Asymptotic solutions of Hamilton-Jacobi equations with convex Hamiltonians

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Abstract

The following discussion is based on joint work with N. Ichihara, especially on [14]. We are concerned with the Cauchy problem for the Hamilton-Jacobi equation

$$\begin{cases} u_t + H(x, Du) = 0 & \text{in } \mathbb{R}^n \times (0, +\infty), \\ u(\,\cdot\,, 0) = u_0 & \text{on } \mathbb{R}^n, \end{cases} \tag{1}$$

and study the long time behavior of the solution of (1).

We assume throughout the following (A1)–(A5).

(A1) $H \in C(\mathbb{R}^n \times \mathbb{R}^n)$,

(A2) $\inf\{H(x,p) \,|\, x \in B(0,r),\ |p| \geq R\} \longrightarrow +\infty$ as $R \to +\infty$ for every $r > 0$,

(A3) $H(x,p)$ is convex with respect to p for every $x \in \mathbb{R}^n$,

(A4) for each $\phi \in \mathcal{S}_H^-$, there exist $C > 0$ and $\psi \in \mathcal{S}_{H-C}^-$ such that

$$\lim_{|x|\to\infty} (\phi - \psi)(x) = \infty,$$

(A5) $u_0 \in C(\mathbb{R}^n)$.

Here and henceforth, we denote by $\mathcal{S}_H$ (resp., $\mathcal{S}_H^-$, $\mathcal{S}_H^+$) the set of all continuous viscosity solutions (resp., subsolutions, supersolutions) of $H(x,\, Du(x)) = 0$ in $\mathbb{R}^n$. Similarly, given a domain Ω, we denote by $\mathcal{S}_H(\Omega)$ (resp., $\mathcal{S}_H^-(\Omega)$, $\mathcal{S}_H^+(\Omega)$) the set of all continuous viscosity solutions (resp., subsolutions, supersolutions) of $H(x,\, Du(x)) = 0$ in Ω.

Note that hypotheses (A1)–(A5) are not enough to assure the unique solvability of (1) in the sense of viscosity solution. To start our discussion with this generality, we

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define the (unique) solution of (1) as follows. For any $\psi \in C(\mathbb{R}^n)$ and $t \geq 0$, we define the function $T_t\psi$ on $\mathbb{R}^n$ by

$$T_t\psi(x) = \inf\Big\{\int_{-t}^{0} L(\eta(s), \dot\eta(s)),\, ds + \psi(\eta(-t)) \;\Big|\; \eta \in \mathcal{C}([-t,\, 0];\, x)\Big\}, \tag{2}$$

and refer the function $u(x,t) := T_t u_0(x)$ as the *solution* of (1). Here L is the *Lagrangian* of H defined by $L(x,\xi) = \sup\{\xi\cdot p - H(x,p)\}$ for $(x,\xi) \in \mathbb{R}^{2n}$ and $\mathcal{C}([a,\, b]; x)$, with $a < b$, denotes the space of all absolutely continuous functions (called curves) $\eta : [a,\, b] \to \mathbb{R}^n$ (i.e. $\eta \in \mathrm{AC}([a,\, b])$) such that $\eta(b) = x$. Also, $\mathcal{C}((-\infty,\, a]; x)$ denotes the space of all functions $\eta \in C((-\infty,\, 0])$ such that $\eta \in \mathcal{C}([c,\, a]; x)$ for all $c < a$.

We remark here that $T_t\psi(x)$ is well-defined with the following interpretation and $T_t\psi(x) \in [-\infty,\, 0)$ for all $(x,t) \in \mathbb{R}^n \times [0,\, \infty)$. To see this, we fix $\psi \in C(\mathbb{R}^n)$. Note that $L(x,\xi)$ is lower semicontinuous on $\mathbb{R}^{2n}$, due to the assumption that $H \in C(\mathbb{R}^{2n})$, and hence that the function $s \mapsto L(\eta(s),\, \dot\eta(s))$ is Lebesgue measurable on $(-\infty,\, 0)$ for any $\eta \in \mathrm{AC}((-\infty,\, 0])$. Noting that $H(x,\, 0) = -\inf_{\xi\in\mathbb{R}^n} L(x,\xi)$ for all $x \in \mathbb{R}^n$, we observe that, for each $t > 0$, $r > 0$ and $\eta \in \mathrm{AC}([-t,\, 0])$ satisfying $\eta([-t, 0]) \subset B(0,r)$, we have $L(\eta(s),\, \dot\eta(s)) \geq -\max_{x\in B(0,\,r)} H(x,\, 0)$. Therefore, for $\eta \in \mathrm{AC}([-t,\, 0])$, it is natural to set

