Perturbation and Dispersion of Rayleigh Waves in Anisotropic Elasticity

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1 Introduction

Rayleigh waves are elastic surface waves which propagate along the traction-free surface with the phase velocity in the subsonic range and whose amplitude decays exponentially with depth below that surface. Such waves serve as a useful tool in nondestructive characterization of materials. The problem there is what material information we obtain if we could measure accurately Rayleigh waves propagating in any direction on the traction-free surface.

For definiteness, we choose a Cartesian coordinate system such that the material half-space occupies the region $x_3 \leq 0$, whereas the 1- and 2-axis are arbitrarily chosen. Then Rayleigh wave considered here can be described as a time-harmonic solution to the equation of motion with zero body force

$$\rho \frac{\partial^2}{\partial t^2} u_i = \sum_{j,k,l=1}^{3} \frac{\partial}{\partial x_j} \left(C_{ijkl} \frac{\partial u_k}{\partial x_l} \right) \quad \text{in} \;\; x_3 < 0, \qquad i = 1, 2, 3 \tag{1}$$

with the zero-traction boundary condition

$$\sum_{j,k,l=1}^{3} C_{ijkl} \frac{\partial u_k}{\partial x_l} n_j \Big|_{x_3=0} = 0, \qquad i = 1, 2, 3. \tag{2}$$

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Here $\rho > 0$ is the uniform mass density, t is the time, $\boldsymbol{u} = \boldsymbol{u}(\mathbf{x}, t) = (u_1, u_2, u_3)$ is the displacement at the place $\mathbf{x} = (x_1, x_2, x_3)$ at time t, $(n_1, n_2, n_3) = (0, 0, 1)$ is the outward unit normal to the surface, and $\mathbf{C} = \mathbf{C}(\mathbf{x}) = \big(C_{ijkl}\big)_{i,j,k,l=1,2,3}$ is the elasticity tensor, which has the physically natural symmetries

$$C_{ijkl} = C_{jikl} = C_{klij}, \qquad i, j, k, l = 1, 2, 3 \tag{3}$$

and satisfies the strong convexity condition

$$\sum_{i,j,k,l=1}^{3} C_{ijkl}\, \varepsilon_{ij}\, \varepsilon_{kl} > 0 \quad \Big((\varepsilon_{ij}) : \text{ any nonzero } 3 \times 3 \text{ real symmetric matrix}\Big)$$

at each $\mathbf{x}$.

First we consider Rayleigh waves propagating along the traction-free surface of a homogeneous elastic half-space. For isotropic elasticity, such waves are well known: Their phase velocity v_R^{Iso} is determined from the secular equation, which is a bi-cubic equation written in terms of the Lamé constants λ and μ (see (6)).

Suppose that the elasticity tensor can be expressed as the sum of its isotropic and its perturbative part. We consider elastic media for which the perturbative part of the elasticity tensor is sufficiently small as compared with the isotropic part. The isotropic part of a given elasticity tensor is itself also an elasticity tensor, which we interpret as a comparative 'unperturbed' isotropic state. The perturbative part then gives the deviation of the elasticity tensor from the comparative isotropic state and represents the anisotropy that the elastic material carries. Here we do not put any restriction on the material symmetry of the perturbative part so that it has 21 independent components.

We investigate the perturbation of the phase velocity v_R of Rayleigh waves, i.e., the shift in v_R from its comparative isotropic value v_R^{Iso}, caused by the perturbative part. In Section 2 we present a perturbation formula for the phase velocity which is correct to within terms linear in the components of the perturbative part. This formula shows explicitly how the perturbative part, to first order of itself, affects the phase velocity of Rayleigh waves. We obtain these formulas by a consistent method on the basis of the Stroh formalism.

Second, we consider Rayleigh waves propagating along the traction-free surface of a vertically inhomogeneous elastic half-space. Here we assume that the elastic tensor depends smoothly only on the depth x_3. The purpose is to derive a high-frequency asymptotic formula for the velocity of Rayleigh waves propagating in various directions along the surface. We seek a time-harmonic solution to (1) and (2) of the form

$$\boldsymbol{u} = (u_1, u_2, u_3) = e^{-\sqrt{-1}\,k(x_1\eta_1 + x_2\eta_2 - vt)}\,\boldsymbol{v}(\eta_1, \eta_2, x_3, v, k), \tag{4}$$

