第１１回

偏微分方程式論札幌シンポジウム

予　稿　集

１９８６．８．１９～８．２２

第１１回偏微分方程式論
札幌シンポジウム

下記の要領でシンポジウムを行ないますので、ご案内申しあげます。

代表者　上　見　練太郎

記

1．　日　時　　１９８６年８月１９日（火）　～　８月２２日（金）

2．　場　所　　北海道大学理学部数学教室　４－５０８室

3．　講　演

８月１９日（火）

９：３０　～　１０：３０　　神　部　　勉（東大理）

渦運動における音波の発生

１１：００　～　１２：００　　儀　我　美　一（北大理）

渦と粘性

１３：３０　～　１４：４５　　＊

１５：００　～　１６：００　　松　村　昭　孝（京大工）

一次元粘性流体の漸近挙動について

８月２０日（水）

９：３０　～　１０：３０　　谷　内　俊　弥（名大理）

Ideal　ＭＨＤ

１１：００　～　１２：００　　岩　崎　敷　久（京大数理研）

Effectively hyperbolic equations の応用について

１３：３０　～　１４：４５　　＊

１５：００　～　１５：３０　　柳　沢　　卓（北大理）

未　　定

１５：４５　～　１６：１５　　横　谷　昌　之（都立大理）

Yang-Mills gradient flows について

8月21日（木）

9：30 ～ 10：30　堤　誉志雄（広島大総科）

Scattering problem for the nonlinear Schrodinger equations

11：00 ～ 12：00　桜井　力（埼玉大理）

Schrodinger 方程式の grazing ray について

13：30 ～ 14：45　＊

15：00 ～ 15：30　坂口　茂（都立大理）

ある準線型楕円型デリクレ問題の解の凸性について

15：45 ～ 16：15　内藤久資（名大理）

調和写像と Eells - Sampson の放物型方程式の大域解の存在について

8月22日（金）

9：30 ～ 10：30　井川　満（阪大理）

波動方程式に対するいくつかの凸な物体の外部問題について

11：00 ～ 12：00　森本芳則（名大工）

Criteria for hypoellipticity

＊この時間は講演者を囲んでの自由な質問の時間とする予定です。

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Vorticity and viscosity

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This is a resume of my joint work with T. Miyakawa and H. Osada [36].

We consider the Navier-Stokes system

$$(1) \qquad \frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = 0, \quad \nabla \cdot u = 0$$

on the whole plane R^2, where u and p represents unknown velocity and pressure, respectively and $\nu > 0$ is the kinematic viscosity. Since the space dimension is two, the vorticity $v = \nabla \times u = \partial u^2/\partial x_1 - \partial u^1/\partial x_2$ is scalar. Moreover, v solves

$$(2a) \qquad \frac{\partial v}{\partial t} - \nu \Delta v + (u \cdot \nabla) v = 0$$

$$(2b) \qquad u(x,t) = \iint_{R^2} \nabla^{\perp} E(x-y) v(x,t) dx$$

where $\nabla^{\perp} = (-\partial/\partial x_2, \partial/\partial x_1)$ and $E(x) = (2\pi)^{-1}\log|x|$. These equations are formally obtained by taking $\nabla\times$ of (1) and using the condition $\nabla\cdot u = 0$. As is well known the vorticity equation (2a)(2b) is formally equivalent to the Navier-Stokes system provided that u is assumed to decay to zero at space infinity.

We consider the initial value preblem for (1) or (2a)(2b) assuming only that initial vorticity $v(x,0)$ is a finite Radon measure. A typical example is N-point sources of votex, i.e.,

$$v(x,0) = \sum_{j=1}^{N}\alpha_j\delta(x-z_j). \tag{3}$$

Here z_j is a point on which j-th point source is located and α_j is a real number describing the strength of the source; δ is a Dirac measure supported at zero. One naive question is whether such point sources of vortex are smoothed out because of viscosity. In other words do solutions for (1) or (2a),(2b) exist globally-in-time and smooth for $t > 0$ even if $v(x,0)$ is a finite measure? When initial vorticity consists only one point source carried at zero (i.e. $N = 1$, $z_1 = 0$), we know an exact solution of (2a),(2b)

$$v(x,t) = \frac{\alpha_1}{4\pi\nu t}\exp\left(\frac{-|x|^2}{4\nu t}\right)$$

which is a constant multiple of the fundamental solution of the heat equation. For a general initial data we claim that a smooth solution exists globally in time. As anticipated, the viscosity smoothes singular vorticities.

Theorem ([36]). *Suppose that* $v(x,0)$ *is a finite Radon measure on* R^2. *Then there is a global solution* $v(x,t)$, $u(x,t)$ *to* (2a),(2b) *or* (1) *such that* v *and* u *are smooth for* $t > 0$ *and* $v(x,t)$ *converges to* $v(x,0)$ *under the weak topology of measures as* t *tends to zero*.

In [3] Benfatto, Esposito and Pulvirenti prove similar results under more stringent assumptions. They assume $v(x,0)$ is expressed by (3) and $|\alpha_j|$ is small compared with ν. Our results need no assumptions on particular forms or smallness of initial vorticity.

The main mathematical difficulty is that the initial energy on D

$$\iint_D |u(x,0)|^2 \, dx$$

is not necessarily finite even if D is a bounded domain. If the initial energy is finite, it is classical that there is a global classical solutions to (1) (cf. [16,17,30]).

To construct such a solution we approximate initial vorticity by smooth functions and solve (2a),(2b) with approximate initial data. It is not difficult to construct a global solution for smooth data. We expect that solutions with approximate initial data converge to a true solution for the original problem. To carry out this process we need a priori estimates.

Lemma ([36]). *Suppose that* $v(x,0)$ *is smooth and*

$\iint_{R^2} |v(x,0)|dx \leq m$. Let $\Gamma_u(x,t;y,s)$ is a fundamental solution to (2a), regarding u is a known function. Then,

$$c(t-s)^{-1}\exp\left(\frac{-|x-y|^2}{c(t-s)}\right) \leq \Gamma_u(x,t;y,s) \leq C(t-s)^{-1}\exp\left(\frac{-|x-y|^2}{C(t-s)}\right)$$

with c and $C > 0$ depending only on m.

Estimates of fundamental solutions independent of the regularity of coefficients are obtained by Aronson [1] for linear parabolic equations of divergence form (see also [2]). Osada [25] extends the estimate for non-divengence form which includes (2a) as a typical example. The above a priori estimates enable us to carry out our original idea.

For uniqueness of the solution we do not know much. We show the uniqueness when $v(x,0)$ is small. In particular, if $v(x,0)$ is absolutely continuous with respect to Lebesgue measure, we can assert the uniqueness.

Our references include those of the paper [36] for the reader's convenience.

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一次元粘性流体の漸近挙動

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最近一次元粘性流体を記述する方程式系の解の $t \to +\infty$ での漸近挙動について多くの結果が得られつつある。ここでは初期値問題に話題を絞ってその結果を概観する。

1. Burgers方程式.

どの様な結果を目標としているのか、次のBurgers方程式の初期値問題に例をとり述べる。

$$(1.1) \quad u_t + u u_x - \mu u_{xx} = 0, \quad x \in \mathbb{R}^1,\ t \geq 0,\ \mu > 0 : \text{定数},$$

$$(1.2) \quad u(0,x) = u_0(x) \in \mathcal{B}^2,$$

$$\lim_{x \to \pm\infty} u_0(x) = u_\pm, \quad (u_\pm \in \mathbb{R}).$$

(1.1),(1.2)に対する基本的仕事は Hopf [1], Ilin & Oleinik [2] 参照。解の漸近挙動は対応する次の非粘性方程式に対するRiemann問題の解と密接に関る。

$$(1.3) \quad \begin{cases} u_t + u u_x = 0, \\ u(x,0) = u_0^R(x) \equiv u_\pm \quad (x \gtrless 0). \end{cases}$$

(1.3)の解としては次の三つの場合がある。

Case 1. $u_\pm = \bar{u}$	Case 2. $u_+ > u_-$	Case 3. $u_+ < u_-$
$u = \bar{u}$: 自明解	$u = u^R(x/t)$: 希薄波解,	$u = u^S(x - st)$: 衝撃波解,
	$u^R(\xi) = \begin{cases} u_-, & \xi \leq u_- \\ \xi, & u_- \leq \xi \leq u_+ \\ u_+, & \xi \geq u_+ \end{cases}$	$s = (u_+ + u_-)/2$ (Rankine-Hugoniot), $u^S(\xi) = \begin{cases} u_- & \xi < 0 \\ u_+ & \xi > 0 \end{cases}$

もとの(1.1), (1.2)の解の漸近形としては、Case 1, Case 2 では粘性項の影響が弱いと予想されそれぞれ $\bar{u}$ と $u^R(x/t)$ が予想されるが、Case 3 では μ がいかに小さくとも平滑化により $u^S(x-st)$ に対応して次の(1.1)の衝撃波形進行波解が漸近形となることが予想される。

$$u(x,t) = U(x-st-\alpha), \quad (\alpha \in \mathbb{R}), \quad s = (u_+ + u_-)/2,$$

$$U(\xi): \begin{cases} -sU' + UU' - \mu U'' = 0 \\ \lim_{\xi \to \pm\infty} U(\xi) = u_\pm, \quad \lim_{\xi \to \pm\infty} U'(\xi) = 0. \end{cases}$$

(この場合、$U(\xi) = (u_+ + u_-)/2 - ((u_+ - u_-)/2) \tanh((u_+ - u_-)\xi/4\mu)$).

基本的な結果

Case 1. $u_0 - \bar{u}, u_{0,x} \in L^2 \Rightarrow {}^{\exists 1}$ global sol. u s.t.

$$\lim_{t \to +\infty} \sup_x |u(t,x) - \bar{u}| = 0.$$

Case 2. $u_0 - u_0^R \in L^2, u_{0,x} \in L^2 \Rightarrow {}^{\exists 1}$ global sol. u s.t.

$$\lim_{t \to \infty} \sup_x |u(t,x) - u^R(x/t)| = 0.$$

Case 3. $u_0 - u_0^R \in L^2 \cap L^1, u_{0,x} \in L^2 \Rightarrow {}^{\exists 1}$ global sol. u & ${}^{\exists 1}\alpha \in \mathbb{R}$

s.t. $\int u_0(x) - U(x-\alpha)\, dx = 0$,

$$\lim_{t \to \infty} \sup_x |u(t,x) - U(x-st-\alpha)| = 0.$$

Remark 1. [2] によれば、これらの結果はより一般な形

(1.4) $\quad u_t + f(u)_x - (\mu(u) u_x)_x = 0$

でも f, μ に適当な条件を附し成立する。

Remark 2. これらの結果は一般形(1.4)を含めて最大値原理で示すことが出来る。

Remark 3. (漸近の速さについて). Hopf, Cole の変換による解の陽な表現と熱方程式の基本解の性質より次の decay 評価を得る。

Case 1 のとき. さらに $u_0 - \bar{u} \in L^1$ を仮定すると

$$\| u(t) - \bar{u} \|_{L^2} \le C t^{-1/4}, \quad \| u(t) - \bar{u} \|_{L^\infty} \le C t^{-1/2},$$

であり, 又この decay order は最善である. さらに $\int_{-\infty}^{x} u_0 - \bar{u}\, dy \in L^1 (x \le 0)$, $\int_{x}^{+\infty} u_0 - \bar{u}\, dy \in L^1 (x \ge 0)$ 等を仮定し, 0次モーメントが等しい同様の初期値を持つ他の解 v との差を考えると ($\int_{-\bar{u}} u_0\, dx = \int_{-\bar{u}} v_0\, dx$).

$$\| u(t) - v(t) \|_{L^2} \le C t^{-3/4}, \quad \| u(t) - v(t) \|_{L^\infty} \le C t^{-1}$$

が分る. ここで v を $\int u_0 - \bar{u}\, dx = M_0$ として

$$\begin{cases} v(x,t) = \bar{u} + c((1+t),\ x - \bar{u}(1+t)), \\ c(t,x) = \sqrt{\mu}\, t^{-1/2} \dfrac{(e^{M_0/2\mu} - 1) e^{-y^2}}{\sqrt{\pi} + (e^{M_0/2\mu} - 1)\int_y^{+\infty} e^{-\xi^2} d\xi}, \\ y = x / \sqrt{4\mu t}. \end{cases}$$

と取ることにより見易い形で次の Level の漸近形まで分る。一般形(1.4)の場合の次の Level での漸近形の議論は small data なら Kawashima [3] に含まれるが large data では?