$$\int_{-t}^{0} L(\eta(s),\, \dot\eta(s))\, ds = \infty$$

if $s \mapsto L(\eta(s),\, \dot\eta(s))$ is not integrable on $(-t,\, 0)$. With this interpretation, $T_t\psi(x)$ is well-defined for all $(x,t) \in \mathbb{R}^n \times [0,\, \infty)$. Next we see that for all $(x,t) \in \mathbb{R}^n \times [0,\, \infty)$,

$$T_t\psi(x) \leq \int_{-t}^{0} L(x,\, 0)\, ds + \psi(x) = L(x,0)t + \psi(x) < \infty.$$

Now, let us consider an example where $T_t u_0(x) = -\infty$ for some (x,t). Let $n = 1$, $H(p) = (1/2)|p|^2$ and $u_0(x) = -|x|^2$. Then the Lagrangian L of H is given by $L(\xi) = (1/2)|\xi|^2$. Consider the curve $\eta \in \mathcal{C}((-\infty,\, 0];\, 0)$ given by $\eta(s) = cs$, with $c > 0$, and observe that for any $t > 0$,

$$T_t u_0(0) \leq \int_{-t}^{0} L(\dot\eta(s))\, ds + u_0(\eta(-t)) = \frac{c^2 t}{2} - (ct)^2 = \frac{c^2 t}{2}(1 - 2t),$$

which implies that $T_t u_0(0) = -\infty$ if $t > 1/2$. We recall that if the function $u(x,t) := T_t u_0(x)$ is continuous on an open set $U \subset \mathbb{R}^n \times (0,\, \infty)$, then u is a viscosity solution of $u_t + H(x, Du) = 0$ in U.

Our objective is to investigate the long-time behavior of the solution of (1). More precisely, we are concerned with the convergence of the form

$$u(x,t) + at - \phi(x) \longrightarrow 0 \quad \text{in } \; C(\mathbb{R}^n) \quad \text{as } t \to \infty \tag{3}$$

for some $a \in \mathbb{R}$ and $\phi \in C(\mathbb{R}^n)$, where $C(\mathbb{R}^n)$ is equipped with the topology of locally uniform convergence. Note that if u satisfies (1) in the viscosity sense, then the function $\phi(x) - at$, which we call an *asymptotic solution* of (1), enjoys the following stationary Hamilton-Jacobi equation in the viscosity sense:

$$H(x, D\phi) = a \qquad \text{in } \mathbb{R}^n. \tag{4}$$

Thus a natural question related to (3) is the *additive eigenvalue problem* for H which seeks for a pair $(a, \phi) \in \mathbb{R} \times C(\mathbb{R}^n)$ such that ϕ is a solution of (4), i.e., $\phi \in \mathcal{S}_{H-a}$. The additive eigenvalue problem appears in ergodic control, in which it is called the ergodic control problem, and in homogenization, in which it is called the cell problem.

A standard approach to (3) is first to solve the additive eigenvalue problem for H and then to try to prove the convergence (3) for each fixed solution of the additive eigenvalue problem for H. However, to simplify our presentation, we will deal only with the latter step in the above approach and investigate if (3) holds for a fixed $a \in \mathbb{R}$. We assume moreover that $a = 0$. Indeed, convergence (3) with general a is equivalent to (3) with $a = 0$ once H and u are replaced by $H - a$ and $u(x,t) + at$, respectively. Then, one of fundamental observations is the following.

