Totally free arrangements of hyperplanes

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Abstract

A central arrangement $\mathcal{A}$ of hyperplanes in an ℓ-dimensional vector space V is said to be *totally free* if a multiarrangement $(\mathcal{A}, m)$ is free for any multiplicity $m : \mathcal{A} \to \mathbb{Z}_{>0}$. It has been known that $\mathcal{A}$ is totally free whenever $\ell \leq 2$. In this article, we will prove that there does not exist any totally free arrangement other than the obvious ones, that is, a product of one-dimensional arrangements and two-dimensional ones.

1 Introduction

Let V be an ℓ-dimensional vector space ($\ell \geq 1$) over $\mathbb{K}$ with a coordinate system $\{x_1, \ldots, x_\ell\} \subset V^*$. Define $S := \mathrm{Sym}(V^*) \simeq \mathbb{K}[x_1, \ldots, x_\ell]$. Let $\mathrm{Der}_{\mathbb{K}}(S)$ be the set of all $\mathbb{K}$-linear derivations of S to itself. Then $\mathrm{Der}_{\mathbb{K}}(S) = \oplus_{i=1}^{\ell} S \cdot \partial_{x_i}$ is a free S-module of rank ℓ. A *central arrangement (of hyperplanes)* in V is a finite collection of linear hyperplanes in V. In this article we assume that every arrangement is central unless otherwise specified. A *multiplicity* m is a function $m : \mathcal{A} \to \mathbb{Z}_{>0}$ and a pair $(\mathcal{A}, m)$ is called a *multiarrangement*. Fix a linear form α_H ($H \in \mathcal{A}$) in such a way that $\ker(\alpha_H) = H$. The *logarithmic derivation module* $D(\mathcal{A}, m)$ associated with $(\mathcal{A}, m)$ is defined by

$$D(\mathcal{A}, m) := \{\theta \in \mathrm{Der}_{\mathbb{R}}(S) \mid \theta(\alpha_H) \in S \cdot \alpha_H^{m(H)} \text{ for all } H \in \mathcal{A}\}.$$

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In general, $D(\mathcal{A}, m)$ is not necessarily a free S-module. We say that $(\mathcal{A}, m)$ is *free* if $D(\mathcal{A}, m)$ is a free S-module. For a fixed arrangement $\mathcal{A}$, a multiplicity m on $\mathcal{A}$ is called *free* if a multiarrangement $(\mathcal{A}, m)$ is free. Define

$$\mathcal{NFM}(\mathcal{A}) := \{m : \mathcal{A} \to \mathbb{Z}_{>0} \mid m \text{ is not a free multiplicity}\}.$$

The following definition was introduced in [4, Definition 5.4].

Definition 1.1
An arrangement $\mathcal{A}$ is called totally free if every multiplicity $m : \mathcal{A} \to \mathbb{Z}_{>0}$ is a free multiplicity, or equivalently $\mathcal{NFM}(\mathcal{A}) = \emptyset$.

When $\mathcal{A}_i$ is an arrangement in $V_i\,(i = 1, 2)$, the *product* $\mathcal{A}_1 \times \mathcal{A}_2$ is an arrangement in $V_1 \oplus V_2$ defined as in [6, Definition 2.13] by

$$\mathcal{A}_1 \times \mathcal{A}_2 = \{H_1 \oplus V_2 \mid H_1 \in \mathcal{A}_1\} \cup \{V_1 \oplus H_2 \mid H_2 \in \mathcal{A}_2\}.$$

Our main theorem is as follows:

Theorem 1.2
An arrangement $\mathcal{A}$ is totally free if and only if it has a decomposition

$$\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_s,$$

where each $\mathcal{A}_i$ is an arrangement in $\mathbb{K}^1$ or $\mathbb{K}^2$.

