Linearity Defects of Face Rings and Their Computation with Macaulay2

Master's Thesis

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INTRODUCTION

The main results of this thesis are joint works with Yanagawa, and most of them can be found in [**17**], except for Proposition 7.6.

Let $A = \bigoplus_{i \in \mathbb{N}} A_i$ be a graded (not necessarily commutative) Noetherian Koszul algebra over a field $K\,(\cong A_0)$. Let M be a finitely generated graded left A-module, and $P_\bullet$ its minimal free resolution. Eisenbud et al. [**5**] defined the *linear part* $\mathrm{lin}(P_\bullet)$ of $P_\bullet$, which is the complex obtained by erasing all terms of degree ≥ 2 from the matrices representing the differential maps of $P_\bullet$ (hence $\mathrm{lin}(P_\bullet)_i = P_i$ for all i). Following Herzog and Iyengar [**11**], we call $\mathrm{ld}_A(M) = \sup\{\, i \mid H_i(\mathrm{lin}(P_\bullet)) \neq 0 \,\}$ the *linearity defect* of M. This invariant and related concepts have been studied by several authors (e.g., [**5, 11, 15, 20, 28**]). We say a finitely generated graded A-module M is *componentwise linear* (or, *(weakly) Koszul* in some literature) if $M_{\langle i \rangle}$ has a linear free resolution for all i. Here $M_{\langle i \rangle}$ is the submodule of M generated by its degree i part M_i. Then we have

$$\mathrm{ld}_A(M) = \min\{\, i \mid \text{the } i^{\text{th}} \text{ syzygy of } M \text{ is componentwise linear} \,\}.$$

Let $E = \bigwedge\langle y_1, \ldots, y_n\rangle$ be an exterior algebra. Though for a finitely generated E-module N, $\mathrm{proj.dim}_E(N) = \infty$ holds in most cases, in [**5**, Theorem 3.1] Eisenbud et al. showed that the linearity defect of any finitely generated graded E-module is always finite. If $n \geq 2$, then we have $\sup\{\, \mathrm{ld}_E(N) \mid N \text{ is a finitely generated graded } E\text{-module} \,\} = \infty$. However Herzog-Römer and Yanagawa proved that if $J \subset E$ is a *monomial ideal*, i.e. an ideal generated by elements of the form $y_{i_1} \wedge y_{i_2} \wedge \cdots \wedge y_{i_t}$, then $\mathrm{ld}_E(E/J) \leq n-2$ if $n \geq 3$ (cf. [**20, 28**]).

Let Δ be a simplicial complex Δ on the vertex $[n] := \{1, \cdots, n\}$. It is well known that any monomial ideal of $E = \bigwedge\langle y_1, \ldots, y_n\rangle$ is always of the form $J_\Delta := (\, \prod_{i \in F} y_i \mid F \notin \Delta \,)$, and similarly any radical monomial ideal of a polynomial ring $S = K[x_1, \ldots, x_n]$ can be always written as $I_\Delta := (\, \prod_{i \in F} x_i \mid F \notin \Delta \,)$. In fact, we have $\mathrm{ld}_S(S/I_\Delta) = \mathrm{ld}_E(E/J_\Delta)$ due to Yanagawa (see Theorem 5.16), and so we set $\mathrm{ld}(\Delta) := \mathrm{ld}_S(S/I_\Delta) = \mathrm{ld}_E(E/J_\Delta)$. Main result of this thesis is the following:

Theorem 0.1. *Let Δ be a simplicial complex on $[n]$ with $n \geq 4$. Then $\mathrm{ld}(\Delta) = n-2$ if and only if Δ is an n-gon, i.e. the* geometric realization $|\Delta|$ *of Δ is homeomorphic to a circle $\mathbb{S}^1$.*

The organization of the thesis is as follows: Section 1 devotes to an introduction of some definition and basic facts in commutative algebra. Most of the facts there hold in more general condition, (e.g. over a Notherian local ring), but we restricted ourself to arguments over a positively graded algebra over a field K. The contents of this section and a more detailed exposition of them can be found in standard textbooks like Matsumura [**13**] or Bruns-Herzog [**3**].

Section 2 is a brief introduction of the concept "face ring" and some basic facts. In general, the word "face ring" can possibly contain an affine semigroup ring and so on, but in this thesis we treat only with Stanley-Reisner rings and exterior face rings (see Section 2 for their definition). Good references for the section and more detailed discussion are [**3**],[**14**] and [**21**].

Let S be a polynomial ring over a field K. In [**24**], Yanagawa introduced the concept of squarefree S-modules for more systematic argument of the theory around Stanley-Reisner rings. The greatest advantage of them is, the author guesses, that the category of squarefree S-modules becomes abelian with some useful properties. For example, this property enable us to argue using sheafs or derived categories ([**25**],[**26**],[**28**]). On the other hand, for an exterior algebra E over K, Römer [**18**] introduced the notion of squarefree E-module as a generation of an exterior face ring, which is the exterior version of a Stanley-Reisner ring. In Section 3, we introduce these squarefree modules, their properties, and the functors due to Römer [**18**] between the category of squarefree S-modules $\mathfrak{Sq}(S)$ and the category $\mathfrak{Sq}(E)$ of squarefree E-modules, which gives an equivalence $\mathfrak{Sq}(S) \cong \mathfrak{Sq}(E)$. We also mention the *Alexander dual functor* given by Römer [**18**]. It is a duality functor of $\mathfrak{Sq}(S)$ which is generalization of Alexander dual.

Section 4 devotes to an introduction of the relation between squarefree modules and Bernstein-Gel'fand-Gel'fand (abbr. BGG) correspondence, which is an equivalence $\mathfrak{D}^b(\mathfrak{mod}_{\mathbb{Z}}S) \cong \mathfrak{D}^b(\mathfrak{mod}_{\mathbb{Z}}E)$. Actually, by the similar way as the $\mathbb{Z}$-graded case, we can get the $\mathbb{Z}^n$-graded version of this correspondence $\mathfrak{D}^b(\mathfrak{mod}_{\mathbb{Z}^n}S) \cong \mathfrak{D}^b(\mathfrak{mod}_{\mathbb{Z}^n}E)$, and the functors ${}^*\mathscr{R}, {}^*\mathscr{L}$, giving its correspondence, can be written by the composition of 2 functors $\mathscr{D}, \mathscr{A}$ which have a combinatorial meaning, in the derived category $\mathfrak{D}^b(\mathfrak{Sq}(S))$ (resp. $\mathfrak{D}^b(\mathfrak{Sq}(E))$) of $\mathfrak{Sq}(S)$ (resp. $\mathfrak{Sq}(E)$), which can naturally be seen a full subcategory of $\mathfrak{D}^b(\mathfrak{mod}_{\mathbb{Z}^n}S)$ (resp. $\mathfrak{D}^b(\mathfrak{mod}_{\mathbb{Z}^n}E)$). The functors ${}^*\mathscr{R}, {}^*\mathscr{L}, \mathscr{D}, \mathscr{A}$ are very useful tools for studying squarefree modules and related things ([**26**]).