Case 3 のとき. [2] において一般形も含めて

$$|\Phi(x)| \equiv \left|\int_{-\infty}^{x} u_0(y) - U(y-\alpha)\, dx\right| \le Ce^{-\gamma|x|} \quad (\gamma>0)$$

$$\Rightarrow \quad \|u(t) - U\|_{L^\infty} \le Ce^{-\gamma' t} \quad (\gamma'>0),$$

が分っているが. Burgers 方程式では Nishihara [4] においてより詳しく

$$|\Phi(x)| \le C(1+|x|)^{-\gamma} \Rightarrow \|u(t) - U\|_{L^\infty} \le C(1+t)^{-\gamma},$$

であり、又この decay order は最善であることが示めされている。一般形 (1.4) については $|u-U| \le C(1+t)^{-\gamma/2}$ までしか分らない. しかし small data のときは Kawashima & Matsumura [5] において次が示めされている.

$$(1+|x|)^{\beta/2}|\Phi(x)| \in L^2 \Rightarrow \|u-U\|_{H^1} \le Ct^{-[\beta]/2}.$$

Case 2 のとき ? . $\|u(t) - u^R(x,t)\|_{L^\infty} \le Ct^{-1}$ と思われる。

2. 圧縮性粘性流体の方程式系.

次の Barotropic Model に対する初期値問題を考える.

$$(2.1)\quad \begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = \mu\left(\dfrac{u_x}{v}\right)_x, \end{cases}$$

$$(2.2)\quad (v,u)(0,x) = (v_0,u_0)(x),$$

$$\lim_{x\to\pm\infty}(v_0,u_0)(x) = (v_\pm, u_\pm), \quad (v_\pm>0,\ u_\pm\in\mathbb{R}).$$

ここに、 v：比体積、 u：流速、 $\mu>0$（定数）：粘性係数.
$p=p(v)$：圧力 （ $p'(v)<0$, $p''(v)>0$ for $v>0$ ）.

$(v_\pm, u_\pm)=(\bar{v}, \bar{u})$ （ $\bar{v}>0$, $\bar{u}\in\mathbb{R}$ ）の場合.（Case 1 に当る）

Kanel' [6] の仕事が基本的：

$$\varphi(v) \equiv \int_{\bar{v}}^{v} \left(\int_{\bar{v}}^{s} p(\bar{v}) - p(z)\, dz \right)^{1/2} s^{-1}\, ds \to \infty \quad \text{as} \quad v \to \infty, 0.$$

$(v_0-\bar{v}, u_0-\bar{u}) \in L^2$, $(v_0, u_0)_x \in L^2$, $\inf v_0 > 0$

$\Rightarrow$ $\exists 1$ global sol. (v,u) s.t. $\lim_{t\to\infty} \sup_x |(v-\bar{v}, u-\bar{u})(t,x)| = 0$.

Remark 1 $p = a v^{-\gamma}$ （ $a>0$, $\gamma \geq 1$ ）は $\varphi(v)$ に対する仮定を満す.

Remark 2 Full system

$$(2.3) \begin{cases} v_t - u_x = 0, \\ u_t + p_x = \mu (u_x/v)_x, \\ (e + u^2/2)_t + (pu)_x = (\kappa \theta_x / v + \mu u u_x / v)_x, \end{cases}$$

$$(2.4) \quad (v,u,\theta)(0,x) = (v_0, u_0, \theta_0)(x), \quad (v_0, u_0, \theta_0)(\pm\infty) = (\bar{v}, \bar{u}, \bar{\theta}),$$

（ θ：絶対温度, $p=p(v,\theta)$, $e=e(v,\theta)$：内部エネルギー ）.

に対しては

Kazhikhov [7] が

$p = R\theta/v$, $e = R\theta/(\gamma-1)$

$(v_0-\bar{v}, u_0-\bar{u}, \theta_0-\bar{\theta}) \in H^1$ $\Rightarrow$ $\exists 1$ global sol. (v,u,θ).

（ $R>0$：気体定数, $\gamma>1$：比熱比 ）

を示したが $(\bar{v}, \bar{u}, \bar{\theta})$ への漸近は？ しかし Kawashima-Nishida [8]

はγが1に充分近いときに漸近性も含めて示めした。p, e がより一般のときは data small の下で Kawashima-Okada [9] に漸近性も含めた結果がある。

Remark 3　漸近の速さと次のLevelの漸近形への議論は始めに Barotropic モデルに対して Nishida [10]、full system も含む非常に一般の系で Kawashima [11] の議論がある。いずれも次のLevelの漸近形が簡単な Burgers 方程式の解や熱方程式の解を用いて陽に与えられることが示めされた。ただし small data に対してである。

$(v_+, u_+) \neq (v_-, u_-)$ の場合

Burgers 方程式のときと同様、次の(2.1)に対応する非粘性方程式系に対する Riemann 問題の解に応じて幾つかの場合にさらに分れる。

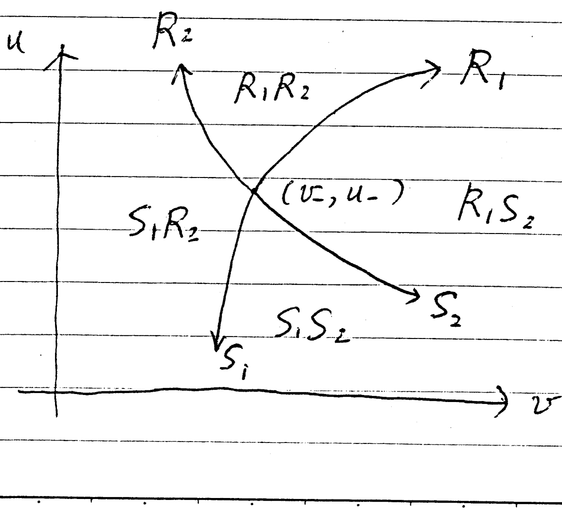
$$(2.5)\quad \begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = 0, \end{cases}$$

$$(2.6)\quad (v,u)(0,x) = (v,u)_0^R(x) \equiv (v_\pm, u_\pm) \quad (x \gtrless 0)$$

まず(2.5)は $v>0$ で特性根

$$\lambda_1(v) = -(-p'(v))^{1/2}, \quad \lambda_2(v) = (-p'(v))^{1/2},$$

を持つ強双曲型である。(v_-, u_-) が与えられたとき (v,u) 平面の (v_-, u_-) の近傍を8つの領域 $R_i(v_-,u_-)$ $(i=1,2)$, $S_i(v_-,u_-)$ $(i=1,2)$, $R_1R_2(v_-,u_-)$, $S_1S_2(v_-,u_-)$, $R_1S_2(v_-,u_-)$, $S_1R_2(v_-,u_-)$ に分ける。



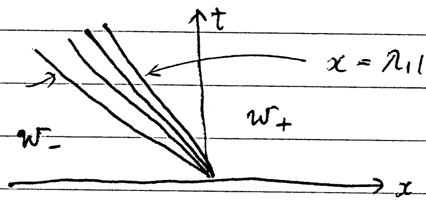
ここに

$$R_i(v_-,u_-)=\left\{(v,u);\ u=u_- - \int_{v_-}^{v}\lambda_i(s)\,ds,\quad \lambda_i(v)\ge\lambda_i(v_-)\right\},$$

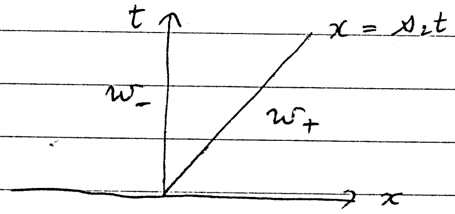
$$S_i(v_-,u_-)=\left\{(v,u);\ \exists s\ \text{s.t.}\ \begin{pmatrix} s(v-v_-)=u_- - u\\ s(u-u_-)=p(v)-p(v_-)\end{pmatrix},\quad \lambda_i(v)\le\lambda_i(v_-)\right\}$$

以下簡単の為 (v,u) をまとめて w と書く。 $w_+\in R_i(w_-)$ のときの Riemann 問題の解を $w_i^R(x/t\,;w_-,w_+)$、 $w_+\in S_i(w_-)$ のときの解を $w_i^S(x-s_it)$ とする。

$w_1^R(x/t\,;w_-,w_+)$



$w_2^S(x-s_2t)$



いま $w_+\in R_1R_2(w_-)$ のときは $\exists\bar{w}\in R_1(w_-)$ s.t. $w_+\in R_2(\bar{w})$ により、Riemann prob. の解は

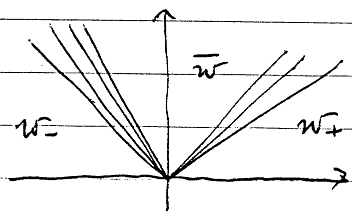
$$(2.7)\qquad w^R(x/t)=w_1^R(x/t\,;w_-,\bar{w})+w_2^R(x/t\,;\bar{w},w_+)-\bar{w}$$

与えられる。他の場合も同様で例えば $w_+\in S_1S_2(w_-)$ なら

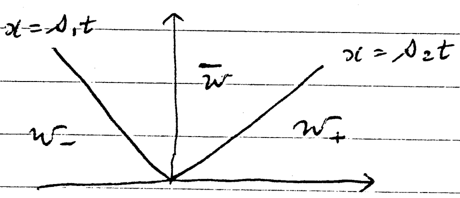
$$(2.8)\qquad w^S(t,x)=w_1^S(x-s_1t\,;w_-,\bar{w})+w_2^S(x-s_2t\,;\bar{w},w_+)-\bar{w}$$

で与えられる。

$w^R(x/t)$



$w^S(t,x)$



$w_+ \in (R_1 \cup R_2 \cup R_1R_2)(w_-)$ の場合 (Case 2 に当る)

$w_0 - w_0^R \in L^2$, $w_{0,x} \in L^2$ を仮定し

$\Phi_0^2 = \|w_0 - w_0^R\|^2 + \|w_{0,x}\|^2 + |w_+ - w_-|^2$ と置く。

次の基本定理は Matsumura-Nishihara [12] による。

Th. 1 For each w_-, $\exists \varepsilon_0 > 0$ s.t. if $w_+ \in (R_1 \cup R_2 \cup R_1R_2)(w_-)$ and $\Phi_0 < \varepsilon_0$, then $\exists^1$ global sol. w of (2.1), (2.2) s.t.

$$\lim_{t\to\infty} \sup_x |w(t,x) - w^R(x/t)| = 0.$$

Remark. 1 $p = av^{-\gamma}$ のときは Kawashima-Matsumura-Nishihara [13] の議論により $|w_+ - w_-|$ small だけで Th. 1 が成立する。
$|w_+ - w_-|$ が大のとき？

Remark 2. [13] ではさらに full system (2.3) に対しても Th. 1 に対応する結果が示され、とくに $p = R\theta/v$, $e = R\theta/(\gamma-1)$ の時には $|w_+ - w_-|$ と $|\gamma - 1|$ small だけで示された。

$w_+ \in (S_1 \cup S_2 \cup S_1S_2)(w_-)$ の場合 (Case 3 に当る)

