Entire solutions associated with front waves to reaction-diffusion equations

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Abstract

We can observe propagation phenomena modeled by reaction-diffusion equations in various fields of materials science, biology and life science. Corresponding to a wave propagation, the equations allow a traveling wave that has a constant profile and a constant speed. Historically, starting from the pioneering work by [5, 15], there are enormous number of papers for the study of traveling wave solutions (for instance see [1], [2], [4], [7], [10], [11], [13], [12], [19], [18], [17], [20], [21], and references therein).

Here we consider a simple model equation, that is a scalar reaction-diffusion equation of one-space dimension

$$u_t = u_{xx} + f(u) \qquad (x \in \mathbb{R}) \tag{1}$$

with the condition

$$f(0) = f(1) = 0, \qquad f'(0) \neq 0, \qquad f'(1) < 0. \tag{2}$$

The condition (2) tells that $u = 0$ and $u = 1$ are nondegenerate constant equilibria and that $u = 1$ is asymptotically stable. In addition to (2), if f satisfies

$$f'(0) > 0, \qquad f(u) \neq 0 \quad (u \neq 0, 1),$$

the equation (1) is called a monostable reaction-diffusion equation while if there is a number $a \in (0, 1)$ such that

$$f'(0) < 0, \quad f(a) = 0, \quad f'(a) > 0, \quad f(u) \neq 0 \quad (u \neq 0, a, 1),$$

(1) is called a bistable reaction-diffusion equation. Typical examples of those cases are the Fisher-KPP equation with

$$f(u) = u(1 - u), \tag{3}$$

and the Nagumo equation with

$$f(u) = u(1 - u)(u - a), \quad 0 < a < 1, \tag{4}$$

respectively.

Putting $u(x,t) = U(x+ct), z = x+ct$, a monotone traveling wave (called front wave) is obtained by solving the equation

$$\begin{cases} U_{zz} - cU_z + f(U) = 0, \qquad U(z) > 0 \qquad (z \in \mathbb{R}), \\ U(-\infty) = 0, \quad U(\infty) = 0. \end{cases} \tag{5}$$

It is known that a monostable equation allows a family of traveling waves with speeds $c \geq c_{min} > 0$; for instance $c_{min} = 2$ if $f(u) = u(1-u)$. On the other hand a bistable reaction-diffusion equation has a unique traveling wave up to translation. For the Nagumo equation the traveling wave is given explicitly as

$$\begin{aligned} U^{01}(x+ct) \quad &:= \quad \frac{\exp\left[x/\sqrt{2} + (1/2-a)t\right]}{1+\exp\left[x/\sqrt{2} + (1/2-a)t\right]} \\ &= \quad \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{x+ct}{2\sqrt{2}}\right), \qquad c = \sqrt{2}\left(\frac{1}{2} - a\right). \end{aligned}$$

We note that the reflected one $U^{10} := U^{01}(-x+ct)$ is also a traveling wave with a monotone decreasing profile.

As for the Nagumo equation there are other exact traveling wave solutions, which are given as

$$\begin{aligned} U^{0a}(x+c_1 t) \quad &:= \quad \frac{a\exp\left[ax/\sqrt{2} - (a-a^2/2)t\right]}{1+\exp\left[ax/\sqrt{2} - (a-a^2/2)t\right]} \\ &= \quad \frac{a}{2} + \frac{a}{2}\tanh\left[\frac{a(x+c_1 t)}{2\sqrt{2}}\right], \quad c_1 = -\frac{2-a}{\sqrt{2}}, \end{aligned}$$

and

$$\begin{aligned} U^{a1}(x+c_2 t) \quad &:= \quad \frac{a+\exp\left[(1-a)x/\sqrt{2} + (1-a^2)t/2\right]}{1+\exp\left[(1-a)x/\sqrt{2} + (1-a^2)t/2\right]} \\ &= \quad \frac{a+1}{2} + \frac{1-a}{2}\tanh\left[\frac{(1-a)(x+c_2 t)}{2\sqrt{2}}\right], \quad c_2 = \frac{1+a}{\sqrt{2}}. \end{aligned}$$

We notice that the former one is a traveling wave connecting $u=0$ to $u=a$ while the latter one connects $u=a$ to $u=1$. More interesting thing is that the Nagumo equation has the following exact solution:

$$u(x,t) = \frac{\exp\left[x/\sqrt{2} + (1/2-a)t\right] + a\exp\left[ax/\sqrt{2} - (a-a^2/2)t\right]}{1+\exp\left[x/\sqrt{2} + (1/2-a)t\right] + \exp\left[ax/\sqrt{2} - (a-a^2/2)t\right]} \tag{6}$$

([14]). This solution is not a traveling wave with a constant profile. In fact it behaves as two traveling waves U^{0a} and U^{a1} propagate from the left axis and right axis respectively until they merge. Then the solution behaves like a single wave U^{01}.

We call an entire solution to (1) if it is a classical solution defined for all x and t. Although an equilibrium solution and a traveling wave solution are examples of entire solutions, the above solution (6) is a different example of an entire solution from those. The solution (6) suggests us an interesting problem how we can show the existence of an entire solution which behaves as two traveling waves for $t << 0$.

The aim of our talk is to introduce the existence theorem for some entire solutions [16]. Applying the theorem, we obtain the similar solution to (6) for a general cubic f. We can also obtain a different type of entire solution than (6).

Finally we note that there is an entire solution which behaves as two fronts U^{01} and U^{10} propagating from the both sides of x-axis and annihilating eventually. This kind of entire solutions are extensively studied in [3], [6], [8], [9], [22].

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