BGG correspondence brings some benefits to the study of linearity defect or squarefree modules, and using them, we can deduce many properties of linearity defect and squarefree modules ([**5, 26, 28, 27**]). In Section 5, we introduce the definition and some known results about linearity defects. We also mention the relation of BGG correspondence with linearity defect, properties yielded from this relation, and some important results of [**17**].

In Section 6, an upper bound of $\operatorname{ld}(\Delta)$ is studied. Though we stated above that there is a uniform bound $\operatorname{ld}(\Delta) \le n-2$ due to Herzog-Römer and Yanagawa, more precisely $\operatorname{ld}(N) \le n$ for $N \in \mathfrak{Sq}(E)$ (hence $\operatorname{ld}(\Delta) \le n$) was shown by Herog and Römer and then this result refined as $\operatorname{ld}(N) \le n-1$ for $N \in \mathfrak{Sq}(E)$ and $\operatorname{ld}(\Delta) \le n-2$, by Yanagawa. In this section, refining these bounds more, we give the following upper bound: for $N \in \mathfrak{Sq}(E)$,

$$\operatorname{ld}_E(N) \le \max\{\, 1, n - \operatorname{indeg}_E(N) - 1 \,\},$$

where $\operatorname{indeg}_E(N) := \min\{\, i \mid N_i \neq 0 \,\}$ (and the same bound holds for $M \in \mathfrak{Sq}(S)$). As a corollary, we obtained the following bound:

$$\operatorname{ld}(\Delta) \le \max\{\, 1, n - \operatorname{indeg}(\Delta) \,\},$$

where $\operatorname{indeg}(\Delta) := \operatorname{indeg}(I_\Delta) = \operatorname{indeg}(J_\Delta)$. Moreover, for $2 \le t \le n-2$, an example of a simplicial complex Δ with $\operatorname{indeg}(\Delta) = t$ which satisfies the equality $\operatorname{ld}(\Delta) = n - \operatorname{indeg}(\Delta)$, is given.

Since there is a uniform bound $\operatorname{ld}(\Delta) \le n-2$ if $n \ge 3$, it is natural to ask which simplicial complex satisfies the equality $\operatorname{ld}(\Delta) = n-2$. Theorem 0.1 gives an answer for this question. Most of Section 7 devotes to a proof of the theorem. As a by-product of the proof, the following lower bound is yielded:

$$\mathrm{ld}(\Delta) \geq \max\{\, \sharp F - \dim \Delta_F - 1 \mid F \subset [n], \Delta_F \text{ is Gorenstein with } \Delta \neq \emptyset, \{\emptyset\} \,\},$$

where we set the value of the right hand of the inequality to be 0 if there is no $F \subset [n]$ such that Δ_F is Gorenstein. In the last of the section, we gave an example which indicates that in the above lower bound the equality does not necessarily hold, and verified that this example is indeed the required, by means of computation with the software system, Macaulay2 [**9**].

In Appendix A, we give 3 programs by distinct methods which compute linearity defects with Macaulay2, and examples of computation using them.

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1. Preliminary

In this section, we shall introduce some basic concepts and results used in this paper. Good references for the section are [**3**] and [**13**].

Let K be a field, and R a finitely generated positively graded commutative K-algebra which is *conencted*, i.e., $R_0 \cong K$ and which is generated by R_1. Henceforth, throughout this thesis, all positively graded K-algebra are assumed to be connected and finitely generated by its component of degree 1. We denote, by $\mathfrak{Mod}_{\mathbb{Z}}R$, the category of $\mathbb{Z}$-graded R-modules and by $\mathfrak{mod}_{\mathbb{Z}}R$ the full subcategory of $\mathfrak{Mod}_{\mathbb{Z}}R$ consisting finitely generated $\mathbb{Z}$-graded R-modules. Following an usual convention, for $M \in \mathfrak{Mod}_{\mathbb{Z}}R$, M_i denotes the i^{th}-*homogeneous component* of M, i.e., the K-vector spaces consisting of homogeneous elements of M of degree i. Thus we can write as $R = \bigoplus_{i\in\mathbb{N}} R_i$ and $M = \bigoplus_{i\in\mathbb{Z}} M_i$ for $M \in \mathfrak{Mod}_{\mathbb{Z}}R$. Of course, a morphism f in $\mathfrak{Mod}_{\mathbb{Z}}R$ is a *degree preserving* R-module homomorphism, i.e., R-module homomorphism $f : M \to N$ with $M, N \in \mathfrak{Mod}_{\mathbb{Z}}R$ satisfying $f(M_i) \subset N_i$ for each i. A graded version of "Hom module" is of the form $\underline{\mathrm{Hom}}_R(M, N) := \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathfrak{Mod}_{\mathbb{Z}}R}(M(-i), N)$, where $M(-i)$ denotes the graded module with grading given by $(M(-i))_j = M_{j-i}$. Though in general, $\underline{\mathrm{Hom}}_R(M, N) \neq \mathrm{Hom}_R(M, N)$ for $M, N \in \mathfrak{Mod}_{\mathbb{Z}}R$, the equality holds if $M \in \mathfrak{mod}_{\mathbb{Z}}R$ (cf. [**3**, Exercise. 1.5.19]). Since we deal only with modules in $\mathfrak{mod}_{\mathbb{Z}}R$, we do not have to distinguish them.

