}$$

$$l_{P^{1,2;m_1,m_2}_{\tau(a)}}(l_1,l_2) = \begin{cases} \tau(a)^2 = a^{-2} \ (\ if \ (l_1,l_2)=(m_1.m_2)\ ), \\ 1 \ \ (otherwise), \end{cases}$$

and

$$l_{P^{1,1,1;m_1,m_2,m_3}_{\tau(a)}}(l_1,l_2) = \begin{cases} \tau(a) = a^{-1} \ (\ if \ \exists(m_i,m_j)=(l_1.l_2)\ ), \\ 1 \ \ (otherwise). \end{cases}$$

So we obtain $P^{2,1;l_1,l_2}_a \not\simeq P^{X;Y}_{\tau(a)}$.

• On $P^{1,2;l_1,l_2}_a \not\simeq P^{X;Y}_{\tau(a)}$ for all $(X;Y) \neq (1,1;m_1,m_2),(3;m),(2,1;m_1,m_2)$.
  In this case,

$$l_{P^{1,2;l_1,l_2}_a}(l_1,l_2) = a^2.$$

On the other hand

$$l_{P^{1,2;m_1,m_2}_{\tau(a)}}(l_1,l_2) = \begin{cases} \tau(a)^2 = a^{-2} \ (\ if \ (l_1,l_2)=(m_1.m_2)\ ), \\ 1 \ \ (otherwise), \end{cases}$$

and

$$l_{P^{1,1,1;m_1,m_2,m_3}_{\tau(a)}}(l_1,l_2) = \begin{cases} \tau(a) = a^{-1} \ (\ if \ \exists(m_i,m_j)=(l_1.l_2)\ ), \\ 1 \ \ (otherwise). \end{cases}$$

So we obtain $P^{1,2;l_1,l_2}_a \not\simeq P^{X;Y}_{\tau(a)}$.

• On $P^{1,1,1;l_1,l_2,l_3}_a \not\simeq P^{1,1,1;m_1,m_2,m_3}_{\tau(a)}$.
  In this case

$$l_{P^{1,1,1;l_1,l_2,l_3}_a}(l_1,l_2) = a,$$

and

$$l_{P^{1,1,1;m_1,m_2,m_3}_{\tau(a)}}(l_1,l_2) = \begin{cases} \tau(a) = a^{-1} \ (\ if \ \exists(m_i,m_j)=(l_1.l_2)\ ), \\ 1 \ \ (otherwise). \end{cases}$$

So we obtain $P^{1,1,1;l_1,l_2,l_3}_a \not\simeq P^{1,1,1;m_1,m_2,m_3}_{\tau(a)}$.

Now we have completed the proof of Lemma 4.4.

5.3. **Proof of the Lemma 4.5.** • On $P_a^{1,1;l_1,l_2}$.

In this case
$$w(P_a^{1,1;l_1,l_2}) = \mathbf{e}_{l_1} + \mathbf{e}_{l_2},$$
and
$$w(P_a^{X;Y}) = 0$$
for all $(X;Y) \neq (1,1;m_1,m_2)$. So we obtain $P_a^{1,1;l_1,l_2} \simeq P_a^{X;Y}$ if and only if $(X;Y) = (1,1;l_1,l_2)$.

• On $P_a^{3;l}$.

By Theorem 2.1 and Lemma 4.2, We obtain $P_a^{3;l} \simeq P_a^{X;Y}$ if and only if $(X;Y) = (3;l)$.

• On $P_a^{2,1;l_1,l_2}$.

The case $P_a^{2,1;l_1,l_2} \not\simeq P_a^{2,1;m_1,m_2}$ if $(l_1,l_2) \neq (m_1,m_2)$.

If $a \neq \tau(a)$, then
$$l_{P_a^{2,1;l_1,l_2}}(l_1,l_2) = a^2,$$
and
$$l_{P_a^{2,1;m_1,m_2}}(l_1,l_2) = 1 \neq a^2.$$
So we obtain $P_a^{2,1;l_1,l_2} \not\simeq P_a^{2,1;m_1,m_2}$ if $(l_1,l_2) \neq (m_1,m_2)$.