Proposition 1. *Let*

$$u_0^-(x) := \sup\{\, \phi(x) \mid \phi \in \mathcal{S}_H^-, \quad \phi \le u_0 \quad \text{in } \mathbb{R}^n\},$$
$$u_\infty(x) := \inf\{\, \psi(x) \mid \psi \in \mathcal{S}_H, \quad \psi \ge u_0^- \quad \text{in } \mathbb{R}^n\}.$$

Under the additional assumption

(A6) $-\infty < u_0^-(x) \le u_\infty(x) < \infty$ *for all* $x \in \mathbb{R}^n$,

we have

$$\liminf_{t\to\infty} u(x,t) = u_\infty(x) \quad \text{for all } x \in \mathbb{R}^n,$$

The condition (A6) is equivalent to saying that

$$\{\phi \in \mathcal{S}_H^- \mid \phi \le u_0 \ \text{ in } \mathbb{R}^n\} \ne \emptyset \quad \text{and} \quad \{\phi \in \mathcal{S}_H \mid \phi \ge u_0^- \ \text{ in } \mathbb{R}^n\} \ne \emptyset.$$

Thus our purpose here is to show the convergence:

$$u(\cdot, t) \longrightarrow u_\infty \quad \text{in } C(\mathbb{R}^n) \ \text{ as } t \to \infty. \tag{5}$$

Asymptotic problems of this type has been studied intensively in the last decade. As one of the most typical cases, it was proved that if H satisfies (A1)–(A3) and $H(x,p)$ is $\mathbb{Z}^n$-periodic with respect to x and is strictly convex with respect to p, then for each $\mathbb{Z}^n$-periodic initial function $u_0 \in \mathrm{BUC}(\mathbb{R}^n)$ there exists a solution $(a, \phi) \in \mathbb{R} \times C(\mathbb{R}^n)$

of the additive eigenvalue problem for H such that (3) is valid and the constant a is determined independently of u_0. We refer to the literatures [2, 4, 5, 8, 9, 18, 19, 20] and references therein for more details. Remark that [2, 19] deal with non-convex Hamiltonians whereas most of others are concerned only with convex ones.

It has also been of interest in recent years on the long-time behavior of viscosity solutions to (1) that are not necessarily spatially periodic. As far as non-periodic solutions are concerned, the above (A1)-(A6) are insufficient to obtain the convergence (3) even if we admit strict convexity for H in any sense (see [3, 12]). The papers [1, 11, 12, 15] deal with some situations in which the solution of (1) has indeed the desired convergence of the form (3) for a suitable (a, ϕ).

Motivated by these earlier results, given a point $z \in \mathbb{R}^n$, we will introduce three general criteria (C1), (C2) and (C3), each of which, together mostly with (A1)–(A6) above, guarantees the pointwise convergence of the solution u of (1):

$$u(z,t) \longrightarrow u_\infty(z) \quad \text{as } t \longrightarrow \infty. \tag{6}$$

We then apply these criteria to obtain general convergence results of the form (5) and apply them to several examples. Our results cover most of existing results, and, on the other hand, involve a few observations which seem to be new.

One of our new observations is concerned with strict convexity for H. As pointed out in several literatures (see e.g. [2]), it is necessary in some situations to require a sort of strict convexity for H, as a genuine nonlinearity for H, so that the solution of (1) converges to an asymptotic solution as $t \to \infty$. We use condition $(\mathrm{A7})_+$ or $(\mathrm{A7})_-$ which guarantees, respectively, strict convexity of $H(x,p)$ in p at the zero level-set of H "upward" or "downward" (see below). We point out here that some of our convergence results involving $(\mathrm{A7})_-$ are not covered by the previous results. Another important feature here is in the observations of a "switch-back" motion of nearly optimal curves for the variational formula (2) for $T_t u_0$ in some situations. In one-dimensional case we have already encountered such a switch-back motion (see [13]), and here we investigate with greater generality the cases where such switch-back motions appear.

We use the dynamical approach in our investigations here which is based on tools from weak KAM theory (see e.g. [8, 10, 6, 7]) such as extremal curves, Aubry sets and representations formulas for solutions. We state our general criteria for the convergence to asymptotic solutions in terms of extremal curves, which seems inevitable to attain further generality.

Now, we introduce our criteria for the pointwise convergence. We begin with existence of extremal curves.