For $M \in \mathfrak{mod}_{\mathbb{Z}}R$, we can construct the exact complex starts with M as follows: choose a minimal homogeneous system $f_1^0, \cdots, f_{b_0}^0$ of generators of M, and let the degree of f_i be d_i^0 for each i. Next, let $\partial_0 : \bigoplus_i R(-d_i^0) \to M$ be the R-homomorphism in $\mathfrak{mod}_{\mathbb{Z}}R$ given by $R(-d_i^0) \ni 1 \mapsto f_i^0$, and set $\Omega_1(M) := \ker \partial_0$. In turn choose a minimal homogeneous system $f_1^1, \cdots, f_{b_1}^1$ with degree $d_1^1, \cdots, d_{b_1}^1$ of generators of $\Omega_1(M)$, and define $\partial_1 : \bigoplus_i R(-d_i^1) \to \Omega_1(M)$ by $R(-d_i^1) \ni 1 \mapsto f_i^1$. Continuing this way, we obtain the exact sequence

$$\longrightarrow \bigoplus_{i=1}^{b_t} R(-d_i^t) \longrightarrow \cdots \longrightarrow \bigoplus_{i=1}^{b_1} R(-d_i^{b_1}) \longrightarrow \bigoplus_{i=1}^{b_0} R(-d_i^{b_0}) \longrightarrow M \longrightarrow 0. \tag{1.1}$$

A complex $P_\bullet : \cdots \to P_i \xrightarrow{\partial} P_{i-1} \to \cdots$ of R-modules is said to be *minimal* if it satisfies that $\partial(P_i) \subset \mathfrak{m}P_{i-1}$ for each i, where $\mathfrak{m} := \bigoplus_{i\geq 1} R_i$ is the graded maximal ideal of R. By construction, the above sequence (1.1) is minimal.

Definition 1.1. The acyclic complex $\cdots \to \bigoplus R(-d_i^{b_1}) \to R(-d_i^{b_0}) \to 0$, given by truncating the above sequence (1.1), is called a *minimal graded free resolution* of M, and $\Omega_i(M)$ the i^{th}-*syzygy module* of M (in the resolution).

It is well known that a minimal graded free resolution of M is uniquely determined up to an isomorphism (because of our assumption of R), and hence so is its syzygy modules. Note that putting the same shifts together, a minimal graded free resolution of M is of the following form:

$$\cdots \longrightarrow \bigoplus_{j\in\mathbb{Z}} R(-j)^{\beta_{i,j}(M)} \longrightarrow \cdots \longrightarrow \bigoplus_{j\in\mathbb{Z}} R(-j)^{\beta_{1,j}(M)} \longrightarrow \bigoplus_{j\in\mathbb{Z}} R(-j)^{\beta_{0,j}(M)} \longrightarrow 0.$$

The integer $\beta_i(M) := \sum_j \beta_{i,j}(M)$ (resp. $\beta_{i,j}(M)$) is called the i^{th}- (resp. $(i, j)^{th}$-*graded*) *betti number* of M. The projective dimension $\operatorname{proj.dim}_R(M)$ of M is, as is well known, equal to $\max\{\, i \mid \beta_i(M) \neq 0 \,\}$.

The following invariant is very important and studied well in commutative algebra:

Definition 1.2. Let $M \in \mathfrak{mod}_{\mathbb{Z}} R$. The value $\sup\{\, j - i \mid \beta_{i,j}(M) \neq 0 \,\}$ is called the *(Castelnuovo-Mumford) regularity of* M, and denoted by $\operatorname{reg}_R(M)$. We set the regularity of zero module to be $-\infty$.

Regularity can be characterized by the concept "linear resolution", which is defined as follows:

Definition 1.3. Let $M \in \mathfrak{mod}_{\mathbb{Z}} R$. We say M has a linear resolution if M has a system of generators with the same degrees, say d, and has a free resolution of the form:

$$\cdots \longrightarrow R(-i-d)^{\beta_{i,i+d}(M)} \longrightarrow \cdots R(-1-d)^{\beta_{1,1+d}(M)} \longrightarrow R(-d)^{\beta_{0,d}(M)} \longrightarrow 0.$$

Proposition 1.4. *The following are equivalent: for* $M \in \mathfrak{mod}_{\mathbb{Z}} R$,

(1) *M is q-regular, i,e.,* $\operatorname{reg}_R(M) \leq q$*;*
(2) *$M_{\geq q}$ has a linear resolution, where $M_{\geq q}$ denotes the trancated module* $\bigoplus_{i \geq q} M_i$.

When R is $\mathbb{Z}^n$-graded and $M \in \mathfrak{mod}_{\mathbb{Z}^n} R$, we can construct a minimal $\mathbb{Z}^n$-graded free resolution of M, similarly. Generally, it is of the form:

$$\cdots \longrightarrow \bigoplus_{\mathbf{a} \in \mathbb{Z}^n} R(-\mathbf{a})^{\beta_{i,\mathbf{a}}(M)} \longrightarrow \cdots \longrightarrow \bigoplus_{\mathbf{a} \in \mathbb{Z}^n} R(-\mathbf{a})^{\beta_{1,\mathbf{a}}(M)} \longrightarrow \bigoplus_{\mathbf{a} \in \mathbb{Z}^n} R(-\mathbf{a})^{\beta_{0,\mathbf{a}}(M)} \longrightarrow 0,$$

and we call $\beta_{i,\mathbf{a}}(M)$ $(i, \mathbf{a})^{th}$-$(\mathbb{Z}^n$-$)$*graded betti number of* M.

Koszul Complexes. Let $L \in \mathfrak{Mod}_{\mathbb{Z}} R$, and $f \in \underline{\operatorname{Hom}}_R(L, R)$. For simplicity, we set $\bigwedge^F \mathbf{x} := x_{i_1} \wedge \cdots \wedge x_{i_j}$ for $F = \{i_1, \cdots, i_j\}$ with $i_1 < \cdots < i_j$. Defining $\bigwedge^t L \to \bigwedge^{t-1} L$ by

$$\bigwedge^F \mathbf{x} \mapsto \sum_{i=1}^{t} (-1)^{i+1} f(x_i) \bigwedge^{F \setminus \{i\}} \mathbf{x},$$

where $F = \{1, \cdots, t\}$ and $x_i \in L$ for all $i \in F$, we obtain the complex

$$\mathcal{K}(f) : \cdots \longrightarrow \bigwedge^t L \xrightarrow{d_f} \bigwedge^{t-1} L \longrightarrow \cdots \bigwedge^2 L \xrightarrow{d_f} L \xrightarrow{f} R \longrightarrow 0.$$

Note that if L is free with basis $e_1, \cdots, e_n$, each $\bigwedge^t L$ is also free with basis $\bigwedge^F \mathbf{e}$, where F spans all the subsets $\{i_1, \cdots, i_t\} \subset \{1, \cdots, n\}$ with $i_1 < \cdots < i_t$. Thus $\mathcal{K}(f)$ is then a *free complex*, i.e. a complex whose components are free modules.

