# Inverse boundary value problem for elliptic equation with complex coefficient

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Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^2$. We consider the equation

$$\begin{cases} -\Delta u + Vu = 0, & \text{in } \Omega, \\ u = f, & \text{on } \partial\Omega. \end{cases} \tag{1}$$

We assume that $V(x)$ is a complex valued $L^p(\Omega)$ $(p > 2)$ function and that 0 is not a Dirichlet eigenvalue for the operator $-\Delta + V$ on $\Omega$. Let $\alpha = (p-2)/p$. Then, for each $f \in C^{1,\alpha}(\partial\Omega)$ there exists a unique solution $u \in C^{1,\alpha}(\overline{\Omega})$ of (1) with the boundary value $f$ on $\partial\Omega$. Here $C^\alpha(\overline{\Omega})$ is a usual Hölder space and $C^{1,\alpha}(\overline{\Omega})$ is the space of functions $u \in C^1$ with $\partial_j u \in C^\alpha$.

The Dirichlet-to-Neumann map (DN map) $\Lambda_V$ corresponding to $V$ is defined by

$$\Lambda_V f := \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega},$$

where $\nu$ is the outer unit normal on $\partial\Omega$ and $u$ is the solution of (1) with the Dirichlet data $u = f$ on $\partial\Omega$.

**Inverse boundary value problem is:** determine $V$ from the knowledge of $\Lambda_V$.

If $V$ is uniquely determined by $\Lambda_V$, can we calculate $V$ from the knowledge of $\Lambda_V$? We call it the reconstruction problem.

In the multidimensional case $\mathbb{R}^n$ $(n \geq 3)$, global uniqueness was proved by Sylvester and Uhlmann [18]. Nachman [17] gave a solution of the reconstruction problem.

In the two dimensional case, global uniqueness was resolved by Bukhgeim [3]. He also gave an inversion formula for smooth potentials in terms of boundary measurements with a special boundary condition.

In this talk, we shall discuss the reconstruction problem in the two dimensional case.

Let $W^{m,p}(\Omega)$ be the usual Sobolev space of order $m$ in $L^p(\Omega)$.

**Theorem 1.** *We assume that $V \in W^{1,p}(\Omega)$ for some $p > 2$. Then there exists $M = M(p, \Omega)$ such that if $\|V\|_{W^{1,p}(\Omega)} \leq M$, then there is a reconstruction scheme to identify $V(x)$ for any $x \in \overline{\Omega}$ from $\Lambda_V$.*

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