$w_0 - w_0^R \in L^2 \cap L^1$, $(w_{0,x} \in L^2$ を仮定する。

衝撃波解 $w_i^S(x - s_i t; w_-, w_+)$ に対応する (2.1) の衝撃波形進行波解を $W_i(x - s_i t - d_i; w_-, w_+)$ $(d_i \in R)$ とする。このとき (2.8) から漸近形は

$$W(x,t) = W_1(x - s_1 t - d_1; w_-, \bar{w}) + W_2(x - s_2 t - d_2; \bar{w}, w_+) - \bar{w}$$

と予想される。いま $w_+ \in S_1S_2$ であれば常に $\exists^1 (d_1, d_2)$ s.t.

$$(2.9)\quad \int_{-\infty}^{+\infty} w_0(x) - W(x,0)\, dx = 0$$

と出来る。このとき

$$(2.10)\quad \phi_0(x) \equiv \int_{-\infty}^{x} w_0(y) - W(0,y)\, dy \in L^2$$

を仮定する。 $w_+ \in S_i(w_-)$ $(i=1 \text{ or } 2)$ のときは常に(2.9)と出来ない為

$$(2.9)'\quad \int_{+\infty}^{+\infty} w_0(x) - W(x,0)\, dx = 0 \quad \text{for } \exists \alpha_i \in \mathbb{R}^1$$

を仮定する。このときも同様に(2.10)を仮定する。このとき次の定理が Matsumura-Nishihara [14], Liu [15] の議論から従う。

Th. 2. For each w_-, $\exists \varepsilon_0 > 0$ s.t. if $w_+ \in (S_1 \cup S_2 \cup S_1S_2)(w_-)$, (2.9), (2.10) and $|w_+ - w_-| + \|\phi_0\|_{H^2} < \varepsilon_0$, then $\exists 1$ global sol. w of (2.1), (2.2) s.t.

$$\lim_{t\to\infty} \sup_x |w(t,x) - W(t,x)| = 0.$$

Remark 1. $p = av^{-\gamma}$ のとき。 $w_+ \in S_1$ or S_2 であれば [14] の議論により $(\gamma-1)|w_+ - w_-| + \|\phi_0\|_2 < \varepsilon_0$ で Th. 成立。これは特に $\gamma = 1$ であれば shock の大きさがどんなに大でもよい事になる。 $w_+ \in S_1S_2$ のときは $(\gamma-1)|w_+ - w_-| + |w_+ - \bar{w}||\bar{w} - w_-| + \|\phi_0\|_2 < \varepsilon_0$ で成立。 ϕ_0 が大きいとき?

Remark 2. (2.9)の下では、Kawashima-Matsumura [5] により、full system の場合にも Th. 成立。 $p = R\theta/v$, $e = R\theta/(\gamma-1)$ のときは Remark 1 に対応する結果成立。

Remark 3. $w_+ \in S_1$ or S_2 であり、$\int w_0 - W\, dx = 0$ と出来ない場合も full system も含めて Liu [16] が Th. を示した。
例えば $w_+ \in S_2(w_-)$ のとき shift α_1 は

$$\int w_0 - W(x-\alpha_2)\,dx = \alpha_2(w_+ - \bar{w}) + \int w_0 - W(x)\,dx$$
$$= \beta_1 r_1(w_-)$$

として決る（[15]）。ここに $r_1(w_-)$ は $\lambda_1(\bar{w}_-)$ に対する右固有ベクトルである。

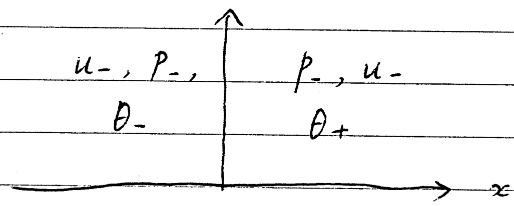
最後に

i) $w_+ \in R_1S_2(w_-)$ （S_1R_2 も同様）のとき漸近形は

$$W = w_1^R(x/t\,;\,w_-, \bar{w}) + W_2(x - s_2 t - \alpha_2, \bar{w}, w_+) - \bar{w}$$

と予想されるが、？

ii) full system (2.3) のときは特性根 0 があるため、Riemann 問題の基本的な解として接触不連続解



がある。この場合に対応する漸近挙動？

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SÉMINAIRE ÉQUATIONS AUX DÉRIVÉES PARTIELLES 1985 - 1986

NONLINEAR EFFECTIVELY HYPERBOLIC EQUATIONS

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Exposé n°XII

25 Février 1986

§ 0. INTRODUCTION :

We shall give some notes about the effective hyperbolicity. We think there are two ways to explain a notion. The first one is to find out an equivalent notion from a different point view. And the other one is to give some interesting examples of applications of the notion. For the first one, we have a proposition for single partial differential operators that to be effectively hyperbolic is equivalent to be strongly hyperbolic. The strong hyperbolicity means the stability of solvability (well posedness) of the Cauchy problem under changing lower terms. The effective hyperbolicity defined later is a geometrical notion on the principal symbols of partial differential operators. The sufficient part of the above proposition, that is, the Cauchy problem is well posed if it is effectively hyperbolic, is very important for applications and does not always require that partial differential operators are single and even linear. So, we may replace the effective hyperbolicity for the strict hyperbolicity used usually and frequently because the previous notion is wider than the later.

Many partial differential operators in applications are non linear. So, we have to extend the results in the linear cases to ones in the non linear cases in order to find out interesting examples. In the present stage, this extension is very easy because we know already a very famous abstract theorem, so called the Nash-Moser implicit function theorem.
Here, we explain that this theorem is also applicable to our cases. We shall remark we needed some minor change of expression of the Nash-Moser implicit function theorem in order to obtain a sharper result for the non linear Cauchy problem making it possible to apply directly to the Monge-Ampère equation.

§ I. RESULTS OF LINEAR CASES :

Let P_m be a polynomial of homogeneous order m in $\xi \in \mathbb{R}^{n+1}$ with coefficients C^∞-functions in $x \in \mathbb{R}^{n+1}$. We assume it is normal in ξ_o,

$$P_m = \xi_o^m + \dots .$$

Définition 1. We call P_m effectively hyperbolic (on an open set in x) if P_m is hyperbolic in ξ_o and if at the critical points of the characteristics

$\{P_m = \nabla P_m = 0\}$, the fundamental matrices F of P_m have non zero real eigenvalues, where the fundamental matrix F is defined by

$$F = \begin{pmatrix} P_{m_{\xi x}} , & P_{m_{\xi\xi}} \\ -P_{m_{xx}} , & -P_{m_{x\xi}} \end{pmatrix} , \quad P_{m_{\xi x}} = \partial_\xi \, \partial_x \, P_m \quad , \text{ etc...}$$

Let us consider a system of partial differential operator P of order m with a diagonal principal part, namely,

$$P = P_m I + Q \, , \tag{1.1}$$

where the lower term $Q = (q_{ij})$ is a system such that order $q_{ij} \leqslant \alpha_i - \alpha_j + m-1$ with respect to a multi-index α of integers.

<u>Theorem</u> 1 . Let P_m in (1.1) be effectively hyperbolic on a neighborhood of the origin. Then, there exist a conic domain $\Omega = \{x; x_o + \lambda|x'| < 0 \, , \, \lambda > 0\}$ and a constant $\varepsilon_o > 0$ such that for all ε $(0 \leqslant \varepsilon < \varepsilon_o)$, the Cauchy problem

$$\begin{cases} Pu = f & \text{on } \{x > 0\} \cap \Omega_\varepsilon \\ u = 0 & \text{on } \{x_o \leqslant 0\} \cap \Omega_\varepsilon \end{cases} \tag{1.2}$$

has a unique C^∞-solution u on Ω_ε for any C^∞- datum f on Ω_ε supported on $\{x_o \geqslant 0\}$, where $\Omega_\varepsilon = \Omega + (\varepsilon, 0, \ldots, 0)$. Moreover, for some suitably fixed ℓ , they satisfy the estimate for all $s \geqslant 0$

$$\| u \|_s \leqq C_s (\| f \|_{s+\ell} + \| a \|_{s+\ell} \| f \|_\ell) \, , \tag{1.3}$$

where $\| . \|_s$ is the Sobolev norms on Ω_ε and a stands for coefficients of the partial differential operator P . We should remark that the constants C_s are uniform on P belonging to a suitable neighborhood of a fixed effectively hyperbolic operator in the space of hyperbolic operators with the type (1.1) on a neighborhood of the origin.

When we usually call the Cauchy problem (1.2) well posed, we do not require the estimate (1.3), especially, the existence of the constant ℓ independent of s . However, almost all cases which we know wellposed, have the type (1.3) of estimates, and this is important to use the Nash-Moser implicit function theorem. So, we introduce a notation for convenience.

Let us consider the problem (1.2) for a set $\mathcal{P}$ of general system P of partial differential operators.

Definition 2. We call a set $\mathcal{P}$ strongly wellposed if the conclusion of Theorem 1 holds for any element P of $\mathcal{P}$, especially, if ε_o, Ω and constants C_s, ℓ of the estimate (1.3) are common on $\mathcal{P}$.

Remark. Let P be a $r \times r$ system of partial differential operators. Define the principal symbol P_{pr} as usual. Assume that det P_{pr} is effectively hyperbolic. This is reduced to Theorem 1 so that it is strongly wellposed.

§ II. NONLINEAR CASES :

We consider a nonlinear system ;

$$(2.1) \qquad \begin{aligned} Pu &= (P_i)_{i=1,\dots,r} \\ P_i &= P_i(x, \eta_{j\alpha}; |\alpha| \leqslant \bar{\alpha}_i - \bar{\beta}_j, \ j = 1,\dots,r)\Big|_{\eta_{j\alpha} = \partial^\alpha u_j} \\ &= P_i(x, \partial^\alpha u), \\ u &= (u_1, \dots, u_r), \end{aligned}$$

where $\bar{\alpha}$ and $\bar{\beta}$ are multi-indices of integers.
The linearization DP of this system is given by

$$DP\ \varphi = \Big(\sum_{j,\beta} \frac{\partial}{\partial \eta_{j\beta}} P_i(x, \partial^\alpha u) \partial^\beta \varphi_j \Big)_{i=1,\dots,r}$$

and the principal symbol $P_{pr} = (P_{ij})$

$$P_{ij} = \sum_{\beta; |\beta| = \bar{\alpha}_i - \bar{\beta}_j} \frac{\partial}{\partial \eta_{j\beta}} P_i(x, \partial^\alpha u) \xi^\beta .$$

Now we extend the strong well posedness to this type of system.

Definition 3. We call a nonlinear system P (2.1) strongly wellposed if there exist two linear system Q and R satisfying the following conditions ;

1) Coefficients of Q and R are also functions of (x, η_β) and $(x, \eta_\beta, \zeta_\beta)$, respectively, where we put the unknown function $\partial^\beta u$ in η_β and the parameter function $\partial^\beta h$ in ζ_β.

2) The linearization DP are decomposed as

$$DP = Q + R \quad \text{with } h = Pu .$$

3) There exists a neighborhood U of $u = 0$ in $C^\infty \cap \{u;\ u=0 \text{ at } x_o < 0\}$

such that $\{Q;\ u \in U\}$ is strongly wellposed in the sense of Definition 2.

4) $R \equiv 0$ at $h \equiv 0$.

We shall later explain by an example the reason why we do not define simply as {DP} are strongly well posed. Roughly speaking, we assume the existence of a parametrix of DP toward $Pu = 0$. Under this assumption we can conclude the following unique extension theorem of a solution.

Theorem 2. Let a nonlinear operator P(2.1) be strongly wellposed. If there exists a neighborhood Ω_o of the origin such that $u = 0$ is a solution of $Pu = 0$ at $\{x_o < 0\} \cap \Omega_o$, then there exists uniquely a C^∞ solution u of $Pu = 0$ on a neighborhood Ω of the origin such that $u = 0$ on $\{x_o < 0\} \cap \Omega$.