where k is a wave number, $\eta = (\eta_1, \eta_2, 0)$ is the direction of wave propagation, v is phase velocity and $\boldsymbol{v}$ is a complex vector function which decays exponentially as $x_3 \longrightarrow -\infty$. In Section 3 we will develop a procedure with which, for each direction of propagation, we express each term of the asymptotic expansion of Rayleigh-wave velocity v_R for large k in terms of C_{ijkl} $(1 \le i, j, k, l \le 3)$ at $x_3 = 0$ and their x_3-derivatives at $x_3 = 0$. This expresses the frequency-dependence of the Rayleigh-wave velocity, or the dispersion of the Rayleigh-wave velocity, caused by vertical inhomogeneity of the elasticity tensor. In nondestructive characterization of materials, by measuring the dispersion of the Rayleigh-wave velocity for various propagation directions, we obtain some information on C_{ijkl} and their x_3-derivatives at $x_3 = 0$.

The project in Section 3 is still in progress. As a partial result we give the first two terms of the asymptotic expansion of Rayleigh-wave velocity for large k when the material has an orthorhombic symmetry. Future extension is to study the perturbation of each term of the asymptotic expansion of Rayleigh-wave velocity caused by the deviation of the elasticity tensor from its comparative 'unperturbed' isotropic state.

2 Perturbation of Rayleigh-wave velocity

Suppose that the elasticity tensor $\mathbf{C} = \left(C_{ijkl}\right)_{i,j,k,l=1,2,3}$ is independent of $\mathbf{x}$ and has the form

$$\mathbf{C} = \mathbf{C}^{\mathrm{Iso}} + \mathbf{A}, \tag{5}$$

where $\mathbf{C}^{\mathrm{Iso}}$ is the isotropic part of $\mathbf{C}$,

$$\mathbf{C}^{\mathrm{Iso}} = \left(C^{\mathrm{Iso}}_{ijkl}\right)_{i,j,k,l=1,2,3}, \qquad C^{\mathrm{Iso}}_{ijkl} = \lambda\,\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj})$$

with the Lamé moduli λ and μ, and $\mathbf{A}$ is the perturbative part of $\mathbf{C}$,

$$\mathbf{A} = \left(a_{ijkl}\right)_{i,j,k,l=1,2,3}.$$

From the symmetries (3) of $\mathbf{C}$ it follows that

$$a_{ijkl} = a_{jikl} = a_{klij}, \qquad i, j, k, l = 1, 2, 3,$$

but we do not assume any other symmetry for $\mathbf{A}$. Hence the perturbative part $\mathbf{A}$ is fully anisotropic and has 21 independent components.

Theorem 1 ([4, 5]) *The phase velocity v_R of Rayleigh waves which propagate along the surface of the half-space $x_3 \leq 0$ in the direction of the 2-axis can be written, to within terms linear in the perturbative part $\mathbf{A} = \left(a_{ijkl}\right)_{i,j,k,l=1,2,3}$, as*

$$v_R \;=\; v_R^{\mathrm{Iso}} - \frac{1}{2\rho\, v_R^{\mathrm{Iso}}} \cdot \Big[\gamma_1(v_R^{\mathrm{Iso}})\, a_{2222} + \gamma_2(v_R^{\mathrm{Iso}})\, a_{2233} + \gamma_3(v_R^{\mathrm{Iso}})\, a_{3333} + \gamma_4(v_R^{\mathrm{Iso}})\, a_{2323}\Big],$$

where

$$\begin{aligned}
\gamma_1(v) &= \frac{(\lambda+2\mu)\Big[-8\mu^2(\lambda+\mu) + 2\mu(5\lambda+6\mu)V - (2\lambda+3\mu)V^2\Big]}{D(v)}, \\
\gamma_2(v) &= \frac{4\lambda(\mu - V)\Big[4\mu(\lambda+\mu) - (\lambda+2\mu)V\Big]}{D(v)}, \\
\gamma_3(v) &= \frac{(\lambda+2\mu-V)\Big[-8\mu^2(\lambda+\mu) + 2\mu(5\lambda+6\mu)V - (2\lambda+3\mu)V^2\Big]}{D(v)} \\
&= \left(1 - \frac{V}{\lambda+2\mu}\right)\gamma_{22}(v), \\
\gamma_4(v) &= \frac{-8\mu(\lambda+2\mu-V)\Big[2\mu(\lambda+\mu) - (\lambda+2\ mu)V\Big]}{D(v)}, \\
D(v) &= (\lambda+\mu)\Big[8\mu^2(3\lambda+4\mu) - 16\mu(\lambda+2\mu)V + 3(\lambda+2\mu)V^2\Big], \\
V &= \rho v^2,
\end{aligned}$$

