Mean curvature flow closes open ends of noncompact surface of rotation

Yukihiro Seki [1]

Graduate School of Mathematical Sciences,
University of Tokyo,
Tokyo 153-8914, Japan

1 Introduction

This is a joint work with Yoshikazu Giga and Noriaki Umeda. We discuss motion of noncompact hypersurfaces Γ_t moved by mean curvature flow, whose initial surface Γ_0 is rotationally symmetric to an axis, say the x_1-axis, and is represented by rotating the graph of a positive function u_0 around the axis. Under the symmetric assumption, the mean curvature flow equation is equivalent with one-dimensional quasilinear parabolic equation

$$u_t = \frac{u_{xx}}{1+(u_x)^2} - \frac{n-1}{u}, \quad x \in \mathbf{R},\ t > 0, \tag{1.1}$$

with initial data

$$u(x,0) = u_0(x) > 0, \quad x \in \mathbf{R}. \tag{1.2}$$

1.1 Derivation of the equation

The surfaces Γ_t remain rotationally symmetric to the axis so long as they exist as is proved in [1, Theorem 4.3a]. Namely, we may assume that the hypersurfaces are given of the form

$$\Gamma_t = \{x = (x_1, x_2, ..., x_n) \in \mathbf{R}^n |\ r = u(x_1, t)\}$$

with some function u, where $r = \left(\sum_{j=2}^n x_j^2\right)^{1/2}$ denotes the distance from the x_1-axis to Γ_t. We call these hypersurfaces **axisymmetric surfaces**.

Although the derivation of equation (1.3) was already done in [10, 2] by the radial distance u of the surface to its axis of rotation, we introduce another way to derive the equation based on level set method (c.f.[4]), which

[1]Supported by Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists.

describes a hypersurface as the zero level set of an auxiliary function. Under the symmetric assumption, we may take the auxiliary function as

$$\phi(x,t) := -r + u(x_1,t),$$

so that the surfaces are represented as $\Gamma_t = \{x \in \mathbf{R}^n |\ \phi(x,t) = 0\}$. Here we observe that $|\nabla\phi| = (1+u_{x_1}^2)^{1/2}$ does not vanish on Γ_t. With this function, a unit normal vector field $\mathbf{n}$ of Γ_t is given by $\mathbf{n} = -\nabla\phi/|\nabla\phi|$, so that we may compute V and H respectively as

$$\begin{aligned} V &= \frac{dx(t)}{dt}\cdot\mathbf{n} = \frac{\phi_t}{|\nabla\phi|} = \frac{u_t}{(1+u_{x_1}^2)^{1/2}}, \\ H &= -\nabla\cdot\mathbf{n} = \nabla\cdot\Big(\frac{\nabla\phi}{|\nabla\phi|}\Big) = \frac{u_{x_1x_1}}{(1+u_{x_1}^2)^{3/2}} - \frac{1}{(1+u_{x_1}^2)^{1/2}}\frac{n-1}{r}, \end{aligned}$$

where $x(t)$ is a C^1-curve such that $\phi(x(t),t) = 0$. (Here and henceforth we do not take average of principal curvatures to define mean curvature.) Putting $V = H$, we get

$$u_t = \frac{u_{x_1x_1}}{1+(u_{x_1})^2} - \frac{n-1}{u}. \tag{1.3}$$

Since the spatial independent variable of unknown function u is essentially one dimension, we may and shall denote x_1 by x in the following for simplicity. We now arrive at the Cauchy problem (1.1)-(1.2).

1.2 Quenching problem

The Cauchy problem (1.1)-(1.2) has a unique positive classical solution u locally in time, but the solution is forced to reach zero in finite time as long as bounded initial data are concerned. This fact is readily seen if one compares u with the explicit solution $v_M(t) = \sqrt{2(n-1)(T(M)-t)}$ with $M = \sup_{x\in\mathbf{R}} u_0(x)$ and $T(M) = M^2/2(n-1)$.

Once a solution reaches zero, the equation (1.1) does not make sense and hence the solution cannot be extended globally in time as a classical solution. For a given initial datum u_0, we set

$$T(u_0) = \sup\{t > 0; \inf_{x\in\mathbf{R}} u(x,t) > 0\} < \infty$$

and call it the *quenching time* of u. It is immediate that

$$\liminf_{t\nearrow T(u_0)} \inf_{x\in\mathbf{R}} u(x,t) = 0.$$

A point $a \in \mathbf{R}$ is said to be a *quenching point* (*or pinching point*) of u if there exists a sequence $\{(x_k, t_k)\} \subset \mathbf{R} \times (0, T(u_0))$ such that

$$x_k \to a,\ t_k \nearrow T(u_0) \ \text{ and } \ u(x_k, t_k) \to 0 \quad \text{as } k \to \infty.$$

In other words, a point $a \in \mathbf{R}^N$ is a quenching point if and only if u is not bounded away from zero. Quenching points of u correspond to positions of pinching necks of the surface Γ_t at $t = T(u_0)$.