Corollary 3 . We assume that the principal symbol P_{pr} of P (2.1) are decomposable as well as Definition 3 such that

$$P_{pr} = Q_o + R_o \quad \text{with} \quad h = Pu \quad ,$$

where Q_o and R_o satisfy the conditions for Q and R in 1) and 4) of Definition 3 , respectively, and where Q_o is one of the type of effectively hyperbolic operators treated in Theorem 1 or uniformly reducible to one of this type for example, det Q_o is effectively hyperbolic, for all u of U in 3).
Then, the conclusion of Theorem 2 holds.
This Theorem 2 is no more than a translation from the abstract Nash-Moser's theorem into the category of the Cauchy problem. However, it requires the improvement of expression of the Nash-Moser's theorem, because we assume a weaker condition than as usual, namely, we assume the existence of parametrices of the linearization DP instead of the existence of exact inverses.

We follow the expression of the Nash-Moser's theorem by L. Hörmander [1] . $\{\mathbb{H}^s\}_{s\geq 0}$ is a Banach scale, in other words, interpolation spaces by means of a smoothing operator.

Let $\Phi(u)$ be a nonlinear operator on $\mathbb{H}^\infty = \bigcap_s \mathbb{H}^s$. Assume the existence of the first and second derivatives in u and their estimates as similar as (2.2). The different point is to assume that the right (left, resp.) parametrix ψ of the first derivative $D\Phi$ exists on a neighborhood of the origin of (u,Φ) such that

$$D\Phi.\psi(\psi D\Phi, \text{ resp.}) = I + \Theta \quad ,$$

where Θ depends on u and Φ , and satisfies

(2.2) $$\|\Theta\varphi\|_s \leqslant C_s[(1+\|u\|_s)\|\Phi\|_m\|\varphi\|_m + (1+\|u\|_m)(\|\Phi\|_{s+m}\|\varphi\|_m + \|\Phi\|_m\|\varphi\|_{s+m})]$$

for a fixed m and for all s .

Then, there exist neighborhood V , W of $u = 0$ and $\Phi = 0$ such that

$$\Phi(V) \cap W = \emptyset \quad \text{or} \quad \Phi(V) \ni 0$$
$$(\Phi(0) = 0 \Rightarrow \Phi^{-1}(0) \cap V = \{0\}, \text{ resp.}) .$$

Therefore we can conclude the existence and the uniqueness only for $\Phi(u) = 0$. In the proof, we need only an addition of the error terms in the argument of the existence of Θ , which is estimated as well as other error terms are. If considering the scale basing on the Sobolev spaces or the Hölder spaces of functions on Ω_ε supported on $\{x_0 \geqslant 0\}$, then the strong wellposedness assures the existence of the right and left parametrix ψ . The non essential cases will be excluded since the norm $\|\Phi(0)\|_s$ on Ω_ε tends to 0 as ε tends to 0 .

§ III. AN EXAMPLE :

The Monge-Ampère equation

$$u_{xx}u_{yy} - (u_{xy})^2 = f(x,y)$$

is elliptic if $f > 0$, strictly hyperbolic if $f < 0$ and of Tricomi type if $\nabla f \neq 0$ at the points $f = 0$ and if it is kowalevskian. They are very classical and well known. So we treat more singular cases. Since we are treating the Cauchy problems, we consider the case where $f \leqslant 0$. In this case, the equation is hyperbolic if it is kowalevskian. More generally, we consider

$$\Phi = \det A - f = 0 ,$$

where $A = \partial^2 u + C(x,u,\partial u)$, C is a symmetric matrix and f is also a function in $(x,u,\partial u)$. A typical example is the Gauss curvature $K(x)$ of a hypersurface $\{y = u(x)\}$, $x \in \mathbb{R}^n$

$$\det(\partial^2 u) = K(x)\ (1+|\partial u|^2)^{(n+2)/2} .$$

Now, we prepare some notation to state the result.

w.r.t. $A_{ij} = (a_{k\ell})_{k,\ell\neq i,j}$ = minor matrix w.r.t. a_{ij} .

$A^{co} = (a^{co}_{ij})$ $a^{co}_{ij} = (-1)^{i+j} \det A_{ji}$: cofactor matrix .

The equation is

$$(3.1) \qquad \begin{cases} \Phi = \det A - f = 0 \\ \text{the initial condition at } x_o = 0 . \end{cases}$$

If we take the Fréchet derivative in u , then

$$D\Phi\varphi = Tr(A^{co}\partial^2\varphi) + \text{lower term}$$
$$= A^{co}(\partial)\,\varphi + \dots \quad .$$

We assume the following (3.2-4).

(3.2) $\qquad f \leqslant 0 \quad$ always.

(3.3) $\qquad A_{oo}(u)\big|_{x_o = 0} > 0$.

(3.4) $\quad L^2 f'(x,u,\partial u) < 0$ at $\{f = 0\} \cap \{x_o = 0\}$, where L is a vector field defined by

$$L = \sum_{j=o}^{n} a^{c}_{o_j}{}^{o}\left(\frac{\partial}{\partial x_j}\right) .$$

Theorem 4. Let $\tilde{u}$ be a formal solution of (3.1) at $x_o = 0$ and satisfy (3.2-4) on a neighborhood of x = 0 . Then there exists a unique C^∞ solution u of $\Phi(u) = 0$ on a neighborhood of x = 0 such that $u - \tilde{u}$ is flat at $x_o = 0$ $(\partial^\alpha(u-\tilde{u})\big|_{x_o=o} = 0, \forall\alpha\ )$.

Example. $u = \frac{1}{2}x^2 - \frac{1}{12}y^4$.

$$\Phi(u) = u_{xx}u_{yy} - (u_{xy})^2 + y^2 = 0 ,$$
$$D\Phi(u)\varphi = \partial_y^2\varphi - y^2\partial_x^2\varphi ,$$
$$f = -y^2 \text{ and } L = \frac{\partial}{\partial y} .$$

In fact, let us put eigenvalues and eigenprojections of A^{co} by θ_j and p_j (i = 0,...,n), and λ_j are eigenvalues of A. Then $\theta_j = \pi_{k\neq j}\lambda_k$ and eigen spaces of θ_j and λ_j are the same, so their projection is p_j . The assumption (3.3) gives us the existence of n positive eigenvalues, so we denote them by $\lambda_j > 0$ (j = 1,...,n).

Since $\det A = \lambda_o \dots \lambda_n = f \leqslant 0$ at $u = \tilde{u}$ and $x_o = 0$, so $\lambda_o \leqslant 0$, there. Hence λ_o his near non positive axis if u is near $\tilde{u}$, that is, remainding eigenvalue λ_o is separated from the others λ_j $(j \geqslant 1)$.

Therefore $\theta_o = \lambda_1 \dots \lambda_n$ and P_o are defined smoothly. So A^{co} is written as

$$A^{co} = \Sigma_{j=o}^{n} \theta_j P_j = \lambda_1 \dots \lambda_n P_o + \dots + \lambda_o \dots \lambda_{n-1} P_n$$

$$= \theta_o P_o + \lambda_o \dots \lambda_n (\Sigma_{j=1}^{n} \lambda_j^{-1} P_j)$$

$$= \theta_o P_o + (\det A)\, E$$

$$= \theta_o P_o + f\, E + \Phi\, E \ , \ E = \Sigma_{j=1}^{n} \lambda_j^{-1} P_j \ .$$

Then we can decompose

$$A^{co}(\partial) = Q_o(\partial) + R_o(\partial)$$

as

$$Q_o(\partial) = \theta_o P_o(\partial) + f\, E(\partial)$$

and

$$R_o(\partial) = \Phi . E(\partial).$$

Here $Q_o(\partial) \sim L^2 + f$. (an elliptic operator with respect to the transversal directions to L).
Since $f \leqslant 0$, and $L^2 f \neq 0$ at $f = 0$, so Q_o is effectively hyperbolic. And also $R_o(\partial) = 0$ at $\Phi = 0$. Therefore $A^{co}(\partial)$ satisfies the conditions of Corollary 3. Using some speciality of the Monge-Ampère equation, we conclude the existence theorem from the unique extension theorem.

Other examples of effectively hyperbolic operators will be found in N. Iwasaki [2] , where you can see the further informations about this note.

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On a Certain Mixed Problem for the Equations of Ideal Magneto-Hydrodynamics

Taku Yanagisawa

§ 1 Introduction and Results.

We consider the following mixed problem for the equations of ideal magneto-hydrodynamics:

$$(1.1)\quad \begin{cases} \text{(a)}\ \rho_p(\partial_t + (u\cdot\nabla))p + \rho\ \mathrm{div}\ u = 0 \\ \text{(b)}\ \rho(\partial_t + (u\cdot\nabla))u + \nabla p + \mu\ H\times \mathrm{curl}\ H = 0 \\ \text{(c)}\ \partial_t H - \mathrm{curl}\ (u\times H) = 0 \qquad \text{in } [0,T]\times\Omega, \end{cases}$$

$$(1.2)\quad (p,\ u,\ H)|_{t=0} = (p_0,\ u_0,\ H_0) \quad \text{in } \Omega,$$

$$(1.3)\quad u\cdot n = 0,\ H\times n = g \quad \text{on } [0,T]\times\Gamma.$$

Here Ω is a bounded domain in R^3 with C^∞ boundary Γ and $n(x) = {}^t(n_1,\ n_2,\ n_3)$ denotes the unit outward normal at $x\in\Gamma$. Pressure $p(t,x)$, velocity $u(t,x) = {}^t(u_1,\ u_2,\ u_3)$ and the magnetic field $H(t,x) = {}^t(H_1,\ H_2,\ H_3)$ are unknowns. μ is the permeability, which we assume to be constant. We suppose that density $\rho>0$ is a smooth known function of $p>0$ i.e. $\rho = \rho(\ p\ )$ and $\rho_p = \partial\rho/\partial p$. $g(t,x)={}^t(g_1,\ g_2,\ g_3)$ is a given function on $[0,T]\times\Gamma$. $\partial_t=\partial/\partial t, \partial_i=\partial/\partial x_i\ (i=1,2,$

∂_t), $\nabla=(\partial_1, \partial_2, \partial_3)$, $(u\cdot\nabla)=\sum_{i=1}^{3} u_i\cdot\partial_i$ and $\cdot$, $\times$ denote scalar and vector product, respectively. T is a positive constant.

We assume that the initial data p_0 and H_0 satisfy

(1.4) $\inf_{x\in\Omega}\{ \rho(p_0), \rho_p(p_0) \} \geqq c_1,$

(1.5) $\operatorname{div} H_0 = 0 \quad \text{in } \Omega.$

These assumptions guarantee the equations (1.1) to be a quasi-linear symmetric hyperbolic system. We further assume that the normal component of H_0 on Γ satisfy

(1.6) $\inf_{x\in\Gamma} | H_0\cdot n | \geqq c_2.$

Here c_1 and c_2 are positive constants.

Our aim is to show a short-time existence theorem for the mixed problem (1.1)-(1.3):

Theorem. For an integer $m \geqq 3$, assume that $g\in\dot{Y}_m(T)$ and the initial data $(p_0, u_0, H_0)\in H^m(\Omega)$ satisfy the assumptions (1.4), (1.5), (1.6) and the compatibility conditions of order m.

Then there exists a positive constant T_0 depending only on $\|(p_0, u_0, H_0)\|_m$, $\|g\|_{Y_m(T)}$, c_1, c_2, m, Ω such that the mixed problem (1.1), (1.2) and (1.3) has a unique solution which belongs to $X_m(T_0)$.

As to the definitions of the compatibility conditions and the function spaces $X_m(T)$ and $\dot{Y}_m(T)$, see (2.3), (2.4), (2.5) and (2.6) in §2.

We remark that this result can be extended to the non-isentropic case without essential modifications of the proof.