Ziegler showed in [13, Corollary 7] that $(\mathcal{A}, m)$ is a free multiarrangement whenever $\ell \leq 2$. Note that

$$D(\mathcal{A}_1 \times \mathcal{A}_2, m) \simeq S \cdot D(\mathcal{A}_1, m|_{\mathcal{A}_1}) \oplus S \cdot D(\mathcal{A}_2, m|_{\mathcal{A}_2})$$

holds true as shown in [3, Lemma 1.4]. Thus

$$\mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_s$$

is known to be totally free if each $\mathcal{A}_i$ is an arrangement in $\mathbb{K}^1$ or $\mathbb{K}^2$. Theorem 1.2 asserts that the converse is also true. In the next section we will prove Theorem 1.2 in a stronger form: we will show that $\mathcal{A}$ is decomposed into one-dimensional arrangements and two-dimensional ones if $\mathcal{NFM}(\mathcal{A})$ is a finite set.

Recall that the *intersection lattice* $L(\mathcal{A})$ is the set $\{X = H_1 \cap \cdots \cap H_s \mid H_i \in \mathcal{A},\ s \geq 0\}$ with the reverse inclusion ordering as in [6, Definition 2.1]. Then Theorem 1.2 implies:

Corollary 1.3
Whether an arrangement $\mathcal{A}$ is totally free or not depends only on its intersection lattice $L(\mathcal{A})$.

Let $\mathcal{A}$ be a nonempty central arrangement and $H_0 \in \mathcal{A}$. Define the *deletion* $\mathcal{A}'$ and the *restriction* $\mathcal{A}''$ as in [6, Definition 1.14]:

$$\mathcal{A}' := \mathcal{A} \setminus \{H_0\}, \quad \mathcal{A}'' := \{H_0 \cap H \mid H \in \mathcal{A}'\}.$$

Because of the characterization in Theorem 1.2, the total freeness is stable under deletion and restriction:

Corollary 1.4
Any subarrangement or restriction of a totally free arrangement is also totally free.

A multiarrangement was introduced and studied by Ziegler in [13]. The third author proved in [10] and [11] that the freeness of a simple arrangement is closely related with the freeness of Ziegler's canonical restriction. Recently the first and second authors and Wakefield developed a general theory of free multiarrangements and introduced the concept of free multiplicity in [3] and [4]. Several papers including [1], [2], [5] and [12] studied the set of free multiplicities for a fixed arrangement $\mathcal{A}$. The main theorem (Theorem 1.2) in this article shows that the set of free multiplicities (or $\mathcal{NFM}(\mathcal{A})$) imposes strong restrictions on the original arrangement $\mathcal{A}$.

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2 Proof of Theorem 1.2

First we review a necessary condition for a given multiarrangement to be free in Theorem 2.1.

Let $(\mathcal{A}, m)$ be a multiarrangement. When $(\mathcal{A}, m)$ is free, there exists a homogeneous basis $\theta_1, \ldots, \theta_\ell$ for $D(\mathcal{A}, m)$. The set $\exp(\mathcal{A}, m)$ of *exponents* is defined by $\exp(\mathcal{A}, m) := (\deg \theta_1, \ldots, \deg \theta_\ell)$, where $\deg(\theta_i) := \deg \theta_i(\alpha)$ for some linear form α with $\theta_i(\alpha) \neq 0$.

Define $L(\mathcal{A})_2 := \{X \in L(\mathcal{A}) \mid \operatorname{codim}_V(X) = 2\}$ and $\mathcal{A}_X := \{H \in \mathcal{A} \mid X \subset H\}$. For $X \in L(\mathcal{A})_2$ the multiarrangement $(\mathcal{A}_X, m|_{\mathcal{A}_X})$ is free with exponents $(d_1^X, d_2^X, 0, \ldots, 0)$. Define the *second local mixed product* $LMP_2(\mathcal{A}, m)$ as in [3, Definition 4.3] by

$$LMP_2(\mathcal{A}, m) := \sum_{X \in L(\mathcal{A})_2} d_1^X d_2^X.$$

If $\mathcal{B}$ is a subarrangement of $\mathcal{A}$, then it is easy to see that