In the above, $f \in \underline{\operatorname{Hom}}_R(L, R)$ is arbitrary, but we need only the following case: given a sequence $\mathbf{x} := x_1, \cdots, x_n$ of elements of R with degrees $d_1, \cdots, d_n$ respectively, we can consider the R-linear map $f : \bigoplus_{i=1}^n R(-d_i) \to R$ by the assignment $R(-d_i) \ni 1 \mapsto x_1$. In this situation, we set $\mathcal{K}(\mathbf{x}) := \mathcal{K}(x_1, \cdots, x_n)$ to be the complex $\mathcal{K}(f)$ given as above.

Definition 1.5. With the above notation, the complex $\mathcal{K}(\mathbf{x})$ is called the *Koszul complex* with respect to $\mathbf{x} = x_1, \cdots, x_n$.

We can compute certain Tor-modules by Koszul complex.

Proposition 1.6. *Let* $\mathbf{x}$ *be a sequence in* R *consisting of homogeneous elements, and* $I = (\mathbf{x})$. *Then there is the isomorphism*

$$H_i(\mathbf{x}) := H_i(\mathcal{K}(\mathbf{x})) \cong \mathrm{Tor}_i^R(R/I, R).$$

The following is a criterion of the acyclicity of $\mathcal{K}(\mathbf{x})$.

Proposition 1.7. *Let* R *be a positively graded* K*-algebra,* $\mathfrak{m}$ *its unique maximal ideal, and* $I \subset \mathfrak{m}$ *an ideal generated by* $\mathbf{x} := x_1, \cdots x_n$. *Then the following are equivalent:*

(1) $H_i(\mathbf{x}) = 0$ *for all* $i > 0$*;*
(2) $H_1(\mathbf{x}) = 0$*;*
(3) $\mathbf{x}$ *is an* R*-regular sequence.*

When R is a polynomial ring $K[\mathbf{x}] = K[x_1, \cdots, x_n]$ with indeterminates $x_1, \cdots, x_n$ over a field K, $x_1, \cdots, x_n$ is a regular sequence, and moreover we have $R/(x_1, \cdots, x_n) = K$. Hence in this case, $\mathcal{K}(\mathbf{x})$ gives a minimal free resolution of K.

Corollary 1.8. *If* $R = K[\mathbf{x}]$, *then* $\mathcal{K}(\mathbf{x})$ *is a minimal free resolution of* K.

There is an important relation between $\operatorname{depth}_R M := \operatorname{grade}(\mathfrak{m}, M)$, the depth of M, i.e. the maximal length of M-regular sequence in $\mathfrak{m}$ (in general $\operatorname{depth}_R M \leq \dim M$), and $\operatorname{proj.dim}_R M$, due to Auslander and Buchsbaum:

Theorem 1.9 (Auslander-Buchsbaum)**.** *Let* R *be a positively graded* K*-algebra and let* $0 \neq M \in \mathfrak{mod}_{\mathbb{Z}} R$. *If* $\operatorname{proj.dim}_R M < \infty$, *then*

$$\operatorname{proj.dim}_R M + \operatorname{depth}_R M = \operatorname{depth}_R R.$$

Let R be a positively graded K-algebra. For a complex $C_\bullet$ of graded R-modules with the differential map ∂ and an integer i, $C_\bullet[i]$ denotes the complex given by translating $C_\bullet$ by i, that is, the complex with the differential map $(-1)^i\partial$ such that the component of homological degree j is C_{i+j}.

For a homogeneous element x of degree d, $C_\bullet \otimes_R \mathcal{K}(x)$ gives the "mapping cone" of the multiplication map $C_\bullet(-d) \to C_\bullet$ by x, that is, the following holds:

Proposition 1.10. *Let* R, x *be as above. For every complex* $C_\bullet$ *of graded* R*-modules, there exists an exact sequence*

$$0 \longrightarrow C_\bullet \longrightarrow C_\bullet \otimes_R \mathcal{K}(x) \longrightarrow C_\bullet(-d)[-1] \longrightarrow 0,$$

which induces the long exact sequence of homology

$$\cdots \longrightarrow H_i(C_\bullet) \longrightarrow H_i(C_\bullet \otimes_R \mathcal{K}(x)) \longrightarrow H_{i-1}(C_\bullet(-d)) \xrightarrow{x} H_{i-1}(C_\bullet) \longrightarrow \cdots.$$

Moreover, if x *is regular on* C_i *for each* i, *then there is an isomorphism* $H_\bullet(C_\bullet \otimes_R \mathcal{K}(x)) \cong H_\bullet(C_\bullet/xC_\bullet)$.

Local Cohomologies. Let R be a positively graded K-algebra, and $\mathfrak{m}$ be its unique graded maximal ideal. For $M \in \mathfrak{mod}_{\mathbb{Z}} R$, we define $H^i_{\mathfrak{m}}(M) := \varinjlim \operatorname{Ext}^i_R(R/\mathfrak{m}^k, M)$. $H^i_{\mathfrak{m}}(M)$ is naturally $\mathbb{Z}$-graded.

Definition 1.11. $H^i_{\mathfrak{m}}(M)$ is called the i^{th}*-local cohomology of* M.

Local cohomology is a useful tool to compute dimension and depth:

Theorem 1.12 (Grothendieck)**.** *For $M \in \mathfrak{mod}_{\mathbb{Z}} R$, we have*

(1) $H^i_{\mathfrak{m}}(M) \neq 0$ *for* $i = \dim M, \operatorname{depth}_R M$*;*
(2) $H^i_{\mathfrak{m}}(M) = 0$ *unless* $\operatorname{depth}_R \leq i \leq \dim M$.

Recall the definition of "Cohen-Macaulayness".

Definition 1.13. $M \in \mathfrak{mod}_{\mathbb{Z}} R$ is said to be *Cohen-Macaulay of dimension d* if $d = \operatorname{depth}_R M = \dim M$.

Now we can characterize Cohen-Macaulayness in terms of local cohomology.

Proposition 1.14. *M is Cohen-Macaulay of dimension d if and only if $H^i_{\mathfrak{m}}(M) = 0$ for $i \neq d$.*

Definition 1.15. Let R be a Cohen-Macaulay positively graded K-algebra of dimension d. A finitely generated graded R-module $C \in \mathfrak{mod}_{\mathbb{Z}} R$ is called a *canonical module*, denoted by ω_R, of R if there exist isomorphisms

$$\operatorname{Ext}^i_R(K, C) \cong \begin{cases} 0 & \text{for } i \neq d; \\ K & \text{for } i = d, \end{cases}$$

in $\mathfrak{mod}_{\mathbb{Z}} R$.