The mixed problem for the compressible Euler equations under the solid-wall boundary condition $u \cdot n=0$ (i.e. (1.1)-(1.3) with $H=0$) is a typical characteristic mixed problem for quasilinear symmetric hyperbolic systems and has been studied by many authors.

However if one considers the effect of the magnetic field, there seems no literature studying well-posedness of the mixed problem for the equations (1.1). This paper intends, from mathematical point of view, to seek some conditions which ensure that this mixed problem will be well-posed. As the boundary conditions for H, we take $H \times n=g$ so that the solution is unique.

The bound of $H_0 \cdot n$ (1.6) is needed to guarantee the rank of the boundary matrix to be 6 on the boundary. We remark here that the assumptions (1.5) and (1.6) force Ω^c to consist of at least two components.

The proof of Theorem proceeds via iteration scheme which involves the following steps. At first, according to [10], we modify the equations (1.1) to make the boundary noncharacteristic. We next establish the uniform estimates of the solution of the modified equations under the same initial boundary conditions (1.2) and (1.3). These uniform estimates are achieved by making use of a special structure of the modified equations and the rank of the boundary matrix being 6 on the boundary. Finally by taking a limit of these solutions, we get the solution of the original mixed problem (1.1)-(1.3).

§2. Definition of the function spaces and the compatibility conditions

(2.3) $$X_m(T) = \bigcap_{j=0}^{m} C^j(0,T;\ H^{m-j}(\Omega)),$$

(2.4) $$Y_m(T) = \bigcap_{j=0}^{m} C^j(0,T;\ H^{m-j+1/2}(\Gamma)) \cap C^{m+1}(0,T;\ H^{1/2}(\Gamma)),$$

with norms

$$\|u\|_{X_m(T)} = \sup_{t\in[0,T]} |||u(t)|||_m = \sup_{t\in[0,T]} \left(\sum_{j=0}^{m} \|\partial_t^j u(t)\|_{m-j}\right),$$

$$\|g\|_{Y^m(T)} = \sup_{t\in[0,T]} \left(\sum_{j=0}^{m} \|\partial_t^j g(t)\|_{H^{m-j+1/2}(\Gamma)} + \|\partial_t^{m+1} g(t)\|_{H^{1/2}(\Gamma)}\right),$$

where $u(t)$ stands for $u(t,x)$ freezing t. Moreover we define

(2.5) $$\dot{Y}_m(T) = \{g \in Y_m(T);\ g\cdot n = 0 \text{ on } [0,T]\times\Gamma\}.$$

We say that the initial data (ρ_0, u_0, H_0) satisfy the compatibility conditions of order m for the equations (1.1) and the boundary conditions (1.3), if

(2.6) $$\partial_t^k u(0)\cdot n = 0, \quad \partial_t^k H(0)\times n = \partial_t^k g(0) \quad \text{on } \Gamma,$$

for $k=0,1,\cdots,m-1$.

Here the terms $\partial_t^k u(0)$, $\partial_t^k H(0)$ are calculated from (1.1) and (1.2), and are then expressed by initial data and their derivatives. For instance,

$$\partial_t u(0) = -(u_0\cdot\nabla)u_0 - \rho(p_0)^{-1}(\nabla p_0 + \mu H_0 \times \mathrm{curl}\ H_0),$$

$$\partial_t H_0 = \mathrm{curl}\ (u_0 \times H_0).$$

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Yang-Mills gradient flows について

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Yang-Mills gradient flows とは，次のようなコンパクト・リーマン多様体(向付可)上のコーシー問題のことである。

$$(\star)\begin{cases}\partial_t A = -d_A^* F_A & \text{in } [0,\infty)\times M \\ A(0) = A_0 \in \mathcal{C}(P)\end{cases}$$

ここで，P は M 上の主束(その構造群は G とする)，$\mathcal{C}(P)$ は P 上の接続全体の集合とする。未知函数 A は，$\mathcal{C}(P)$ に値をもつ t の 1-パラメータ・ファミリーである。F_A は A の曲率形式で，d_A^* は A の共変外微分 d_A の共役作用素である。局所座標を用いると

$$A = A_j\,dx^j \qquad (A_j \in \mathfrak{g}\ ;\ \mathfrak{g}\ \text{は}\ G\ \text{のリー環})$$

$$F_A = \frac{1}{2}F_{ij}\,dx^i\wedge dx^j, \qquad F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$$

$$-d_A^* F_A = F_{ij\,;\,i}^{\ A}\,dx^j, \qquad (\nabla_{\partial_i}^A F_A)_{jk} = F_{jk\,;\,i}^{\ A}$$

となるが，記号に関する詳細なことは，伊藤[1]を参考にしていただきたい。

<u>定理</u> コーシー問題(☆)は，すべての $A_0 \in \mathcal{C}(P)$ に対して解ける。

2つの接続 $A_1, A_2 \in \mathcal{C}(P)$ に対して，差 $A_1 - A_2$ はベクトル空間 $\Omega^1(M, \mathfrak{g}_P)$ の元（すなわち，P の無限小ゲージ変換全体を表現することができるベクトル束 $\mathfrak{g}_P$ に値をもつ1次微分形式のことである）であるから，$\partial_t A \in \Omega^1(M; \mathfrak{g}_P)$．また，(☆)の右辺の $d_A^* F_A$ は，$\mathcal{C}(P)$ 上の Yang-Mills 汎函数 $\mathcal{YM}$ の A における第1変分で，$F_A \in \Omega^2(M; \mathfrak{g}_P)$ から，$-d_A^* F_A \in \Omega^1(M; \mathfrak{g}_P)$．故に，方程式(☆)はナンセンスではない。

(☆)において，初期値 A_0 を用いて，未知函数を $B = A - A_0 \in \Omega^1(M; \mathfrak{g}_P)$ に変えると，

$$\begin{cases} \partial_t B = -d_A^* d_{A_0} B - d_A^* F_{A_0} - \frac{1}{2} d_A^* [B \wedge B], \\ B(0) = 0 \end{cases}$$

となるが，右辺の主部 $-d^* dB$ は楕円型微分作用素ではない．しかし，P のゲージ変換の1パラメータ・ファミリー $\chi = \chi(t)$ を用いて，$C = \chi^* A$ とすると

$$\partial_t C = \chi^* \partial_t A + d_C(\chi^{-1} \partial_t \chi)$$

であるから，$\chi^{-1} \partial_t \chi = -d_C^* B$ とすればよいことに気が付く。

<u>定理の証明の概略</u>　まず次のような初期値問題を考える。

$$(\star\star)\begin{cases}\partial_t B - \Delta_{A_0} B = -d_{A_0}^* F_{A_0} - V_{A_0}(B), \\ B(0)=0\end{cases}$$

ここで，$\Delta_{A_0} = -d_{A_0}^* d_{A_0} - d_{A_0} d_{A_0}^*$ は，接続 A_0 及び M のリーマン計量に関するラプラス・ベルトラミ作用素で，M上の対角型強楕円型微分作用素になっている。また，

$$V_{A_0}(B) = B^* F_{A_0} + [B \wedge d_{A_0}^* B] + B^* d_{A_0} B + \frac{1}{2} d_{A_0}^* [B\wedge B] + \frac{1}{2} B^* [B \wedge B]$$

であり，$B^*\omega$ は，$d_{A_0+B}^*\omega = d_{A_0}^*\omega + B^*\omega$ が成り立つような，Bとωの2次形式である。
$(\star\star)$の解Bが見つかったとき，

$$\begin{cases}\chi^{-1}\partial_t \chi = -d_{A_0}^* B, \\ \chi(0) = id\end{cases}$$

となるゲージ変換の1-パラメータ・ファミリー χ を用いて

$$\chi^* A = A_0 + B$$

となるような接続の1-パラメータ・ファミリー A が$(\star)$の解である。

次に，方程式(☆☆)を あるバナッハ空間上の縮小写像に帰着して解こう。(Kato [2] を参照)

$\mathfrak{g}_P$ に値をもつ1次微分形式のベクトル束 $\Lambda^1(M;\mathfrak{g}_P)$ 上の自然な内積を用いて，ソボレフ空間

$$\mathbb{W}^\ell(M;\mathfrak{g}_P) \qquad (\ell \text{ は整数})$$

が定義される。そこで，$T>0$ に対して，

$$\begin{cases} \mathcal{B}^\ell_T(M;\mathfrak{g}_P) \equiv C([0,T];\mathbb{W}^\ell(M;\mathfrak{g}_P)), \\ \|B\|_{\mathcal{B}^\ell_T(M;\mathfrak{g}_P)} \equiv \sup_{0\leq t\leq T} \|B(t)\|_{\mathbb{W}^\ell(M;\mathfrak{g}_P)}, \quad B\in\mathcal{B}^\ell_T(M;\mathfrak{g}_P) \end{cases}$$

とおく。n を M の次元とし，$\ell \geq n+2$ とすれば，$B\in\mathcal{B}^\ell_T(M;\mathfrak{g}_P)$ に対して，$V_{A_0}(B)\in\mathcal{B}^{\ell-1}_T(M;\mathfrak{g}_P)$ であるが，

$$\int_0^t e^{(t-s)\Delta_{A_0}} V_{A_0}(B(s))\,ds \in \mathcal{B}^\ell_T(M;\mathfrak{g}_P)$$

となる。そこで，

$$(\Phi(B))(t) \equiv -\int_0^t e^{(t-s)\Delta_{A_0}}\left(d^*_{A_0}F_{A_0}+V_{A_0}(B(s))\right)ds$$

とおくと，

$$\Phi : \mathcal{B}^\ell_T(M;\mathfrak{g}_P) \longrightarrow \mathcal{B}^\ell_T(M;\mathfrak{g}_P)$$

となって，方程式(☆☆)は

$$B = \Phi(B)$$

と同等になる。$\exp t\Delta_{A_0}$ の性質から，ある定数 C_1, C_2, C_3 があって，

$$\begin{cases} \|\Phi(B)\| \leq T\|d_{A_0}^* F_{A_0}\|_{W^\ell(M;\mathfrak{g}_P)} \\ \qquad\qquad + \sqrt{T}\left(C_1\|B\| + C_2\|B\|^2 + C_3\|B\|^3\right), \\ \\ \|\Phi(B_1) - \Phi(B_2)\| \leq \sqrt{T}\left\{C_1 + C_2(\|B_1\| + \|B_2\|) \right. \\ \qquad\qquad \left. + C_3(\|B_1\| + \|B_2\|)^2\right\}\|B_1 - B_2\| \end{cases}$$

が，B, B_1, $B_2 \in \mathcal{B}_T^\ell(M;\mathfrak{g}_P)$ に対して成り立つ．ただし，$\|\cdot\| = \|\cdot\|_{\mathcal{B}_T^\ell(M;\mathfrak{g}_P)}$ とした．そこで，任意の $R > 0$ に対して，T を十分小さく，たとえば，

$$T = \min\left\{\left(\frac{-(C_1R + C_2R^2 + C_3R^3) + \sqrt{(C_1R + C_2R^2 + C_3R^3)^2 + 4R\|d_{A_0}^* F_{A_0}\|_{W^\ell(M;\mathfrak{g}_P)}}}{2\|d_{A_0}^* F_{A_0}\|_{W^\ell(M;\mathfrak{g}_P)}}\right)^2, \left(\frac{1}{2(C_1 + 2C_2R + 4C_3R^2)}\right)^2\right\}$$

とすれば，Φ は $\left\{B \in \mathcal{B}_T^\ell(M;\mathfrak{g}_P) \;\middle|\; \|B\| \leq R\right\}$ 上の縮小写像になる。

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[1] 伊藤光弘：Yang-Mills方程式 ― インスタントンとモノポールを中心として ―，数学 第34巻 第4号 1985年10月 秋季号 322～337

[2] Tosio Kato : On nonlinear Schrödinger equations, preprint

なお，$A = A(t)$ の $t \uparrow \infty$ での挙動は，まだ何もわかっていない．

Scattering Problem for the Nonlinear Schrödinger Equations

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We consider the asymptotic behavior as $t \to \pm\infty$ of solutions and the scattering theory for the following nonlinear Schrödinger equation with power interaction:

(1) $\quad i\dfrac{\partial u}{\partial t} = -\Delta u + |u|^{p-1}u, \qquad t \in R, \quad x \in R^n,$

(2) $\quad u(0,x) = u_0(x), \qquad x \in R^n.$

Let $U(t)$ be the evolution operator associated with the free Schrödinger equation. For $1 \le q < \infty$, L^q and $\|\cdot\|_q$ denote the standard q-integrable function space on R^n and its norm, respectively. Let Σ denote the Hilbert space

$$\Sigma = \{ v \in L^2 \ ; \ \nabla v \in L^2 \text{ and } xv \in L^2 \}$$

with the norm $\|v\|_\Sigma^2 = \|v\|_2^2 + \|\nabla v\|_2^2 + \|xv\|_2^2$. We put

$$\alpha(n) = \begin{cases} \infty & , n = 1, 2, \\ (n+2)/(n-2), & n \ge 3, \end{cases}$$

$$\gamma(n) = \frac{n + 2 + \sqrt{n^2 + 12n + 4}}{2n} .$$

We note that $1 + \frac{2}{n} < \gamma(n) < 1 + \frac{4}{n} < \alpha(n)$.