$$LMP_2(\mathcal{A}, m) \geq LMP_2(\mathcal{B}, m|_{\mathcal{B}}).$$

Next assume that $(\mathcal{A}, m)$ is free with exponents $(d_1, \ldots, d_\ell)$. Define the *second global mixed product* $GMP_2(\mathcal{A}, m)$ as in [3, Definition 4.5] by

$$GMP_2(\mathcal{A}, m) := \sum_{1 \leq i < j \leq \ell} d_i d_j.$$

Theorem 2.1
If a multiarrangement $(\mathcal{A}, m)$ *is free, then* $GMP_2(\mathcal{A}, m) = LMP_2(\mathcal{A}, m)$.

In fact, Theorem 2.1 is true for any GMP_k and LMP_k $(1 \leq k \leq \ell)$, see [3, Corollary 4.6].

An arrangement $\mathcal{A}$ is said to be *reducible* if $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ for certain arrangements $\mathcal{A}_i$ in V_i $(i = 1, 2)$. We say $\mathcal{A}$ is *irreducible* if it is not reducible.

Lemma 2.2
Let $\mathcal{A}$ *be an irreducible arrangement in* $\mathbb{K}^\ell$ *with* $\ell \geq 2$. *Then there exist* $\ell+1$ *hyperplanes* $H_1, H_2, \ldots, H_{\ell+1}$ *in* $\mathcal{A}$ *satisfying the following conditions:*

$$\begin{aligned} \operatorname{codim}_V H_{i_1} \cap H_{i_2} \cap \cdots \cap H_{i_p} &= p \ \ (1 \leq i_1 < i_2 < \cdots < i_p \leq \ell,\ 1 \leq p \leq \ell), \\ H_1 \cap H_2 \cap \cdots \cap H_{\ell+1} &= \{\mathbf{0}\}. \end{aligned}$$

Proof. When $\ell = 2$ the assertion is obvious. Suppose $\ell \geq 3$. We will prove by an induction on $|\mathcal{A}|$. When $|\mathcal{A}| = \ell + 1$, the arrangement $\mathcal{A}$ itself satisfies the conditions. Suppose $|\mathcal{A}| \geq \ell + 2$. Let $H_0 \in \mathcal{A}$. Let $\mathcal{A}'$ and $\mathcal{A}''$ be the deletion and the restriction respectively. Then either $\mathcal{A}'$ or $\mathcal{A}''$ is irreducible by Tutte [9] (see also [7, Theorem 4.3.1]). If $\mathcal{A}'$ is irreducible, then $\mathcal{A}'$ contains $\ell + 1$ hyperplanes satisfying the conditions. If $\mathcal{A}''$ is irreducible, then $\mathcal{A}''$ contains ℓ hyperplanes $H_0 \cap H_1, \ldots, H_0 \cap H_\ell$ satisfying the conditions. Then $H_0, H_1, \ldots, H_\ell$ in $\mathcal{A}$ satisfy the conditions. □

Recall

$$\mathcal{NFM}(\mathcal{A}) = \{m : \mathcal{A} \to \mathbb{Z}_{>0} \mid m \text{ is not a free multiplicity}\}.$$

Proposition 2.3
If $\mathcal{A}$ *is an irreducible arrangement in* $\mathbb{K}^\ell$ *with* $\ell \geq 3$, *then* $\mathcal{NFM}(\mathcal{A})$ *is an infinite set.*

Proof. Suppose that $\mathcal{NFM}(\mathcal{A})$ is a finite set. Choose $\ell + 1$ hyperplanes $H_1, H_2, \ldots, H_{\ell+1}$ in $\mathcal{A}$ satisfying the conditions in Lemma 2.2. Let

$$\mathcal{B} := \{H_1, H_2, \ldots, H_{\ell+1}\}$$

and consider the multiplicity m defined by

$$m(H) = \begin{cases} 1 & \text{if } H \notin \mathcal{B}, \\ k & \text{if } H \in \mathcal{B}, \end{cases}$$

for every positive integer k . Since $\mathcal{NFM}(\mathcal{A})$ is a finite set, the multiarrangement $(\mathcal{A}, m)$ is free whenever k is sufficiently large. Note $|L(\mathcal{B})_2| = \binom{\ell+1}{2}$. By the definition of LMP_2,

$$LMP_2(\mathcal{A}, m) \geq LMP_2(\mathcal{B}, m|_{\mathcal{B}}) = |L(\mathcal{B})_2| k^2 = \binom{\ell+1}{2} k^2.$$