It is well known that a canonical module is uniquely determined up to isomorphisms in $\mathfrak{mod}_{\mathbb{Z}} R$.

Example 1.16. Let S be a polynomial ring $K[x_1, \cdots, x_n]$. Then $\omega_S = S(-n)$. Actually, ω_S is $\mathbb{Z}^n$-graded, and it follows that $\omega_S \cong S(-\mathbf{1})$, where $\mathbf{1} := \{1, \cdots, 1\} \in \mathbb{Z}^n$.

Another important property of local cohomology is the local duality:

Theorem 1.17 (local duality)**.** *Let R be a positively graded K-algebra with $R_0 = K$, and let $M \in \mathfrak{mod}_{\mathbb{Z}} R$. Then for all i, there are natural isomorphisms*

$$\operatorname{Hom}_K(H^i_{\mathfrak{m}}(M), K) \cong \operatorname{Ext}^{d-i}_R(M, \omega_R).$$

Specific rings. Let R be a positively graded K-algebra and $\mathfrak{m}$ its unique maximal ideal. Besides Cohen-Macaulayness, there is the various important concept of rings or modules. Of them, we shall recall the definition of Buchsbaumness and Gorensteinness.

Definition 1.18. $M \in \mathfrak{mod}_{\mathbb{Z}} R$ is said to be *Buchsbaum* if every system of parameters is weekly M-sequence. A positively graded K-algebra R is also said to be *Buchsbaum* if R itself is Buchusbaum as a graded R-module.

Definition 1.19. R is said to be *Gorenstein* if $\operatorname{inj.dim}_R R < \infty$.

There is the following hierarchy:

Theorem 1.20. *R is a polynomial ring $\Rightarrow$ R is Gorenstein $\Rightarrow$ R is Cohen-Macaulay $\Rightarrow$ R is Buchsbaum.*

For Gorensteinness, there are famous criterion:

Proposition 1.21. *Let R be Cohen-Macaulay with canonical module ω_R. Then the following are equivalent:*

(1) *R is Gorenstein;*
(2) $\omega_R \cong R(a)$ *for some* $a \in \mathbb{Z}$.

The following criterion is also well known:

Theorem 1.22 (Auslander-Buchsbaum-Serre)**.** *For a positively graded K-algebra R, the following are equivalent:*

(1) *R is a polynomial ring;*
(2) $\operatorname{proj.dim}_R M < \infty$ *for all* $M \in \mathfrak{mod}_{\mathbb{Z}} R$*;*
(3) $\operatorname{proj.dim}_R K < \infty$.

This theorem yields the finiteness of regularities of finitely generated graded modules over a polynomial ring, as a corollary.

Corollary 1.23. *If R is a polynomial ring, then* $\operatorname{reg}_R(M) < \infty$ *for all* $M \in \mathfrak{mod}_{\mathbb{Z}} R$.

Proof. Since $M \in \mathfrak{mod}_{\mathbb{Z}} R$ and R is Noetherian, its syzygies $\Omega_i(M)$ are all finitely generated for all $i \geq 0$. Moreover the above theorem ensures that $\operatorname{proj.dim}_R(M) < \infty$. Hence all $\beta_{i,j}(M)$ vanishes, expect for finitely many i, j, so that we have the required. □

Let R be a positively graded K-algebra with $R_0 = K$. Then we can write $R = S/I$, where $S := K[x_1, \cdots, x_n]$ is a polynomial ring over a field K, and I is a graded ideal of S. If R is Gorenstein, then there is a symmetry of the graded betti numbers of R as a graded S-module. This symmetry can be inferred from the following proposition:

Proposition 1.24. *If R is Cohen-Macaulay with the above notation, then the following holds;*

(1) $\omega_R \cong \operatorname{Ext}^t_S(R, \omega_S)$, *where* $t = n - \dim R$*;*
(2) *there is a minimal graded free S-resolution* $P_\bullet : 0 \to P_t \to \cdots P_1 \to P_0 \to 0$ *of R, where* $t = n - \dim R$, *and* $\operatorname{Hom}_S(P_\bullet, \omega_S) : 0 \to \operatorname{Hom}_S(P_0, \omega_S) \to \operatorname{Hom}_S(P_1, \omega_S) \to \cdots \to \operatorname{Hom}_S(P_t, \omega_S) \to 0$ *is a minimal graded free S-resolution of ω_R (Note that by Example 1.16* $\operatorname{Hom}_S(P_i, \omega_S)$ *is indeed free).*

By Example 1.16, Proposition 1.21, and the above Proposition, we have:

Proposition 1.25. *Let R, S, I be as above. Assume that R is Gorenstein, and let*

$$P_\bullet : 0 \longrightarrow P_t \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0 \qquad (*)$$

be a minimal graded free S-resolution of R with $t = n - \dim R$ *and* $P_t = \bigoplus_{i=1}^r S(-a_i)$. *We set* $a := \max\{ a_i \mid 1 \leq i \leq r \}$. *Then* $\operatorname{Hom}_S(P_\bullet, S)(-a)$ *is also a minimal graded free resolution of R. In particular, we have* $\beta_{i,j}(M) = \beta_{t-i,a-j}(M)$.

Proof. We denote $\operatorname{Hom}_S(P_i, S)$ by $\tilde{P}_i$. Proposition 1.24 ensures the existence of a minimal graded free resolution as $(*)$, and moreover in conjunction with Example 1.16, implies that $0 \to \tilde{P}_0(-n) \to \tilde{P}_1(-n) \to \cdots \to \tilde{P}_t(-n) \to 0$ is a minimal graded free resolution of ω_R. On the other hand, in our condition it follows that $-\min\{ i \mid [\omega_R]_i \neq 0 \} = -n + a$ (cf [**3**, Example 3.6.15]), and so we have $\omega_R \cong R(-n + a)$, by Proposition 1.21. Therefore shifting the above resolution of ω_R by $n - a$, we have the former of the required assertions. The second is an easy consequence of the former. □

2. Face Rings

There is a new branch, called “combinatorial commutative algebra”, of commutative algebra created by Hochster and Stanley in the middle of 1970’s. In this theory, a ”(symmetric) face ring” (usually called a Stanley-Reisner ring now), which is a quotient ring of a polynomial ring by a “squarefree” ideal, has been one of the central algebraic object ([**3**], [**14**], [**21**]), and recently, its exterior version, a “exterior face ring”, has come to be studied by several authors ([**1**], [**2**], [**18**], [**19**]). This section is devoted to a brief introduction to symmetric and exterior face rings.