Our main results are the following.

<u>Theorem 1.</u> (i) Assume that $1 + \frac{2}{n} < p < \alpha(n)$. Then, for any $u_0 \in \Sigma$ there exist unique $u_\pm \in L^2$ such that

(3) $\|u_\pm - U(-t)u(t)\|_2 \to 0 \quad (t \to \pm\infty)$,

where $u(t)$ is a solution of (1) with $u(0) = u_0$.

(ii) Assume that $1 \le p \le 1 + \frac{2}{n}$. Then, for any non-zero $u_0 \in \Sigma$ there do not exist any $u_\pm \in L^2$ satisfying (3).

Theorem 2. Assume that $\gamma(n) < p < \alpha(n)$.

(i) For any $u_+ \in \Sigma$, there exists a unique $u_0 \in \Sigma$ such that

(4) $\|u_+ - U(-t)u(t)\|_\Sigma \to 0 \quad (t \to +\infty)$,

where $u(t)$ is a solution of (1) with $u(0) = u_0$.

(ii) For any $u_- \in \Sigma$, there exists a unique $u_0 \in \Sigma$ such that

(5) $\|u_- - U(-t)u(t)\|_\Sigma \to 0 \quad (t \to -\infty)$,

where $u(t)$ is a solution of (1) with $u(0) = u_0$.

Theorem 3. Assume that $\gamma(n) < p < \alpha(n)$. For any $u_0 \in \Sigma$, there exist unique $u_\pm \in \Sigma$ such that the solution $u(t)$ of (1) with $u(0) = u_0$ satisfies

(6) $\|u_\pm - U(-t)u(t)\|_\Sigma \to 0 \quad (t \to \pm\infty)$.

Remark 1. (i) If $1 < p < \alpha(n)$, for any $u_0 \in \Sigma$ there exists a unique solution in $C(\mathbf{R};\Sigma)$ of (1)-(2) (see Ginibre and Velo [4, Theorem 3.1] and [5, Proposition 3.5]).

(ii) Theorem 1 shows that $1 + \frac{2}{n}$ is the critical power.

(iii) Theorem 2 implies that if $\gamma(n) < p < \alpha(n)$, the wave operators $W_\pm$: $u_\pm \to u_0$ are well defined as a mapping from Σ to Σ. Theorem 3 implies that Range (W_+) = Range $(W_-) = \Sigma$ and that $W_\pm$ are one to one. Therefore, we can construct the scattering operator $S = W_+^{-1}W_-$: $u_- \to u_+$, when $\gamma(n) < p < \alpha(n)$.

Corollary 4. Assume that $\gamma(n) < p < \alpha(n)$. The wave operators $W_\pm$ are well defined in Σ and are homeomorphisms from Σ onto Σ. Therefore, the scattering operator S is well defined in Σ and is a homeomorphism from Σ to Σ.

There are many papers concerning the asymptotic behavior as $t \to \pm\infty$ of solutions for (1)-(2) (see, e.g., [1], [3-11], [15-18] and [20-22]). Recently the scattering theory for the nonliner Schrödinger equations has been remarkably developed by a series of papers of Ginibre and Velo [4-8]. In [5] they proved Theorems 2 and 3 for $1 + \frac{4}{n} < p < \alpha(n)$. Furthermore, in [7] they proved Theorems 2 and 3 for $1 + \frac{4}{n} < p < \alpha(n)$ and $n \geq 3$ with Σ replaced by H^1. However, in [17, 18] Strauss conjectured that construction of the wave operators and their asymptotic completeness could be brought down from $1 + \frac{4}{n}$ to $\gamma(n)$. Theorems 2 and 3 answer Strauss' conjecture. In addition, Theorem 1 gives us interesting information about the asymptotic behavior as $t \to \pm\infty$ of solutions of (1)-(2) for $p \leq \gamma(n)$. But two important problems still remain open: (1) Can we construct the scattering theory for $1 + \frac{2}{n} < p \leq \gamma(n)$? (2) When $1 < p \leq 1 + \frac{2}{n}$, how does the solution of (1)-(2) behave as $t \to \pm\infty$?

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ある準線型楕円型デリクレ問題の解の凸性について

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最近 非線型楕円型境界値問題の解の凸性に関して多くの結果が多くの人によって得られている。その方法は "Concavity Maximum Principle" という Korevaar ('83 Indiana) によって発見されたある種の最大値の原理による。彼はそこで $\mathbb{R}^{n+1}$ の中のある capillary surface の凸性を示している。また Korevaar ('83 Indiana) と Caffarelli-Spruck ('82 Comm. P.D.E.) はラプラシアンの第1固有関数が log-concave になるという Brascamp-Lieb ('76 J. Funct. Anal.) の結果の "Concavity Maximum Principle" による証明を与えた。さらに Kennington ('85 Indiana) Kawohl ('85 Com. P.D.E.) は "$-\Delta u = u^{\alpha}$ in Ω $u=0$ on $\partial\Omega$ (Ωは凸領域 $0 \leqq \alpha < 1$)" の解 u について $v = u^{\frac{1-\alpha}{2}}$ が concave になることを "Concavity Maximum Principle" を改良して示した。

一方 pseudo-Laplacian $-\mathrm{div}(|\nabla\cdot|^{p-2}\nabla\cdot)$ $(p>1)$ は準線型の楕円型作用素として多くの人に研究されている。

（例えば Díaz（'85 Pitman Research Notes 106）に詳しい。）。
pseudo-Laplacian の特徴は 解の gradient が消えるところで 退化 又は 特異楕円型であることにある。

我々の結果は ラプラシアンを pseudo-Laplacian で置き換えても 同様なことが成り立つことを示すものである。つまり 次の定理である。

定理1：Ω を $\mathbb{R}^n$ $(n \geq 2)$ の滑らかな境界 $\partial\Omega$ をもつ有界な凸領域とする。定数 $p>1$ をひとつ固定する。関数 $u \in W_0^{1,p}(\Omega)$ を次の非線型固有値問題の正値弱解とする。

$$(1) \quad \begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

ここで λ は Poincaré の不等式の定数で

$$\lambda = \inf\left\{ \int_\Omega |\nabla v|^p dx \Big/ \int_\Omega |v|^p dx \;;\; v \in W_0^{1,p}(\Omega) \right\}$$

このとき $v = \log u$ は 凹関数である。

定理2：Ω と p は定理1と同じとする。関数 $u \in W_0^{1,p}(\Omega)$ を次のデリクレ問題の一意正値弱解とする。

$$(2) \quad \begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{q-2} u & \text{in } \Omega. \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

ここで λ は正定数，qは $1 \leqq q < p$ なる定数である。
このとき $v = u^{\frac{p-q}{p}}$ は 凹関数である。

定理1 は先の Korevaar と Caffarelli-Spruck の結果に対応し 定理2は Kennington と Kawohl の結果に対応する。

一般に pseudo-Laplacian の性質上 (1), (2) の解は C^2級ではない。Tolksdorf の結果 ('83 Comm. P.D.E.) によれば $C^{1+\alpha}(\Omega)$ に解は属するが たとえば (2) で $q=1$，Ω が原点中心の ball の時 関数 $u(x) = a|x|^{\frac{p}{p-1}} + b$ $(a<0,\ b>0)$ は (2)の解である。一方 Korevaar たちの "Concavity Maximum Principle" は C^2級の解にしか適用できないので 直接に我々の問題(1)(2)には使えない。

そこで我々は 滑らかな解をもち なおかつ "Concavity Maximum Principle" を適用できる問題で (1)(2) を近似することによって 定理1と定理2を示す。

ところで 定理1と定理2 の解の存在と一意性について少し述べる必要がある。存在は変分の直接法により示せるのだが一意性については問題がある。ラプラシアンの第1固有値が単純である

のは解の強比較定理によっているのであるが我々のpseudo-Laplacianは一般の強比較定理が示されていないので単純性はまだ知られていない。しかし我々は$\partial\Omega$の近くで(1),(2)の解のgradientが消えないという事実をうまく利用することによって次のことを示すことができた。

定理3　Ωを$\mathbb{R}^n$ $(n\geqq 2)$の滑らかな境界$\partial\Omega$をもつ有界な領域とする。しかも$\partial\Omega$を連結とする。(Ωが凸ならば$\partial\Omega$は連結である。)

このとき固有値問題(1)についてλは単純である。

定理4　Ωを$\mathbb{R}^n$ $(n\geqq 2)$の滑らかな境界$\partial\Omega$をもつ有界な領域とする。このとき(2)の解は正負の符号を除いて一意的である。

最後に定理1の証明の概略を述べる。方程式(1)は変分問題

(3) Find $u\in\mathbb{K}$ satisfying

$$\int_\Omega |\nabla u|^p\,dx = \min_{v\in\mathbb{K}}\int_\Omega |\nabla v|^p\,dx \quad (=\lambda)$$

where

$$\mathbb{K}=\{v\in W_0^{1,p}(\Omega)\ ;\ \|v\|_{L^p(\Omega)}=1\}$$

から得られるがこれを次で近似する。

(4) Find $u \in \mathbb{K}$ satisfying

$$\int_\Omega (\varepsilon u^2 + |\nabla u|^2)^{\frac{p}{2}} dx = \min_{v \in \mathbb{K}} \int_\Omega (\varepsilon v^2 + |\nabla v|^2)^{\frac{p}{2}} dx$$

(4)の正値解が(3)の正値解に収束するところに定理3を要する。

あとは(4)の解の滑らかさを Tolksdorf の結果('84 J. Diff. Eqs)を使って示し、(4)の解を u_ε とするとき $v_\varepsilon = \log u_\varepsilon$ のみたす方程式に Korevaar の "Concavity Maximum Principle" を適用する。

— Appendix (Korevaar の Concavity Maximum Principle) —

Ω を $\mathbb{R}^n$ の有界凸領域とし $u \in C^2(\Omega) \cap C(\overline{\Omega})$ を考える。concavity function $C(y, z, t)$ を

$$C(y,z,t) = (1-t)u(y) + t\,u(z) - u((1-t)y + tz)$$

$((y,z,t) \in \Omega \times \Omega \times [0,1])$ によって定める。$C \leqq 0$ は u が concave であることに対応する。このとき、

定理 (Korevaar) $u \in C^2(\Omega) \cap C(\overline{\Omega})$ が

$$\sum a^{ij}(\nabla u) D_{ij} u + b(x, u, \nabla u) = 0 \quad \text{in } \Omega$$