1.3 Known results

We shall recall several known results on mean curvature flow equations for compact hypersurfaces.

General surface
1. Huisken [11].
If initial surface is smooth, compact and convex in $\mathbf{R}^n, n \geq 3$, then the hypersurface Γ_t remains smooth, compact and convex and shrinks to a "round point" in finite time.

2. Gage and Hamilton [3].
The result of [11] still holds for simple convex curves in $\mathbf{R}^2$.

3. Grayson [9].
Even if initial curve is not convex, the solution curve must become convex before it shrinks to a point.

Axisymmetric surface
4. Grayson [10].
When $n \geq 3$, this result of [9] can fail to hold in general. There is an example of surface whose neck pinches before it shrinks to a point. (A barbell-like surface: two spherical surfaces connected by a thin "neck".)

5. Altschuler, Angenent and Giga [1].
There exists a finite sequence $0 = t_0 < t_1 < ... < t_\ell$ such that Γ_t is smooth for $t_{j-1} < t < t_j$. The number of components **can** change only at $t = t_j$, $(j = 1, 2, ..., \ell)$.

1.4 Our aim

We would like to show that any noncompact axisymmetric hypersurface such that the quenching time is "minimal" has no pinching point on the axis of rotation at the quenching time **no matter how thin** necks of the initial surface are, except for flat surfaces, and to characterize such hypersurfaces by initial data. We will give the definition that the quenching time is minimal in the next section.

2 Main results

If the initial data is a positive constant, then the solution of (1.1)-(1.2) coincides with the solution $v_m(t)$ of the corresponding ordinary differential equation

$$v' = -\frac{n-1}{v}, \ t > 0; \quad v(0) = m, \tag{2.1}$$

that is,

$$v_m(t) = \sqrt{2(n-1)(T(m)-t)} \quad \text{with} \quad T(m) = \frac{m^2}{2(n-1)}. \tag{2.2}$$

The function v_m provides the vanishing cylinder with diameter $v_m(t)$ for each time $t < T_m$. In what follows, m is chosen as

$$m = \inf_{x \in \mathbf{R}} u_0(x).$$

A simple comparison argument shows that any solution u of (1.1)-(1.2) satisfies

$$u(x,t) \geq v_m(t) \quad \text{in } x \in \mathbf{R} \times (0, T_m).$$

We thus have, in general,

$$T(u_0) \geq T_m.$$

Definition. We say that a solution u of the Cauchy problem (1.1)-(1.2) has a *minimal quenching time* if

$$T(u_0) = T(m).$$

Proposition 2.1. *Suppose that a solution u of the Cauchy problem* (1.1)-(1.2) *quenches at minimal quenching time $T(m)$. Then*

$$\liminf_{x \to -\infty} u(x,t) = v_m(t) \qquad \text{or} \qquad \liminf_{x \to +\infty} u(x,t) = v_m(t)$$

for every $t \in [0, T(m))$ and quenching occurs at space infinity in the sense that there exists a sequence $\{(x_k, t_k)\} \subset \mathbf{R} \times (0, T(m))$ such that

$$t_k \nearrow T(m) \ \text{ and } \ u(x_k, t_k) \to 0 \ \text{ as } \ k \to \infty.$$

We are now in the position to state our main results.

Theorem 2.2. *Let u be a solution of the Cauchy problem* (1.1)-(1.2) *having a minimal quenching time $T(m)$. If $u_0 \not\equiv m$, then there is no quenching point of u. Moreover, there exists a function $u(\cdot, T(m)) \in C^\infty(\boldsymbol{R})$ such that $u(\cdot, t) \to u(\cdot, T(m))$ in the Frechét space $C^\infty(\boldsymbol{R})$ as $t \nearrow T(m)$, and it fulfills $u(x, T(m)) > 0$ in the whole $\boldsymbol{R}$. Furthermore, $\lim_{x\to-\infty} u(x, T(m)) = 0$ and/or $\lim_{x\to+\infty} u(x, T(m)) = 0$.*