First, we shall recall the definition and notation of a simplicial complex and some related things:

Definition 2.1. For a finite set $[n] := \{1, \cdots, n\}$, an (abstract) simplicial complex Δ on the *vertex set* $[n]$ is a subset of the power set $2^{[n]}$ of $[n]$ with the property that $F \subset G, G \in \Delta$ implies that $F \in \Delta$. (We do not assume that $\{i\} \in \Delta$ for each $i \in \Delta$.)

An element $F \in \Delta$ is called a *face of* Δ, and its *dimension* $\dim F$ is defined to be $\sharp F - 1$, where $\sharp F$ denotes the cardinality of F. Note that we distinguish the *empty* simplicial complex $\Delta = \{\varnothing\}$, i.e. the simplicial complex with the only face $\varnothing(\in 2^{[n]})$, from the *void* simplicial complex $\Delta = \varnothing$, i.e. the simplicial complex without faces. Thus the empty set $\varnothing(\in 2^{[n]})$ is a face of any non-void simplicial complex of dimension -1. The dimension $\dim \Delta$ of Δ is defined to be $\max\{\dim F \mid F \in \Delta\}$, and a face $F \in \Delta$ with $\dim F = \dim \Delta$ is called a *facet*. We set the dimension of the void simplicial complex to be $-\infty$.

Let K be a field, $S := K[x_1, \cdots, x_n]$ a polynomial ring over a field K, and $E := \bigwedge\langle y_1, \cdots, y_n\rangle$ an exterior algebra over K. Throughout this thesis, we henceforth fix the notations and follows the below convention:

Notation 2.2. For $F := \{i_1, \cdots, i_t\} \subset [n]$ with $i_1 < \cdots < i_t$, we set $\mathbf{x}^F := \prod_{i \in F} x_i$ and $\bigwedge^F \mathbf{y} := y_{i_1} \wedge \cdots \wedge y_{i_t}$, we often identify F with the vector $\sum_{i \in F} \mathbf{e}_i \in \mathbb{Z}^n$ where $\mathbf{e}_i := \{0, \cdots, 0, 1, 0, \cdots, 0\}$ denotes the i^{th} unit vector. (e.g. $M_F = M_{\{1,0,0,1,0,1\}}$ for $M \in \mathfrak{mod}_{\mathbb{Z}^n} S$ and $F = \{1, 4, 6\} \in [6]$).

Associating with a simplicial complex Δ on $[n]$, we can construct an ideal of S and E respectively as follows:

$$I_\Delta := (\, \mathbf{x}^F \mid F \notin \Delta\,) \subset S \qquad \text{and} \qquad J_\Delta := (\, \textstyle\bigwedge^F \mathbf{y} \mid F \notin \Delta\,) \subset E,$$

which we call the *(symmetric) face* (or *Stanley-Reisner*) *ideal of* Δ and the *exterior face ideal of* Δ. Face rings are quotient rings of S or E by these ideals. We set $I_\Delta = S$ (resp. $J_\Delta = E$) if Δ is void, and $I_\Delta = 0$ (resp. $J_\Delta = 0$) if Δ is the simplex $2^{[n]}$.

Definition 2.3. We call $K[\Delta] := S/I_\Delta$ the *(symmetric) face* (or *Stanley-Reisner*) *ring* of Δ over K, and $K\langle\Delta\rangle := E/J_\Delta$ the *exterior face ring* of Δ over K.

Remark 2.4. As is stated in the beginning of this section, it is most popular to call $K[\Delta]$ (resp. I_Δ) Stanley-Reisner rings (resp. ideals), but we intentionally defined by the name “symmetric” face rings (resp. ideals) in contrast with “exterior” ones. But the reader should note that exterior face rings (or ideals) is called otherwise in some literature; they are also called “indicator algebra”, or “exterior Stanley-Reisner rings”, and so on.

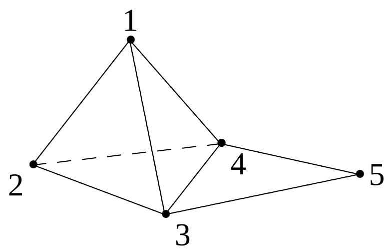
Recall an ideal I (resp. J) of S (resp. E) is called a *monomial ideal* if it is generated by elements of the form $\mathbf{x}^F$ (resp. $\bigwedge^F \mathbf{y}$) for some $F \subset [n]$.

The following is one of the most important reasons for the study of face rings:

Proposition 2.5. *Any* squarefree *ideal of* S*, i.e. a monomial radical ideal, can be written as* I_Δ *for some* Δ*, and similarly, any monomial ideal of* E *as* J_Δ *for some* Δ*.*

Thus the study of face ideals means the study of squarefree ideals of S or monomial ones of E.

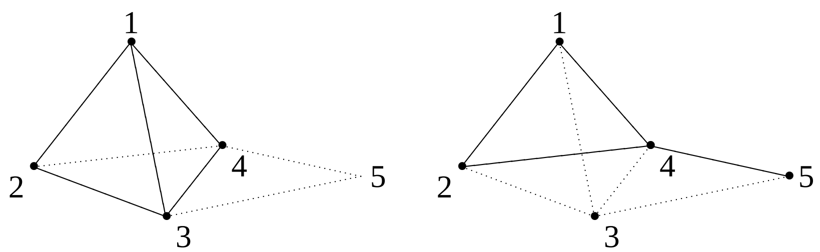
Example 2.6. Let Δ be a simplicial complex on $[5] = \{1, \cdots, 5\}$ consists of the shell of a tetrahedron and a triangle as follows:



Then the facets of Δ are $\{1, 2, 3\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 4\}, \{3, 4, 5\}$, and $\dim \Delta = 2$. The symmetric face ideal and exterior one are $I_\Delta = (x_1x_5, x_2x_5, x_1x_2x_3x_4)$ and $J_\Delta = (y_1 \wedge y_5, y_2 \wedge y_5, y_1 \wedge y_2 \wedge y_3 \wedge y_4)$

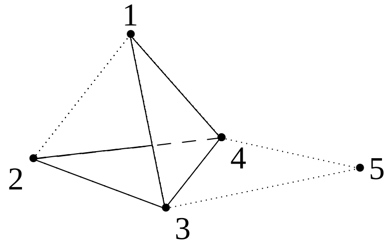
For a simplicial complex Δ on $[n]$, the following 2 operators are elementary: the *star* $\mathrm{st}_\Delta F := \{ G \in \Delta \mid F \cup G \in \Delta\}$ of F in Δ and the *link* $\mathrm{lk}_\Delta F := \{ G \in \Delta \mid F \cup G \in \Delta, F \cap G = \varnothing \}$ of F in Δ.