をみたす。ここで $[a^{ij}(\nabla u)]$ は正値対称行列であり、

$b = b(x, u, \nabla u)$ は (x, u) について jointly concave である。

このとき もし C が正の値をとれば 正の最大値は $\partial(\Omega \times \Omega) \times [0,1]$ でとられる

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調和写像とEells-Sampsonの放物型方程式について

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以下では，(M, g), (N, h)はcompact Riemann多様体とする．

Definitions

$f : M \longrightarrow N$; smooth map に対して，energy density e(f), および Total energy E(f) を次のように定義する．

$$e(f) = \frac{1}{2}|df|^2 = \frac{1}{2} g^{ij} h_{\alpha\beta}(f) \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\beta}{\partial x^j}$$

ここで　$df \in T^*M \otimes f^{-1}TN$

$$E(f) = \int_M e(f)\, d\mu_M$$

ここで $d\mu_M$ はRiemann計量から定まる標準的なvolume element

$f : M \longrightarrow N$; smooth map が 調和写像(harmonic map)であるとは，f が汎関数 E の停留点であること．

この定義により，次の命題をえる．

Proposition

$f : M \longrightarrow N$ がharmonic mapであることと f が次の微分方程式を満たす事は同値．

$$\Delta f = 0$$

ここで，

$$\Delta f = \mathrm{Trace}\, \nabla df = \left(\Delta_M f^\alpha + g^{ij}\, {}^N\Gamma^\alpha_{\beta\gamma}(f) \frac{\partial f^\beta}{\partial x^i} \frac{\partial f^\gamma}{\partial x^j}\right)$$

また，1964年に，J.Eellsと J.H.Sampson は，harmonic map の存在に関する，基本的な次の結果を得た．

<u>Theorem</u> (J.Eells and J.H.Sampson [ES])

(M,g),(N,h)をcompact Riemann 多様体， ${}^{N}K \leq 0$ (N の断面曲率) とすると，$C^{\infty}(M,N)$ の各 homotopy 類には，少なくとも一つの harmonic map が存在する．

なお一意性については，P.Hartman [H] により，ある種の一意性がえられている．この定理の証明の中で，Eells-Sampson は，次の半線形放物型方程式の初期値問題をあつかった．

$$(*)\quad \begin{cases} \dfrac{\partial f}{\partial t} = \Delta f & \text{on } M \times [0,\omega) \\ f = f_0 & \text{on } M \times 0 \end{cases}$$

ここでは，(*) を Eells-Sampson の放物型方程式と呼ぶ．

一方，エネルギー積分 E(f) の第二変分公式は，1975年に R.T.Smith によって得られている．

<u>Theorem</u> (R.T.Smith [Sm])

E(f) の harmonic map f での Hessian は，

$$H_f(v,w) = \int_M (\Delta^f v - \mathrm{Trace}\ {}^{N}R(df,v)df, w)\, d\mu_M$$

v, w は $f^{-1}TN$ の section

である．

<u>Definition</u>

$J_f v = \Delta^f v - \mathrm{Trace}\ {}^{N}R(df,v)df$ を

f の Jacobi operator とよぶ

f の index , nullity とは，

Index(f) = { J_f の negative definite subspace の最大次元}

$\mathrm{Null}(f) = \mathrm{Dim}(\ker(J_f))$

harmonic map f が安定とは
$\mathrm{Index}(f) = \mathrm{Null}(f) = 0$,
であること.

ここで，次の問題を考えよう.

<u>問題</u>

f : M ⟶ N を安定な harmonic map とするとき,(*) は f に十分”近い”初期値から ω＝＋∞ までの解をもち，f に収束するだろうか?

この問題に関して，次の定理をえた.

<u>Theorem</u>

f : M ⟶ N を安定な harmonic map とするとき，ε>0 が存在して，

$$\|f-f_0\|_{H^m(M,N)} < \varepsilon,\ \|f-f_0\|_{C^0(M,N)} < i_M \quad (m > \tfrac{1}{2}\dim M + 2)$$

であれば，(*) は ω＝＋∞ までの解を，

$$B = C^{1/2}([0,\infty);H^{m-1}(M,N)) \cap C([0,\infty);H^m(M,N))$$

にもち，その解は f に$H^m(M,N)$で収束する.
i_M は M の単射半径.

証明の方針は，(*) を $f^{-1}TN$ - valued の方程式に直すことによって，
が十分小さい時に，

$$(**)\quad \begin{cases} \dfrac{\partial u}{\partial t} = -J_f u + \varphi(u)\Delta^f u + P_f u & \text{on } M\times[0,\omega) \\ u = u_0 & \text{on } M\times 0 \end{cases}$$

と直し，(**) の解をBで存在することを逐次近似によって証明する.

さて， f として特に M 上の恒等写像 id_Mを考えよう.

id_M は isometry であり，特に harmonic map である．

id_M での Jacobi operator は，

$$J_{id_M} = \Delta v - Ric(v)$$

Ric(・) は M の Ricci 曲率．

であって，M が非負な Ricci 曲率（ただし，Ric ≠ 0 ）をもてば，id_M は安定な harmonic map となる，したがって定理は特に，M = N，f = id_M で成り立つ．

Remark

L. Simon 〔Si〕 によって，同様の問題が N が analytic Riemann 多様体のときに，扱われている．

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Criteria for Hypoellipticity

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Main Results. Let $P = p(x, D_x)$ be a differential operator of order $m \geq 1$ with coefficients in $C^\infty(R^n_x)$, that is,

$$p(x, D_x) = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha, \quad a_\alpha(x) \in C^\infty(R^n_x),$$

where for multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $D_x^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ and $D_j = -i\partial_{x_j}$.

We say that P is *hypoelliptic in* R^n if for any $u \in D'(R^n)$ and for any open set Ω of R^n, $Pu \in C^\infty(\Omega)$ implies $u \in C^\infty(\Omega)$. Let Λ and $\log \Lambda$ be pseudodifferential operators with symbols $\langle\xi\rangle$ and $\log \langle\xi\rangle$, respectively, where $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$. We write $p^{(\alpha)}_{(\beta)}(x,\xi) = \partial_\xi^\alpha D_x^\beta p(x,\xi)$ for multi-indices α and β. We set $\|u\|_s = \|\Lambda^s u\|$ for real s and $u \in C_0^\infty(R^n)$, where $\|.\|$ denotes the usual L^2 norm.

Theorem 1. *Assume that for any* $\varepsilon > 0$ *and any compact set* K *of* R^n *there exists a constant* $C_{\varepsilon,K}$ *such that for any* $u \in C_0^\infty(K)$

(1) $$\|(\log \Lambda)^m u\| \leq \varepsilon \|Pu\| + C_{\varepsilon,K}\|u\|,$$

(2) $$\sum_{0<|\alpha+\beta|<m} \|(\log \Lambda)^{|\alpha+\beta|} P^{(\alpha)}_{(\beta)} u\|_{-|\beta|} \leq \varepsilon\|Pu\| + C_{\varepsilon,K}\|u\|,$$

where $P^{(\alpha)}_{(\beta)} = p^{(\alpha)}_{(\beta)}(x, D_x)$. *Then* P *is hypoelliptic in* R^n. *Furthermore we have*

(3) $$\mathrm{WF}\ Pv = \mathrm{WF}\ v \quad \textit{for any} \quad v \in \mathcal{D}'(R^n).$$

Corollary 1. *Let P be a differential operator of second order with C^∞-coefficients, that is,*

$$P = \sum_{j,k} a_{jk}(x) D_j D_k + \sum_j i b_j(x) D_j + c(x) .$$

We assume that

$$\begin{cases} a_{jk} \text{ and } b_j \text{ are real valued,} \\ \sum a_{jk}(x)\xi_j\xi_k \ge 0 \quad \text{for all } (x,\xi) \in R^{2n} \end{cases}$$

If for any $\varepsilon > 0$ and any compact set K of R^n the estimate

$$(4) \qquad \|(\log \Lambda)^2 u\| \le \varepsilon\|Pu\| + C_{\varepsilon,K}\|u\|, \qquad u \in C_0^\infty(K)$$

holds with a constant $C_{\varepsilon,K}$ then we have (3).

Corollary 2. *Let P be the same as in Corollary 1. If for any $\varepsilon > 0$ and any compact set K of R^n the estimate*

$$(5) \qquad \|(\log \Lambda)u\|^2 \le \varepsilon \operatorname{Re}(Pu, u) + C_{\varepsilon,K}\|u\|^2, \quad u \in C_0^\infty(K)$$

holds with a constant $C_{\varepsilon,K}$ then we have (3).

Remark. The estimate (4) easily follows from (5).

We discuss the necessity of hypoellipticity for second order differential operators given in Corollary 1. The estimate (4) is not always necessary for hypoellipticity. We have a counter example given by Fediǐ [Fd], $\mathcal{A}_0 \equiv D_1^2 + \exp(-1/|x_1|^\delta)D_2^2$, $\delta > 0$. Indeed, this example does not satisfy (4) when $\delta \geq 1$, while it is hypoelliptic for any $\delta > 0$. However, the estimate (4) is necessary for a class of operators to be hypoelliptic.

The result concerning the necessity of (4) can be discussed for some class of operators of higher order. Let m be even positive integer and let P_0 be a differential operator of the form

(6) $$P_0 = D_t^m + \mathscr{A}(x, D_x) \quad \text{in} \quad R_t \times R_x^n ,$$

where $\mathscr{A}(x, D_x)$ is a differential operator of order m with C^∞-coefficients. We assume that $\mathscr{A}(x, D_x)$ is formally self-adjoint in R_x^n and bounded from below, that is, there exists a real c_0 such that $(\mathscr{A}(x, D_x)u, u) \geq c_0 \|u\|^2$ for $u \in C_0^\infty(R_x^n)$.

Theorem 2. *Let P_0 be the above operator. Assume that P_0 is hypoelliptic in $R_t \times R_x^n$. Then for any $x_0 \in R_x^n$ there exists a neighborhood ω of x_0 such that for any $\varepsilon > 0$ the estimate*

(7) $$\|(\log \Lambda)^{m/2} u\|^2 \leq \varepsilon \operatorname{Re}(P_0 u, u) + C_{\varepsilon, K} \|u\|^2, \quad u \in C_0^\infty(R_t \times \omega)$$

holds with a constant C_ε. Here Λ, of course, denotes $\langle D_t, D_x \rangle = (1 + D_t^2 + |D_x|^2)^{1/2}$.

Remark 1. When $m = 2$ the estimate (4) follows from (7). In fact, for any compact set K of $R_t \times R_x^n$, let K' be the projection of K to R_x^n and take the partition of unity $\sum \varphi_j^2(x) = 1$ over K'. Since $\operatorname{Re}([P_0, \varphi_j]u, \varphi_j u)$ is majorated by a constant times of $\|u\|^2$, we have (7) for $u \in C_0^\infty(R_t \times K')$, which implies (5) and hence (4).

Remark 2. Almost the same result as Theorem 2 was obtained independently by [Hs]. Proof of Theorem 2 is performed by the similar method as in [Me], where nonanalytic hypoellipticity was studied for operators of the same form as (6).

As an application of Theorems we shall consider the hypoellipticity of degenerate elliptic opertors of the following form;

(8) $$L_0 = D_t^{2\ell} + D_x^{2\ell} + g(x)D_y^{2\ell} \quad \text{in} \quad R^3,$$

where $\ell = 1, 2, \ldots$ and $g(x)$ is C^∞ function such that $g(x) > 0$ ($x \neq 0$) and $g(0) = 0$. When $\ell \geq 2$ we assume that for any $j > 0$

(9) $$|D_x^j g(x)| \leq C_j g(x)^{1-\sigma j} \quad \text{in a neighborhood of} \quad x = 0 ,$$

where σ is a number satisfying

(10) $$0 < \sigma < 1/2\ell^2 .$$

It is clear that a function $\exp(-1/|x|^\delta)$, $\delta > 0$, satisfies (9) for any $\sigma > 0$.