We can actually obtain a necessary and sufficient condition on initial data for a solution of the Cauchy problem (1.1)-(1.2) to have a minimal quenching time, making use of the technique developed in [15, 14] for related blow-up problems. We shall consider the following conditions on initial data:

$$\begin{aligned}&\text{There exists a sequence } \{x_k\} \subset \boldsymbol{R} \text{ such that } x_k \to \infty \text{ and}\\ &\quad u_0(x + x_k) \to m \quad \text{a.e.} \quad \text{as } k \to \infty.\end{aligned} \tag{2.3}$$

$$\begin{aligned}&\text{There exists a sequence } \{x_k\} \subset \boldsymbol{R} \text{ such that } x_k \to -\infty \text{ and}\\ &\quad u_0(x + x_k) \to m \quad \text{a.e.} \quad \text{as } k \to \infty.\end{aligned} \tag{2.4}$$

Theorem 2.3. *A solution of the Cauchy problem* (1.1)-(1.2) *has a minimal quenching time if and only if u_0 satisfies conditions* (2.3) *or* (2.4).

3 Tools

We shall recall some basic tools obtained in [1] in the restricted form convenient to our aim. In what follows, the half interval $(0, \infty)$ is denoted by $\mathbf{R}_+$.

Lemma 3.1.(Altschuler-Angenent-Giga [1]; gradient bound.) *Let u be a solution of* (1.1) *in $(a, b) \times (0, T)$ for some $-\infty < a < b < \infty$. Then there is a function $\sigma : \boldsymbol{R}_+ \times \boldsymbol{R}_+ \to \boldsymbol{R}$ such that*

$$|u_x(x, t)| \leq \sigma(t, u(x, t)) \tag{3.1}$$

holds for all $a < x < b, 0 < t < T$. The function σ has the form $\sigma(t, u) = \exp\left(\rho(u)/t\right)$ with a positive continuous function ρ on $\boldsymbol{R}_+$ and depends only on $\sup u(x, 0)$ and $b - a$. Moreover, if u solves the equation in $\boldsymbol{R} \times (0, T)$, then (3.1) *holds in $\boldsymbol{R} \times (0, T)$ and σ depends only on $\sup u(x, 0)$.*

Lemma 3.2.(Altschuler-Angenent-Giga [1, Single-Point Pinching Lemma]). *Let u be a solution of* (1.1) *in* $(a, b) \times (0, T)$ *for some* $-\infty < a < b < \infty$*. If the solution u of* (1.1) *is monotone increasing (or decreasing) with respect to x in* (a, b)*, then for any subinterval* $(c, d) \Subset (a, b)$*, there is a constant* $\delta > 0$ *such that*

$$u(x,t) \geq \delta \quad \text{in } (c,d) \times (0,T).$$

4 Related studies

Our problem is closely related with a blow-up problem ([7, 8, 16, 15, 14]) for nonlinear parabolic equation

$$u_t = \Delta\phi(u) + f(u), \qquad x \in \mathbf{R}^N, \ t > 0, \tag{4.1}$$

with initial data

$$u(x,0) = u_0(x), \qquad x \in \mathbf{R}^N. \tag{4.2}$$

Typical nonlinear terms are $\phi(u) = u^m$ and $f(u) = u^p$ with $m > 0, p > 1$ being constants. Our results and proofs of the present study are the same with those of [15, 14] in spirit.

In the semilinear case $\phi(u) = u$, if u_0 is not a constant and takes its maximum at infinity, then the solution of (4.1)-(4.2) blows up only at space infinity: See [12, 5] for one-dimensional problem; [7, 8, 16] for the Cauchy problem in $\mathbf{R}^N$ (See also [6].) The notion of "blow-up direction" was originally introduced in [8]. For a solution u of (4.1)-(4.2) blowing up at $t = T(u_0)$, we say that a direction $\psi \in S^{N-1}$ is a blow-up direction if there exists a sequence $\{(x_n, t_n)\} \subset \mathbf{R}^N \times (0, T(u_0))$ such that

$$|x_n| \to \infty, \ \frac{x_n}{|x_n|} \to \psi, \ t_n \nearrow T(u_0), \ \text{ and } \ u(x_n, t_n) \to \infty \ \text{ as } \ n \to \infty.$$

It is shown in [8] that the blow-up directions are characterized by initial data. The authors of [15, 14] generalized the results of [7, 8] to the quasilinear case. The definition of "minimal blow-up time" (or "the least (possible) blow-up time") was originally given in [15]. The notion of minimal quenching time introduced in the present note is an analogue of this notion. The authors of [15, 14] obtained a necessary and sufficient condition on initial data for a solution to have a minimal blow-up time, which is close to Theorem 2.3.

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