Example 2.7. Let Δ be a simplicial complex as in Example 2.6, and set $F := \{1, 3\}, v := \{3\}$. Then $\mathrm{st}_\Delta F$ consists of two triangles $\{1, 2, 3\}, \{1, 3, 4\}$, and $\mathrm{lk}_\Delta v$ of the edge $\{4, 5\}$ and the frame of the triangle $\{1, 2, 4\}$.



For a simplicial complex Δ on $[n]$, consider the subset $\Delta^\vee$ of $2^{[n]}$ consisting of $F \subset [n]$ with $F^c := [n] \setminus F \notin \Delta$, and thus $\Delta^\vee = \{ F \mid F^c \notin \Delta \}$. Then it is easy to verify that it is again a simplicial complex. We call this simplicial complex the *Alexander dual* of Δ.

Example 2.8. The Alexander dual of the simplicial complex given in Example 2.6 consists of two triangles $\{1,3,4\}, \{2,3,4\}$ and the vertex $\{5\}$:



The name "dual" comes from the following duality:

Proposition 2.9 (Alexander's duality)**.** *Let K be a field, and $\Gamma \subset \Delta \subset 2^{[n]}$ simplicial complexes. Then we have*

$$\tilde{H}_i(\Delta, \Gamma; K) \cong \tilde{H}^{n-2-i}(\Gamma^\vee, \Delta^\vee; K) \cong \tilde{H}_{n-2-i}(\Gamma^\vee, \Delta^\vee; K),$$

for all i, and in particular, $\tilde{H}_i(\Delta; K) \cong \tilde{H}^{n-3-i}(\Delta^\vee; K) \cong \tilde{H}_{n-3-i}(\Delta^\vee; K)$.

Given a simplicial complex Δ on $[n]$ and its face $F \subset [n]$, we denote, by Δ_F, the *restriction to F* of Δ which are defined as $\Delta_F := \{ G \in \Delta \mid G \subset F \}$.

Proposition 2.10. *Let Δ be a simplicial complex on $[n]$. Then it follows that*

$$\tilde{H}^{i-2}(\mathrm{lk}_{\Delta^\vee} F; K) \cong \tilde{H}_{\sharp F^c - i - 1}(\Delta_{F^c}; K)$$

for all $F \in \Delta^\vee$.

We can compute the dimensions of face rings combinatorially.

Proposition 2.11. *Let Δ be a simplicial complex on $[n]$. For $F \subset [n]$, we set $\mathfrak{p}_F := (x_i \mid i \notin F) \subset S$. Then we have $\mathrm{Ass}(K[\Delta]) = \{ \mathfrak{p}_F \mid F$ is a facet of $\Delta \}$. In particular, we have $\dim K[\Delta] = \dim \Delta + 1$.*

For a property $\mathscr{P}$ of rings, it is customary that the condition that the face ring $K[\Delta]$ of a simplicial complex Δ possesses the property is expressed by saying "Δ has $\mathscr{P}$". For instance, we say Δ is Cohen-Macaulay (over K) if so is $K[\Delta]$. (Though most properties we will deal with may depend on the field K, we omit the word "over K" unless necessity). Henceforth, we tacitly follows this custom.

The i^{th} *skeleton* $\Delta^{(i)}$ of a simplicial complex Δ is the subcomplex of Δ defined by $\{ F \in \Delta \mid \dim F \leq i \}$. The depth of $K[\Delta]$ can be calculate by investigating whether their skeletons are Cohen-Macaulay or not.

Proposition 2.12. *We have* $\mathrm{depth}_S(K[\Delta]) = \max\{ i \mid \Delta^{(i)}$ *is Cohen-Macaulay*$\} + 1$.

The following formula, due to Hochster, is well known, but in Section 7, we need explicit correspondence between $[\mathrm{Tor}_S^\bullet(K[\Delta], K)]_F$ and reduced cohomologies of Δ_F, and so we will introduce its proof.

Theorem 2.13 (Hochster)**.** *For a simplicial complex Δ on $[n]$, it follows that $\beta_{i,F}(K[\Delta]) = \dim_K \tilde{H}_{\sharp F - i - 1}(\Delta_F; K)$, and hence that*

$$\beta_{i,j}(K[\Delta]) = \sum_{F \subset [n], \sharp F = j} \dim_K \tilde{H}_{j-i-1}(\Delta_F; K),$$

where $\beta_{i,j}(K[\Delta])$ is the $\mathbb{Z}$-graded betti number of $K[\Delta]$.

Proof. Let $\mathcal{K}_\bullet := \mathcal{K}(x_1, \cdots, x_n) = S \otimes_K \bigwedge S_1$ be the Koszul complex with respect to $x_1, \ldots, x_n$. Note that by Corollary 1.8 $\mathcal{K}(x_1, \cdots, x_n)$ is a minimal graded free resolution of K. Then we have

$$[\mathrm{Tor}_S^i(K[\Delta], K)]_F = H_i([K[\Delta] \otimes_S \mathcal{K}_\bullet]_F) = H_i([K[\Delta] \otimes_K \bigwedge S_1]_F)$$

for $F \subset [n]$. Furthermore, the basis of the K-vector space $[K[\Delta] \otimes_K \bigwedge S_1]_F$ is of the form $\mathbf{x}^G \otimes \bigwedge^{F\setminus G}\mathbf{x}$ with $G \in \Delta_F$. Thus the assignment

$$\varphi^i : \tilde{C}^{i-1}(\Delta_F; K) \ni e_G^* \longmapsto (-1)^{\alpha(G,F)}\mathbf{x}^G \otimes \bigwedge^{F\setminus G}\mathbf{x} \in [K[\Delta] \otimes_K \bigwedge S_1]_F$$

with $G \in \Delta_F$ gives the isomorphism $\varphi^\bullet : \tilde{C}^\bullet(\Delta_F; K)[-1] \longrightarrow [K[\Delta] \otimes_K \bigwedge S_1]_F$ of chain complexes, where $\tilde{C}^{i-1}(\Delta_F; K)$ is the $(i-1)^{\text{st}}$ term of the augmented cochain complex of Δ_F over K, e_G is the basis element of $\tilde{C}_{i-1}(\Delta_F; K)$ corresponding to G, and e_G^* is the K-dual base of e_G. Here we set $\alpha(A, B) := \sharp\{(a, b) \mid a > b, a \in A, b \in B\}$ for $A, B \subset [n]$. Thus we have the isomorphism

$$\bar{\varphi} : \tilde{H}^{i-1}(\Delta_F; K) \longrightarrow [\mathrm{Tor}_S^{\sharp F - i}(K[\Delta], K)]_F, \tag{2.1}$$

and hence the assertion follows. □

There are the well-known criteria of Cohen-Macaulayness, Buchsbaumness, and Gorensteinness of simplicial complexes.