Theorem 2. *Let $\phi \in \mathcal{S}_H$ and $z \in \mathbb{R}^n$. Then there exists a curve $\gamma \in \mathcal{C}((-\infty, 0]; z)$*

such that for all $t > 0$,

$$\int_{-t}^{0} L(\gamma(s),\, \dot\gamma(s))\, ds = \phi(z) - \phi(\gamma(-t)). \tag{7}$$

A curve $\gamma \in \mathcal{C}((-\infty, 0]; z)$ is said to be *extremal* for ϕ at z if it satisfies (7) for all $t > 0$. We denote by $\mathcal{E}_z(\phi)$ the set of all extremal curves for ϕ at z. Also, we use the notation: $\mathcal{E}(\phi) := \bigcup_{x\in\mathbb{R}^n} \mathcal{E}_x(\phi)$ and $\mathcal{E} := \bigcup_{\phi\in\mathcal{S}_H} \mathcal{E}(\phi)$.

We fix any $\gamma \in \mathcal{E}_z$ and introduce the first criterion

(C1) $\lim_{t\to\infty} (u_0 - u_\infty)(\gamma(-t)) = 0.$

Note that $u_0^- \le u_0$ in $\mathbb{R}^n$ by definition and that

$$\liminf_{t\to\infty} (u_\infty - u_0^-)(\gamma(-t)) = 0.$$

Hence condition (C1) is equivalent to the condition

$$\lim_{t\to\infty} (u_0 - u_0^-)(\gamma(-t)) = 0. \tag{8}$$

Theorem 3. *Under condition (C1), the convergence (6) holds.*

Next, we introduce our second criterion.

(C2) For each $\varepsilon > 0$ there exists a $\tau > 0$ such that for any $t > 0$ and for some $\eta \in \mathrm{AC}([-t,\, 0])$,

$$\eta(-t) = \eta(0) = \gamma(-\tau) \quad \text{and} \quad \int_{-t}^{0} L(\eta(s),\, \dot\eta(s))\, ds < \varepsilon.$$

Theorem 4. *Under condition (C2), the convergence (6) holds.*

Our third criterion is the following.

(C3) For any $\varepsilon > 0$, there exists a $\tau > 0$ and for each $t \ge \tau$, a $\sigma(t) \in [0,\, \tau]$ such that

$$u_\infty(\gamma(-t)) + \varepsilon > u(\gamma(-t), \sigma(t)).$$

Note that the above inequality is equivalent to the condition that there is an $\eta \in \mathcal{C}([-\sigma(t),\, 0];\, \gamma(-t))$ such that

$$u_\infty(\gamma(-t)) + \varepsilon > \int_{-\sigma(t)}^{0} L(\eta(s),\, \dot\eta(s))\, ds + u_0(\eta(-\sigma(t)).$$

In our next theorem, condition (C3) is used together with one of the conditions $(\mathrm{A7})_\pm$ on H, which are certain strict convexity requirements on H. We set $Q := \{(x,p) \in \mathbb{R}^{2n} \mid H(x,p) = 0\}$ and

$$S := \{(x,\xi) \in \mathbb{R}^{2n} \mid (x,p) \in Q, \quad \xi \in D_2^- H(x,p) \ \text{ for some } p \in \mathbb{R}^n\},$$

where $D_2^- H(x,p)$ stands for the subdifferential of H with respect to the p-variable.

(A7)$_+$ (resp. **(A7)$_-$**) There exists a modulus ω satisfying $\omega(r) > 0$ for $r > 0$ such that for all $(x,p) \in Q$, $\xi \in D_2^- H(x,p)$ and $q \in \mathbb{R}^n$,

$$H(x, p+q) \geq \xi \cdot q + \omega((\xi \cdot q)_+) \qquad (\text{resp.} \quad \geq \xi \cdot q + \omega((\xi \cdot q)_-)), \tag{9}$$

where $r_\pm := \max\{\pm r, 0\}$ for $r \in \mathbb{R}$.

We remark here that condition (A7)$_+$ has already been used in [2, 12] to replace the strict convexity of $H(x, \cdot)$ in order to get the convergence (6) and also that condition (A7)$_-$ has been discussed in [13] when $n = 1$.

Theorem 5. *Assume that (C3) and either of (A7)$_+$ or (A7)$_-$ are satisfied. Then (6) holds.*

We will apply the criteria above, to obtain more specific conditions on (H, u_0) for the convergence (5).

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