Proposition 1. *Let* L_0 *be the above operator. If* $g(x)$ *satisfies*

(11) $$\lim_{x \to 0} |x||\log g(x)| = 0$$

then L_0 *is hypoelliptic in* R^3. *Assume in addition that* $xg'(x) \geq 0$, *that is,* g *is monotone in* R^+ *and* R^-, *respectively. Then the condition (11) is also necessary for* L_0 *to be hypoelliptic in* R^3.

Remark. For the function $\exp(-1/|x|^{\delta})$, $\delta > 0$, the condition (11) means $\delta < 1$. The proposition in the case $\ell = 1$ was first proved by Kusuoka-Strook [K-S] in a little weak form. Their proof is based on the Malliavin calculus, which is a theory of stochastic differential equations.

Unfortunately, when $\ell \geq 2$ we can not apply directly Theorem 1 to the proof of Proposition 1, because it is quite hard to check the hypothesis (2) for L_0, more precisely, to show

$$\|(\log \Lambda)\, D_x^{2\ell-1} u\| \leq \varepsilon \|L_0 u\| + C_{\varepsilon,K}\|u\|, \quad u \in C_0^\infty(K).$$

So we need the following amelioration of Theorem 1 under an additional assumption.

Theorem 3. *Assume that the principal symbol of* $p_m(x,\xi)$ *of* P *satisfies*

(12) $$p_m(x,\xi) \neq 0 \quad \text{for} \quad x' \neq 0, \quad \text{where} \quad x = (x', x'').$$

Then the conclusion (3) of Theorem 1 still holds even if the estimate (2) is replaced by

(13) $$\sum_{\substack{0<|\alpha+\beta|<m \\ \alpha=(0,\alpha'')}} \|(\log \Lambda)^{|\alpha+\beta|} P_{(\beta)}^{(\alpha)} u\|_{-|\beta|}$$
$$\leq \varepsilon \|Pu\| + C_{\varepsilon,K}\|u\| \quad , \quad u \in C_0^\infty(K).$$

The hypoelliptic operator $\mathscr{A}_0$ given by [Fd] with $\delta \geq 1$ is not covered by Corollary 2 (nor 1) as stated in the preceding. To cover this exceptional example we give another criterion of hypoellipticity.

Theorem 4. *Assume that the principal symbol of* P *satisfies (12). If for any compact set* K *of* R^n *there exist a* $\kappa_0 > 0$ *and a constant* C_K *such that*

$$(14)\qquad \|u\| + \sum_{\substack{0<|\alpha+\beta|\leq m \\ \alpha=(0,\alpha'')}} \|P^{(\alpha)}_{(\beta)} u\|_{\kappa_0 - |\beta|}$$

$$\leq C_K(\|Pu\| + \|u\|_{-1}), \qquad u \in C_0^\infty(K),$$

then we have (3).

A little historical survey.

Constant coeffecients case 《 i.e. $P = p(D_x) = \sum a_\alpha D_x^\alpha$ 》.

Theorem 5 (Hörmander[H_1]). *It is necessary and sufficient for* $p(D_x)$ *to be hypoelliptic in* R^n *that*

$$\operatorname{Im} \zeta \to \infty \quad \text{if} \quad |\zeta| \to \infty \quad \text{on the surface} \quad \{ \zeta \in \mathbf{C}^n ; p(\zeta) = 0 \}.$$

Corollary. *If* $p(D_x)$ *is hypoelliptic in* R^n *then* $p(\xi) \neq 0$ for large $|\xi|$, $\xi \in R^n$.

Variable coefficients case.

Let L be a differential operator of the form

$$(15)\qquad L = -\sum_{j=1}^{r} X_j^2 + X_0 \quad ,$$

where X_j are real vector fields, that is, $X_j = \sum_{k=1}^{n} a_{jk}(x)\partial_{x_k}$ for real-valued $a_{jk}(x) \in C^\infty(R^n)$.

Theorem 6 (Hörmander [H_2]). *L is hypoelliptic in R^n if*

(*) $\begin{cases} \text{the vector fields } X_0,\dots, X_n \text{ and their repeated} \\ \text{commutators span the tangent space at each point in } R^n_x . \end{cases}$

Example. $\mathscr{A}_1 \equiv D_1^2 + x_1^2 D_2^2$ is hypoelliptic because with $X_1 = \partial_{x_1}$ and $X_2 = x_1\partial_{x_2}$ we have $[X_1, X_2] = \partial_{x_2}$. Note that the symbol of $\mathscr{A}_1$, $\sigma(\mathscr{A}_1) \equiv \xi_1^2 + x_1^2\xi_2^2$ vanishes on $\xi_1 = 0$, $\xi_2 \to \infty$ when $x_1 = 0$. Similarly, $\mathscr{A}_k \equiv D_1^2 + x_1^{2k}D_2^2$ is hypoelliptic since $[X_1, [X_1, \dots [X_1, X_2] \dots] = k!\partial_{x_2}$ if $X_1 = \partial_{x_1}$ and $X_2 = x_1^k\partial_{x_2}$.

The key point of the proof of Hömander's theorem is to derive the following subelliptic estimate from (*): For some $\varepsilon > 0$

(16) $$\|u\|_\varepsilon^2 \le C(\,\mathrm{Re}(Lu, u) + \|u\|^2), \quad u \in C_0^\infty(R^n),$$

where for the brevity we consider the simple case $X_0 \equiv 0$.
If $L = \mathscr{A}_k$ then $\varepsilon = 1/(k+1)$ (see Fefferman-Phong [F-P]). By using (16) we can prove that if $u \in H_s^{loc}(\Omega)$ and $Lu \in H_s^{loc}(\Omega)$ then $u \in H_{s+\varepsilon}^{loc}(\Omega)$. The repetition of ε-regularity up gives the hypoellipticity of L.

It is clear that infinitely degenerate elliptic operators $\mathscr{A}_0$ and L_0 with $\ell = 1$ don not satisfy Hörmander's criterion (*). As understood from the fact that, for $\mathscr{A}_k$, ε tends to 0 when

$k \to \infty$, infinitely degenerate elliptic operators $\mathcal{A}_0$ and L_0 with $\ell = 1$ do not satisfies the subelliptic estimate (16) for any $\varepsilon > 0$. Hence, neither Hörmander's theorem nor his method apply to these infinitly degenerate operaors.

Proof of Proposition 1.

For the brevity we shall consider the second order case and only prove the sufficiency of (11) when $g(x) = \exp(-1/|x|^{\delta})$. (Concerning the necessity of (11) see $[M_2]$.) By Corollary 2 it suffices to show that $L_0 = D_t^2 + D_x^2 + \exp(-1/|x|^{\delta})D_y^2$ satisfies (5) when $\delta < 1$. Note that

(17) $$\mathrm{Re}(L_0 u, u) = \|D_t u\|^2 + \|D_x u\|^2 + (\exp(-1/|x|^{\delta})D_y^2 u, u).$$

We use the following lemma given by Fefferman [Ff]:

Lemma. *Assume that* $V(x) \geq 0$, C^{∞} *on a cube* Q *in* R^n. *Suppose that there exists a* $c > 0$ *such that*

(18) $$m(\{x \in Q \; ; \; V(x) \geq c(\mathrm{diam}\ Q)^{-2}\}) \geq c|Q|,$$

where $m(.)$ *denotes the lebeague measure. Then for* $u \in C^1$ *we have*

(19) $$\int_Q \{|\nabla u(x)|^2 + V(x)|u(x)|^2\}dx \geq c'(\mathrm{diam}\ Q)^{-2}\int_Q |u(x)|^2 dx.$$

The constant c' *depends only on* n *and* c.

In [Ff], it is assumed that $V(x)$ is polynomial and $(\mathrm{Av}_Q\ V) \geq (\mathrm{diam}\ Q)^{-2}$. The proof in [Ff] is still valid for the above hypothese.

Set $V(x) = \exp(-1/|x|^{\delta})\eta^2$. (In what follows we assume $x \in R^1$.) In view of (17), for the proof of (5) it suffices to show that for any $k > 0$ there exists a $M_k > 0$ such that for $u \in C_0^\infty(R^1)$

$$(20) \qquad \int\{|\nabla_x u(x)|^2 + V(x)|u(x)|^2\}dx \geq c(k\log|\eta|)^2\int|u(x)|^2dx,$$

$$\text{if} \quad |\eta| \geq M_k \quad ,$$

where c is a constant independent of k and η. Of course, c may epend on the interval $[-R,R]$ containing supp u.

Let $\{Q_j\}_{j=1}^{\infty}$ be pairwise disjoint intervals such that $\cup Q_j = R^1$, diam $Q_j = (k\log|\eta|)^{-1}$ and $Q_1 = \{x ; |x| \leq (2k\log|\eta|)^{-1}\}$. When $|x| \geq (4k\log|\eta|)^{-1}$ we have

$$\exp(-1/|x|^{\delta})\eta^2 \geq \exp(2\log|\eta| - (4k\log|\eta|)^{\delta}).$$

Then, if $|\eta| \geq M_k$ for a sufficently large M_k, we have

$$(21) \qquad \exp(-1/|x|^{\delta})\eta^2 \geq |\eta| \geq (\text{diam } Q_j)^{-2}$$

$$\text{for } |x| \geq (4k\log|\eta|)^{-1}.$$

Thus, for each Q_j we have (18). By th above lemma we get (19) and so (20).

The above argument can apply to more degenerate elliptic operators than L_0. For example, let $\tilde{L}$ be a differential operator of the same form as (8) with $\ell = 1$ and $g(x)$ equal to $\exp(-1/|x|^{\delta})\sin^2(1/|x|)$. If $0 < \delta < 1$, $\tilde{L}$ is hypoelliptic.

In fact, for any Q_j where $\sin(1/|x|)$ vanishes at least twice, we have

$$(22) \qquad m(\{ x \in Q_j \ ; \ \sin^2(1/|x|) \geq 1/2 \}) \geq |Q_j|/4 .$$

For other Q_j contained in $[-R, R]$ we also have

$$(23) \qquad m(\{ x \in Q_j \ ; \ \sin^2(1/|x|) \geq (2\pi R^2 k \log |\eta|)^{-2}\}) \geq |Q_j|/2$$

because $\sin t \geq 2t/\pi$ for $0 \leq t \leq \pi/2$. If we set $V(x) = \exp(-1/|x|^{\delta})\sin^2(1/|x|)\eta^2$, (22) or (23) together with (21) gives (18) for each Q_j. By Lemma 1 we obtain (20) and so $\tilde{L}$ is hypoelliptic.

We end this abstract by giving a more pathological example. Let $\hat{L}$ be a second order differential operator of the form (8). We assume that $g(x)$ vanishes on a Cantor set E with measure 0 defined as follows: Set $E = I_0 \setminus \bigcup_{j=1}^{\infty} I_j$, where $I_0 = [0, 1]$, I_1 is an open interval with length $1/3$ whoes center is $1/2$. Here I_2 and I_3 are open intervals with length $(1/3)^2$ whoes centers coincide with the center of each conected component of $I_0 \setminus I_1$, respectively. Furthermore, $I_4, \ldots, I_7$ are open intervals with lenghth $(1/3)^3$ whoes centers coincide with the center of each connected component of $I_0 \setminus \bigcup_{j=1}^{3} I_j$, respectively. We define open intervals I_j in this manner, recursively. On each $I_j = (-a+b, a+b)$, $j \geq 1$, we set $g(x) = \exp\{(-1/|x+a-b|^{\delta})+(-1/|x-a-b|^{\delta})\}$ and we set $g(x) = \exp(-1/|x|^{\delta})$ and $\exp(-1/(|x-1|^{\delta})$ for $(-\infty, 0)$ and $(1, \infty)$, respectively. If $0 < \delta < 1$ then $\hat{L}$ is hypoelliptic.

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