and v_R^{Iso} is the velocity of Rayleigh waves in the comparative isotropic medium defined by $\mathbf{C} = \mathbf{C}^{\mathrm{Iso}}$ and $\mathbf{A} = \mathbf{O}$, i.e., $V_R^{\mathrm{Iso}} = \rho\,(v_R^{\mathrm{Iso}})^2$ is the unique solution to the cubic equation

$$V^3 - 8\mu V^2 + \frac{8\mu^2(3\lambda+4\mu)}{\lambda+2\mu}V - \frac{16\mu^3(\lambda+\mu)}{\lambda+2\mu} = 0 \tag{6}$$

in the range $0 < V < \mu$.

Remarks Only four components $a_{2222}, a_{2323}, a_{2233}$ and a_{3333} of the perturbative part $\mathbf{A}$ can influence the first order perturbation of the phase velocity v_R. The perturbation formula above do not agree totally with the result in [1]. In [5], where the initial stress is also taken into account, an argument is given to support our present result. According to our first-order formula above and the transformation formula for fourth-order tensors, we shall see that the anisotropy-induced velocity shifts of Rayleigh waves, taken in totality of all propagation directions on the free surface, carry information only on 13 components of the perturbative part $\mathbb{A}$ of the elasticity tensor [5].

In the homogeneous medium where the elasticity tensor $\mathbf{C}$ is independent of $\mathbf{x}$, the surface-wave solution to (1) in the half-space $x_3 \leq 0$ which decays exponentially as $x_3 \longrightarrow -\infty$ and has direction of propagation $\eta = (\eta_1, \eta_2, 0)$, phase velocity v and wave number k can be expressed in the form

$$\boldsymbol{u} = (u_1, u_2, u_3) = \sum_{\alpha=1}^{3} e^{-\sqrt{-1}\,k(x_1\eta_1 + x_2\eta_2 + p_\alpha x_3 - vt)}\, c_\alpha\, \boldsymbol{a}_\alpha(\eta_1, \eta_2, v), \tag{7}$$

where p_α ($\operatorname{Im} p_\alpha > 0, \alpha = 1,2,3$) are Stroh's eigenvalues, $\boldsymbol{a}_\alpha$ ($\alpha = 1,2,3$) are linearly independent vectors in $\mathbb{C}^3$ and c_α ($\alpha = 1,2,3$) are arbitrary complex constants. The boundary traction

$$\boldsymbol{t} = \left(\sum_{j,k,l=1}^{3} C_{ijkl}\, \frac{\partial u_k}{\partial x_l}\, n_j \right)_{i\downarrow 1,2,3} \Bigg|_{x_3=0} \tag{8}$$

pertaining to the solution (7) can be written in the form

$$\boldsymbol{t} = -\sqrt{-1}\,k \sum_{\alpha=1}^{3} e^{-\sqrt{-1}\,k(x_1\eta_1 + x_2\eta_2 - vt)}\, c_\alpha\, \boldsymbol{l}_\alpha(\eta_1, \eta_2, v). \tag{9}$$

It follows that $\begin{bmatrix} \boldsymbol{a}_\alpha \\ \boldsymbol{l}_\alpha \end{bmatrix} \in \mathbb{C}^6$ ($\alpha = 1,2,3$) are linearly independent eigenvectors of Stroh's eigenvalue problem associated with the eigenvalues p_α ($\alpha = 1,2,3$) (see, for example, [4]).*

The surface impedance matrix $\mathbf{Z}(v, \eta)$, which maps the boundary displacement $\boldsymbol{u}|_{x_3=0}$ linearly to the boundary traction (8) is given by

$$\mathbf{Z}(v, \eta) = -\sqrt{-1}\,[\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3]\,[\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]^{-1}, \tag{10}$$

where $[\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3]$ and $[\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]$ denote 3×3 matrices which consist of the column vectors $\mathbf{l}_\alpha$ and $\mathbf{a}_\alpha$ respectively. It is proved that $\mathbf{Z}(v, \eta)$ is Hermitian.

From (2) it follows that the phase velocity v_R of Rayleigh waves in the homogeneous medium satisfies

$$\det \mathbf{Z}(v, \eta) = 0 \quad \text{at} \quad v = v_R. \tag{11}$$

Applying the implicit function theorem to (11), we obtain Theorem 1.

