1 Introduction

Consider

\[ V = a(x, y) \kappa + b(x, y) \quad \text{for} \quad (x, y) \in \Gamma_t \subset \mathbb{R}^2, \]

where \( \Gamma_t \) is a simple plane curve, \( V \) denotes the normal velocity on \( \Gamma_t \), \( \kappa \) is the curvature, \( a, b \) are positive functions. This equation is an example of mean curvature flows in heterogeneous media. It can be used to model the motion of an interface. Generally, one can expect to get more information about the interface by studying mean curvature flows instead of reaction diffusion equations. Especially, in the case where an interface reduces to a simple plane curve, the shape of the interface, the relation between the propagation speed and the shape are expected to be described clearly.

In [2], [3], [8], [9], [10] etc., the authors studied traveling waves of \( V = a_0 \kappa + b_0 \): the homogeneous version of (1). We are concerned with traveling waves of (1).

In the last two decades, many authors studied traveling waves of reaction diffusion equations in heterogeneous media (cf. [1], [7], [12] and references therein). Most of the study are concerned with traveling waves with planar-like front (that is, the level set of the front – interface – is bounded oscillation of a hyperplane). Recently, [4], [5], [11] etc. studied traveling waves with curved fronts, but for homogeneous equations. As far as we know, no much is known about traveling waves with curved fronts for heterogeneous equations.

In this talk, we are concerned with graphic curves, that is, for each \( t > 0 \), \( \Gamma_t \) is the graph of a function \( y = u(x, t) \). In this case, equation (1) is equivalent to

\[ u_t = a(x, u) \frac{u_{xx}}{1 + u_x^2} + b(x, u) \sqrt{1 + u_x^2}. \]

In this talk, we assume that both of \( a(x, y) \) and \( b(x, y) \) are smooth, positive and almost periodic in both variables.
Definitions of Traveling waves

(i) Assume that \(a\) and \(b\) are independent of \(u\). A solution of (2) with the form \(u(x, t) = v(x) + ct\) (for some \(v\) and \(c > 0\)) is called a traveling wave (solution).

(ii) If \(a\) and \(b\) are 1-periodic in \(y\), then we call a solution \(u(x, t)\) of (2) as a periodic traveling wave with average speed \(c\) if \(u(x, t + 1/c) = u(x, t) + 1\).

(iii) If \(a\) and \(b\) are almost periodic in \(y\), then we call a solution \(u(x, t)\) of (2) as an almost periodic traveling wave with average speed \(c\) if there exist a continuous map \(W : \mathcal{H}_a \times \mathcal{H}_b \to L^\infty_{\text{loc}}(\mathbb{R})\) and an increasing function \(\xi(t)\) such that

\[
\frac{\xi(t + T) - \xi(t)}{T} \to c \quad \text{as} \quad T \to \infty.
\]

Here \(\mathcal{H}_a := \{a(x, y + s) \mid s \in \mathbb{R}\}^{L^\infty_{\text{loc}}(\mathbb{R}^2)}\) denotes the hull of \(a\). Note that this definition is indeed the analogue of Matano’s definition for traveling wave of reaction diffusion equation in heterogeneous media ([7]).

(iv) A traveling wave is called a curved one if \(u(x, t) - u(0, t)\) is unbounded. Especially, if the graph of \(u(x, t) - u(0, t)\) lies in a bounded neighborhood of a line for \(x > 0\) and lies in a bounded neighborhood of another line for \(x < 0\), then we call it a “V”-like traveling wave.

Our main purpose in this talk is to study the existence of “V”-like traveling waves of (2), as well as the relation among the traveling speed, the shape of the profile and the spatially heterogeneity.

Denote \(a_* := \inf a(x, y), \quad a^* := \sup a(x, y), \quad b_* := \inf b(x, y), \quad b^* := \sup b(x, y)\).

2 Traveling waves of \(V = a(x)\kappa + b(x)\)

In this part we assume that \(a\) and \(b\) are independent of \(y\), and are almost periodic in \(x\). In this case a traveling wave \(u(x, t) = v(x) + ct\) satisfies

\[
c = a(x) \frac{v''}{1 + v'^2} + b(x) \sqrt{1 + v'^2}, \quad x \in \mathbb{R}.
\]

After a further transformation \(\varphi(x) := \arctan v'(x)\), the equation is equivalent to

\[
\varphi'(x) = \frac{c}{a(x)} - \frac{b(x)}{a(x) \cos \varphi(x)}, \quad \varphi(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \quad \forall x \in \mathbb{R}.
\]

For some of our results, we need to restrict \(\alpha\) to the range

\[
\alpha \in \left(\arccos \frac{b_*}{b^*}, \frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, -\arccos \frac{b_*}{b^*}\right) \quad (\Leftrightarrow b_* > b^* \cos \alpha).
\]

Lemma 1. For any \(\alpha\) in the range given by (5), there exists a unique pair \((c, \varphi) \in \mathbb{R} \times C^1(\mathbb{R})\) such that (i) \((c, \varphi)\) satisfies (4), (ii) \(\varphi\) is almost periodic, and (iii) the arithmetic mean of \(\tan \varphi\) is \(\tan \alpha\).
In addition, as a function of $\alpha$, the unique solution $(c, \varphi)$ satisfies the following estimates

\begin{equation}
\arccos \frac{b_M \cos \alpha}{b_m} \leq \text{sgn}(\alpha) \varphi(x) \leq \arccos \frac{b_m \cos \alpha}{b_M} \quad \text{for } x \in \mathbb{R},
\end{equation}

\begin{equation}
0 \leq \text{sgn}(\alpha) \frac{dc}{dx} \leq \frac{3a_M b_M^3}{a_m b_m^3 \cos^4 \alpha}.
\end{equation}

(In the case, $a$ and $b$ are periodic, $\varphi$ is also periodic).

