SPARSE REPRESENTATION OF IMAGES AND DECODING BY LINEAR PROGRAMMING IN THE ℓ_1 NORM

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It is important for engineers and applied scientists to represent data, such as signals or images, in the most parsimonious terms. Let us consider a dictionary D of generating elements $\{d_k\}_{k=1}^{L}$, $d_k \in \mathbb{C}^N$ with unit length. The dictionary D can be viewed as a matrix of size $N \times L$. It is known that highly sparse solutions can be obtained by convex optimization for several interesting dictionaries. More precisely, for a given signal $s \in \mathbb{C}^N$, we seek the sparsest coefficient vector $r \in \mathbb{C}^L$ such that $Dr = s$. That is, we want to solve the ℓ_0 optimization problem:

$$\text{Minimize } \|r\|_0 \text{ subject to } Dr = s,$$

where the ℓ_0 norm is the number of non-zero elements in r. Under certain suitable conditions, the minimizing solution to the ℓ_1 optimization problem:

$$\text{Minimize } \|r\|_1 \text{ subject to } Dr = s$$

is also the minimizing solution to the ℓ_0 optimization problem. By this reason, decoding by linear programming in the ℓ_1 norm has become one of the central problems for sparse representation.

Linear codes of diverse lengths are constructed for decoding transmitted messages by ℓ_1 linear programming. Decoding consists in recovering an input vector $x \in \mathbb{R}^n$ from corrupted oversampled measurements $y = Ax + w$ where $A \in \mathbb{R}^{m\times n}$ is a full rank matrix with $m > n$ and $w \in \mathbb{R}^m$ is a sparse vector. Appropriate random matrices A are empirically constructed by means of singular value decomposition or QR orthonormalization so that the vector x can be recovered numerically to an error smaller than 10^{-5}, provided w is sufficiently sparse. Numerical results on the percentage of zero components in w are obtained for $m = 2n$, $4n$, $6n$, $8n$. For comparison, other less effective forms for A will be presented. In the case $m = 2n$, the above linear code problem can be effectively reduced to an underdetermined problem of the form $y = Bx$ and solved by ℓ_1 linear programming. Applications to cryptography are mentioned.

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