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*When $\begin{bmatrix} \boldsymbol{a}_\alpha \\ \boldsymbol{l}_\alpha \end{bmatrix}$ becomes a generalized eigenvector of Stroh's eigenvalue problem for some α, the forms (7) and (9) are slightly modified.

3 Dispersion of Rayleigh-wave velocity

Suppose that the elasticity tensor $\mathbf{C} = \big(C_{ijkl}\big)_{i,j,k,l=1,2,3}$ depends smoothly only on the depth x_3 below the surface $x_3 = 0$. Our procedure will be divided into three steps.

1. Construct an asymptotic solution to (1) of the form (4) for large k by using the factorization of the principal symbol pertaining to the differential operator in x_3 which is obtained from substitution of the form (4) into (1).

2. Let $\mathbf{Z}(v, \eta, k)$ be the surface impedance matrix, which maps the boundary displacement $\boldsymbol{u}|_{x_3=0}$ linearly to the boundary traction

$$\boldsymbol{t} = \left(\sum_{j,k,l=1}^{3} C_{ijkl} \frac{\partial u_k}{\partial x_l} n_j \right)_{i\downarrow 1,2,3} \Bigg|_{x_3=0} .$$

Determine 3×3 surface impedance matrix $\mathbf{Z}_n(v, \eta)$ $(n = 0, 1, 2, \cdots)$ that appear in the asymptotic formula

$$\mathbf{Z}(v, \eta, k) = k\,\mathbf{Z}_0(v, \eta) + \mathbf{Z}_1(v, \eta) + k^{-1}\mathbf{Z}_2(v, \eta) + k^{-2}\mathbf{Z}_3(v, \eta) + \cdots \quad (12)$$

for large k.

3. Apply the implicit function theorem to $\det \mathbf{Z}(v, \eta, k) = 0$ to obtain the asymptotic formula for the phase velocity v_R of Rayleigh waves for large k:

$$v_R = v_R(\eta, k) = v_0(\eta) + v_1(\eta)\,k^{-1} + v_2(\eta)\,k^{-2} + \cdots . \quad (13)$$

$\mathbf{Z}_0(v, \eta)$ in (12) is the surface impedance matrix for the homogeneous half-space defined by (10) with $\mathbf{C} = \mathbf{C}(0)$. Then $v_0(\eta)$ in (13) is identical to v_R in (11) under $\mathbf{C} = \mathbf{C}(0)$.

Now we shall give an equation for $\mathbf{Z}_1(v, \eta)$ in (12). Define the 3×3 matrices $\mathbf{R}_0 = \mathbf{R}_0(\eta)$, $\mathbf{T}_0$ by

$$\mathbf{R}_0 = \left(\sum_{j=1}^{2} C_{ijk3}(0)\,\eta_j \right)_{i\downarrow k\to 1,2,3}, \qquad \mathbf{T}_0 = \big(C_{i3k3}(0)\big)_{i\downarrow k\to 1,2,3} = {\mathbf{T}_0}^T$$

and put

$$\mathbf{K}_0 = \mathbf{K}_0(v, \eta) = \mathbf{T}_0^{-1}\Big(\mathbf{R}_0^T - \sqrt{-1}\,\mathbf{Z}_0(v, \eta)\Big).$$

Define the 3×3 matrices $\mathbf{Q}'_0 = \mathbf{Q}'_0(\eta)$, $\mathbf{R}'_0 = \mathbf{R}'_0(\eta)$, $\mathbf{T}'_0$ by

$$\mathbf{Q}'_0 = \left(\sum_{j,l=1}^{2} C'_{ijkl}(0)\, \eta_j\, \eta_l \right)_{i \downarrow k \to 1,2,3} = {\mathbf{Q}'_0}^T, \qquad \mathbf{R}'_0 = \left(\sum_{j=1}^{2} C'_{ijk3}(0)\, \eta_j \right)_{i \downarrow k \to 1,2,3},$$
$$\mathbf{T}'_0 = \left(C'_{i3k3}(0) \right)_{i \downarrow k \to 1,2,3} = {\mathbf{T}'_0}^T,$$

where superimposed primes ($'$) on C_{ijkl} denote differentiation with respect to x_3.