**Theorem A.** Assume that $\alpha_1 > 0$ and $\alpha_2 < 0$ satisfy (5). Also assume that $(c, \varphi_1)$ and $(c, \varphi_2)$ are solutions of (4), that $\varphi_1$ and $\varphi_2$ are almost periodic and $M[\tan \varphi_i] = \tan \alpha_i$, $i = 1, 2$. Then, for any small initial value $\varphi(0)$, problem (4) (with the same $c$) admits a unique solution $\varphi$ and the solution satisfies the following (see Figure 1):

(i) $\varphi_2(x) < \varphi(x) < \varphi_1(x)$ for all $x$;

(ii) There exist positive constants $L$ and $\nu$ such that

\[ \varphi_1(x) - Le^{-\nu x} \leq \varphi(x) \leq \varphi_2 + Le^{\nu x} \quad \forall x \in \mathbb{R}. \]

(iii) There exist a unique $x_0$ and a unique $S_0 > 0$ such that the functions defined by

\[ v_1(x) := \int_{x_0}^x \tan \varphi_1(x)dx, \quad v_2(x) := \int_{x_0}^x \tan \varphi_2(x)dx, \quad v(x) := \int_{x_0}^x \tan \varphi(x)dx + S_0 \]

satisfy, for some $\hat{L} > 0$ and the same $\nu$ as above,

\[ \max\{v_1(x), v_2(x)\} < v(x) < \max\{v_1(x), v_2(x)\} + \hat{L}e^{-\nu |x|} \quad \forall x \in \mathbb{R}. \]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{(a) $\varphi(x)$ approaches $\varphi_i(x)$ as $x \to \pm\infty$; (b) "V"-like profile}
\end{figure}

In order to give an explicit estimate for the traveling speed $c$ in terms of $\alpha$, we consider a homogenization problem:

\[ \frac{d}{dx} \varphi^\varepsilon(x) = \frac{c}{a^\varepsilon(x)} - \frac{b^\varepsilon(x)}{a^\varepsilon(x) \cos \varphi^\varepsilon(x)}, \quad \varphi^\varepsilon(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \quad \forall x \in \mathbb{R} \quad (4)_\varepsilon \]

where $a_\varepsilon(x) = a(x/\varepsilon)$, $b_\varepsilon(x) = b(x/\varepsilon)$. For an almost periodic function $h(x)$ we use $M[h]$ to denote its arithmetic mean. Denote

\[ A = \left( M \left[ \frac{1}{a(x)} \right] \right)^{-1}, \quad B = M \left[ \frac{b(x)}{a(x)} \right] A. \]
Theorem B. For any \( \alpha \) satisfying (5), let \((c^\varepsilon, \varphi^\varepsilon)\) be the solution of (4)\(_{\varepsilon}\) as in Lemma 1. Then

\[
\lim_{\varepsilon \searrow 0} \| \varphi^\varepsilon - \alpha \|_{L^\infty(\mathbb{R})} = 0, \quad \lim_{\varepsilon \searrow 0} c^\varepsilon = \frac{B}{\cos \alpha}.
\]

If in addition assume that for some \( L_1, L_2 > 0 \),

\[
\left| \int_0^x \left( \frac{1}{a(x)} - \frac{1}{A} \right) dx \right| \leq L_1, \quad \left| \int_0^x \left( \frac{b(x)}{a(x)} - \frac{B}{A} \right) dx \right| \leq L_2 \quad \text{for } x \in \mathbb{R},
\]

then there exists a positive constant \( Q \) depending only on \( L_1, L_2, a_m, a_M, b_m, b_M \) and \( \alpha \) such that

\[
\left\| \varphi^\varepsilon(\cdot) - \alpha \right\|_{L^\infty(\mathbb{R})} + \left| c^\varepsilon - \frac{B}{\cos \alpha} \right| \leq Q\varepsilon \quad \forall \varepsilon > 0.
\]

3 Traveling waves of \( V = a(x, y)\kappa + b \)

In this part, for some technical reasons, we study curvature flow \( V = a(x, y)\kappa + b \), where \( a \) is almost periodic in \( y \), \( b \) is a positive constant.

Theorem C. For any given \( c > b \), equation (2) (with \( b(x, u) \equiv b \)) has an almost periodic traveling wave \( U(x, t) \in C^{2+\nu, 1+\nu/2}(\mathbb{R}^2) (\nu \in (0, 1)) \). For each \( t \in \mathbb{R} \), the graph of \( U(x, t) \) is a "V"-like curve:

\[
\varphi(x, a_\star) \leq U(x, t) - ct \leq \varphi(x, a_\star^\ast) + (a_\star^\ast - a_\star)S \quad \text{for } x \in \mathbb{R},
\]

where \( S = S(b, c) > 0 \) is a constant, both \( \varphi(x, a_\star) \) and \( \varphi(x, a_\star^\ast) + (a_\star^\ast - a_\star)S \) approach the same line \( x\sqrt{c^2 - b^2}/b - a_\star S \) as \( x \to \infty \), approach \( -x\sqrt{c^2 - b^2}/b - a_\star S \) as \( x \to -\infty \) (see Figure 2).

(When \( a(x, y) \) is periodic in \( y \), the traveling wave is also a periodic one).

![Fig. 2 "V"-like profile with asymptotic straight wings](image)

References