Let $|\mathcal{A}| = n$. Then

$$\sum_{d \in \exp(\mathcal{A}, m)} d = (k-1)(\ell+1) + n$$

and thus

$$GMP_2(\mathcal{A}, m) \leq \binom{\ell}{2} \left\{ \frac{(k-1)(\ell+1)+n}{\ell} \right\}^2 = \frac{(\ell+1)^2(\ell-1)}{2\ell} k^2 + Ak + B$$

with some constants A and B. By Theorem 2.1 we have

$$\binom{\ell+1}{2} k^2 \leq LMP_2(\mathcal{A}, m) = GMP_2(\mathcal{A}, m) \leq \frac{(\ell+1)^2(\ell-1)}{2\ell} k^2 + Ak + B$$

whenever k is sufficiently large. This is a contradiction because

$$\binom{\ell+1}{2} - \frac{(\ell+1)^2(\ell-1)}{2\ell} = \frac{\ell+1}{2\ell} > 0.$$

□

We now prove the following theorem which is stronger than Theorem 1.2.

Theorem 2.4
The following four conditions for a central arrangement $\mathcal{A}$ are equivalent:
(1) $\mathcal{A}$ is totally free, i. e., $\mathcal{NFM}(\mathcal{A})$ is empty,
(2) $\mathcal{NFM}(\mathcal{A})$ is a finite set,
(3) $\mathcal{A}$ has a decomposition

$$\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_s,$$

where each $\mathcal{A}_i$ is an arrangement in $\mathbb{K}^1$ or $\mathbb{K}^2$,
(4) every subarrangement of $\mathcal{A}$ is free.

Proof. The implications (1) $\Rightarrow$ (2), (3) $\Rightarrow$ (4) and (3) $\Rightarrow$ (1) are obvious. Thus it is enough to prove that (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (3).

(2) $\Rightarrow$ (3): Suppose that $\mathcal{NFM}(\mathcal{A})$ is a finite set. Decompose $\mathcal{A}$ into

$$\mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_s$$

such that each $\mathcal{A}_i$ is irreducible. Since

$$D(\mathcal{A}, m) \simeq S \cdot D(\mathcal{A}_1, m|_{\mathcal{A}_1}) \oplus S \cdot D(\mathcal{A}_2, m|_{\mathcal{A}_2}) \oplus \cdots \oplus S \cdot D(\mathcal{A}_s, m|_{\mathcal{A}_s})$$

holds by [3, Lemma 1.4], each $\mathcal{A}_i$ is an irreducible arrangement and $\mathcal{NFM}(\mathcal{A}_i)$ is a finite set. Thus Proposition 2.3 shows that each arrangement $\mathcal{A}_i$ is in $\mathbb{K}^1$ or $\mathbb{K}^2$.

(4) $\Rightarrow$ (3): Decompose $\mathcal{A}$ into irreducible arrangements. Then each of the irreducible arrangements satisfies the assumption (4). Therefore we may assume that $\mathcal{A}$ is irreducible from the beginning. Suppose $\ell \geq 3$. Then, by Lemma 2.2, there exist $\ell + 1$ hyperplanes $H_1, H_2, \ldots, H_{\ell+1}$ in $\mathcal{A}$ satisfying the conditions in Lemma 2.2. Then the arrangement $\mathcal{B} = \{H_1, H_2, \ldots, H_{\ell+1}\}$ is a generic arrangement [6, Definition 5.22] which is known to be non-free (e.g., [8]). This is a contradiction and thus we may conclude $\ell \leq 2$. $\square$

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