If $a = \tau(a)$, then by Lemma 4.1 we can assume $\alpha = \{a\}$. So we can use Gibson's S_o invariant. In this case
$$S_o(P_a^{2,1;l_1,l_2}) = (\emptyset, \cdots, \emptyset, \{\overset{l_1}{\check{\mathbf{e}}_{l_2}}\}, \emptyset, \cdots, \emptyset),$$
and
$$S_o(P_a^{2,1;m_1,m_2}) = (\emptyset, \cdots, \emptyset, \{\overset{m_1}{\check{\mathbf{e}}_{m_2}}\}, \emptyset, \cdots, \emptyset).$$
So we obtain $P_a^{2,1;l_1,l_2} \not\simeq P_a^{2,1;m_1,m_2}$ if $(l_1,l_2) \neq (m_1,m_2)$.

The case $P_a^{2,1;l_1,l_2} \not\simeq P_a^{1,2;m_1,m_2}$.

If $a \neq \tau(a)$ and $(l_1,l_2) \neq (m_1,m_2)$, then
$$l_{P_a^{2,1;l_1,l_2}}(l_1,l_2) = a^2,$$
and
$$l_{P_a^{1,2;m_1,m_2}}(l_1,l_2) = 1.$$
If $a \neq \tau(a)$ and $(l_1,l_2) = (m_1,m_2)$, then
$$T(P_a^{2,1;l_1,l_2}) = \mathbf{e}_{l_1},$$
and
$$T(P_a^{1,2;l_1,l_2}) = -\mathbf{e}_{l_2}.$$
Since l_1 is not equal to l_2,
$$T(P_a^{2,1;l_1,l_2}) \neq T(P_a^{1,2;l_1,l_2}).$$
If $a = \tau(a)$, then
$$S_o(P_a^{2,1;l_1,l_2}) = (\emptyset, \cdots, \emptyset, \{\overset{l_1}{\check{\mathbf{e}}_{l_2}}\}, \emptyset, \cdots, \emptyset),$$
and
$$S_o(P_a^{1,2;m_1,m_2}) = (\emptyset, \cdots, \emptyset, \{\overset{m_2}{\check{\mathbf{e}}_{m_1}}\}, \emptyset, \cdots, \emptyset).$$

So if $P_a^{2,1;l_1,l_2} \simeq P_a^{1,2;m_1,m_2}$, then $(l_1, l_2) = (m_2, m_1)$. However this contradict the assumption $l_1 < l_2$ and $m_1 < m_2$. By the above, we obtain $P_a^{2,1;l_1,l_2} \not\simeq P_a^{1,2;m_1,m_2}$.

The case $P_a^{2,1;l_1,l_2} \not\simeq P_a^{1,1,1;m_1,m_2,m_3}$.

In this case,

$$lk(P_a^{1,1,1;m_1,m_2,m_3}) = \begin{cases} (1, \cdots, 1, a^2, 1, \cdots, 1) \ (\ if \ a \neq \tau(a) \ ), \\ (1, \cdots, 1) \ \ (\ if \ a = \tau(a) \ ). \end{cases}$$

On the other hand,

$$lk(P_a^{1,1,1;m_1,m_2,m_3}) = (1, \cdots, 1, a, 1, \cdots, 1, a, 1, \cdots, 1, a, 1, \cdots, 1).$$

So we obtain $lk(P_a^{2,1;l_1,l_2}) \neq lk(P_a^{1,1,1;m_1,m_2,m_3})$. This implies $P_a^{2,1;l_1,l_2} \not\simeq P_a^{1,1,1;m_1,m_2,m_3}$.

- On $P_a^{1,2;l_1,l_2}$.

  This case proved similarly as the case on $P_a^{2,1;l_1,l_2}$.

- On $P_a^{1,1,1;l_1,l_2,l_3}$.

  In this case,

$$l_{P_a^{1,1,1;l_1,l_2,l_3}}(i,j) = \begin{cases} a \ (\ if \ (i,j) = (l_1,l_2), (l_1,l_3), (l_2,l_3) \ ), \\ 1 \ \ (otherwise). \end{cases}$$

So we obtain $P_a^{1,1,1;l_1,l_2,l_3} \simeq P_a^{1,1,1;m_1,m_2,m_3}$ if and only if $(l_1, l_2, l_3) = (m_1, m_2, m_3)$.

Now we have completed the proof of Lemma 4.5.

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