Theorem 2 *Hermitian matrix $\mathbf{Z}_1 = \mathbf{Z}_1(v, \eta)$ is the unique solution to the linear system*

$$(\mathbf{K}_0^*)^2\, \mathbf{Z}_1 - 2\, \mathbf{K}_0^*\, \mathbf{Z}_1\, \mathbf{K}_0 + \mathbf{Z}_1\, (\mathbf{K}_0)^2 = \mathbf{Q}'_0 - \mathbf{R}'_0\, \mathbf{K}_0 - \mathbf{K}_0^*\, (\mathbf{R}'_0)^T + \mathbf{K}_0^*\, \mathbf{T}'_0\, \mathbf{K}_0, \tag{14}$$

where $\mathbf{K}_0^*$ is the adjoint matrix of $\mathbf{K}_0$.

In what follows we assume that a half-space $x_3 \le 0$ is occupied by elastic materials whose elasticity tensor $\mathbf{C} = \mathbf{C}(x_3) = \left(C_{ijkl} \right)_{i,j,k,l=1,2,3}$ has an orthorhombic symmetry at each x_3 and that the axes of the orthorhombic symmetry of the medium coincide with the 1-, 2-, and 3-axis of the Cartesian coordinate system. Then possibly non-zero components of $\mathbf{C} = \mathbf{C}(x_3)$ are $C_{1111}, C_{2222}, C_{3333}, C_{1122}, C_{1133}, C_{2233}, C_{2323}, C_{1313}$ and C_{1212} at each x_3. Under the setting above we consider Rayleigh waves which propagate along the surface of the half-space $x_3 \le 0$ in the direction of the 2-axis. Henceforth we set $\eta = (0, 1, 0)$.

It is well known [3] that $v_0(\eta)$ in (13) is the unique solution to

$$\begin{aligned} &C_{3333}(0) C_{2323}(0) \Big(C_{2222}(0) - V \Big) V^2 \\ &\quad - \Big(C_{2323}(0) - V \Big) \Big(C_{3333}(0)\, (C_{2222}(0) - V) - C_{2233}^2(0) \Big)^2 = 0 \end{aligned} \tag{15}$$

in the subsonic range with $V = \rho\, v_0(\eta)^2$.[†]

For $v_1(\eta)$ in (13) we have

Corollary 3

$$v_1(\eta) = - \left. \frac{x_{22}\, z_{33} + x_{33}\, z_{22} - 2 z_{23}\, x_{23}}{\dfrac{\partial}{\partial v} (z_{22}\, z_{33} - {z_{23}}^2)} \right|_{v = v_0(\eta)}.$$

[†]Equation (15) follows also from (11).

Here z_{22}, z_{33} and $\sqrt{-1}\, z_{23}$ are the (2,2), (3,3) and (2,3) components of $\mathbf{Z}_0(v,\eta)$, respectively, which are given by

$$\begin{aligned} z_{22} &= \sqrt{\frac{C_{2323}(0)}{C_{3333}(0)}}\,\frac{P_1}{P_1+P_2}\sqrt{(P_1+P_2)^2-(C_{2233}(0)+C_{2323}(0))^2}, \\ z_{33} &= \sqrt{\frac{C_{3333}(0)}{C_{2323}(0)}}\,\frac{P_2}{P_1+P_2}\sqrt{(P_1+P_2)^2-(C_{2233}(0)+C_{2323}(0))^2}, \\ z_{23} &= \frac{-1}{P_1+P_2}(C_{2323}(0)\,P_1 - C_{2233}(0)\,P_2), \end{aligned}$$

with

$$P_1 = \sqrt{C_{3333}(0)(C_{2222}(0)-V)}, \quad P_2 = \sqrt{C_{2323}(0)(C_{2323}(0)-V)}, \quad V = \rho\, v^2$$

and x_{22}, x_{33} and $\sqrt{-1}\, x_{23}$ are the (2,2), (3,3) and (2,3) components of $\mathbf{Z}_1(v,\eta)$, respectively.

Remarks Writing down equation (14) explicitly, we see that x_{22}, x_{33} and x_{23} are all real-valued and depend only on $C_{2222}(0), C_{2233}(0), C_{3333}(0), C_{2323}(0)$ and the derivatives $C'_{2222}(0), C'_{2233}(0), C'_{3333}(0), C'_{2323}(0)$, from which we conclude that $v_1(\eta)$ depends only on the boundary values of the four components of $\mathbf{C}$ and their derivatives at the boundary. This corollary is an alternative expression of the result in [2].

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