Theorem 2.14 (Reisner). *Let Δ be a simplicial complex on $[n]$. The following are equivalent:*

(1) *Δ is Cohen-Macaulay over K;*
(2) *$\tilde{H}_i(\mathrm{lk}_\Delta F; K) = 0$ for all $F \in \Delta$ and all $i < \dim \mathrm{lk}_\Delta F$.*

Theorem 2.15. *For a simplicial complex Δ of dimension $d-1$, the following are equivalent:*

(1) *Δ is Buchsbaum;*
(2) *Δ is pure and $K[\Delta]_\mathfrak{p}$ is Cohen-Macaulay for all prime ideals $\mathfrak{p} \neq \mathfrak{m}$;*
(3) *$H_\mathfrak{m}^i(K[\Delta])$ has finite length for all $i < d$;*
(4) *$H_\mathfrak{m}^i(K[\Delta])_\mathbf{a} = 0$ for all $\mathbf{a} \neq \mathbf{0}$ and $i < d$;*
(5) *$\tilde{H}_i(\mathrm{lk}_\Delta F; K) = 0$ for all $\emptyset \neq F \in \Delta$ and all $i < \dim \mathrm{lk}_\Delta F$;*
(6) *$H_\mathfrak{m}^i(K[\Delta]) \cong \tilde{H}_{i-1}(\Delta; K)$ for all $i < d$.*

As a corollary, the above criterion of Buchsbaumness yields the following:

Corollary 2.16. *If Δ is a manifold, then Δ is Buchsbaum.*

Let Δ be a simplicial complex on $[n]$. We set $\mathrm{core}[n] := \{i \in [n] \mid \mathrm{st}_\Delta i \neq \Delta\}$, and call the simplicial complex $\Delta_{core[n]}$ the *core of* Δ. Since any Δ can be written as $\Delta = \Delta_{\mathrm{core}[n]} * 2^{[n]\setminus\mathrm{core}[n]}$, we have $K[\Delta] \cong K[\Delta_{\mathrm{core}[n]}] \otimes K[[n] \setminus \mathrm{core}[n]] \cong K[\Delta_{\mathrm{core}[n]}][x_i \mid i \in [n] \setminus \mathrm{core}[n]]$, and hence Δ is Gorenstein if and only if so is $\Delta_{\mathrm{core}[n]}$.

Theorem 2.17. *Given a simplicial complex $\Delta \neq \emptyset$, the following are equivalent:*

(1) *Δ is Gorenstein over K;*

(2) *for all* $F \in \Delta_{\mathrm{core}[n]}$, *we have*

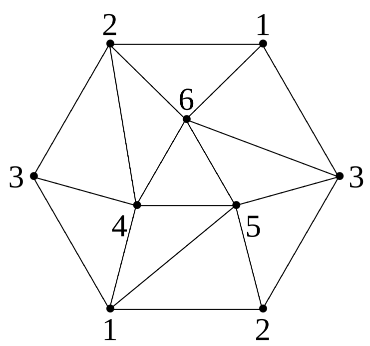
$$\tilde{H}_i(\mathrm{lk}_{\mathrm{core}\,\Delta} F; K) \cong \begin{cases} K & \text{if } i = \dim \mathrm{lk}_{\Delta_{\mathrm{core}[n]}} F, \\ 0 & \text{if } i < \dim \mathrm{lk}_{\Delta_{\mathrm{core}[n]}} F. \end{cases}$$

Definition 2.18. A simplicial complex Δ is said to be *Gorenstein** if Δ is Gorenstein and $\mathrm{core}[n] = [n]$.

Example 2.19. Clearly, $\Delta_{\mathrm{core}[n]}$ is Gorenstein*. Moreover, a simplicial complex Δ whose geometric realization $|\Delta|$ is homeomorphic to the $(d-1)$-dimensional sphere $\mathbb{S}^{d-1}$ (for $m > d$ there indeed exists a triangulation of $\mathbb{S}^{d-1}$ with m vertices (cf. [**3**, Proposition 5.5.10]), is Gorenstein*.

Since it is known that $\tilde{H}_i(\mathrm{lk}_\Delta F; K)$ is a topological invariant of $|\Delta|$ for $F \in \Delta$, we can deduce from these criteria that Cohen-Macaulayness, Buchsbaumness, and Gorensteinness* are a topological properties. But we should note that their property may depends on the field K. We complete this section by presenting the famous example which shows Cohen-Macaulayness may depend on the field K.

Example 2.20. Consider the triangulation of the real projective plane $P^2\mathbb{R}$ with 6 vertices as follows:



Since $P^2\mathbb{R}$ is connected, we have $\tilde{H}_0(\Delta; K) = 0$, but

$$\tilde{H}_1(\Delta; K) = \begin{cases} K & \text{if } \mathrm{char}\, K = 2; \\ 0 & \text{otherwise.} \end{cases}$$

Thus $K[\Delta]$ is Cohen-Macaulay if $\mathrm{char}\, K \neq 2$, but is not if $\mathrm{char}\, K = 2$.

3. Squarefree Modules

In [**24**], Yanagawa introduced the concept of "squarefree S-modules" to study face rings more systematically, and indicated, indeed, that many properties of face rings can be lifted up to this framework. Moreover, he gave some interesting results by means of this concept ([**24, 25, 26**]). On the other hand, Römer defined their exterior version, "squarefree E-modules", and detected, by means of the result of Aramova, Aramov, and Herzog ([**1**]), that there is an equivalence between the category consisting of squarefree S-modules and that of squarefree E-modules. Moreover, by use of these tools, he defined the "generalized Alexander duality functor", which is a generalization of Alexander duality. Our goal in this section is the brief introduction of these concepts. The interested reader should refer to the series of Yanagawa's paper [**24, 25, 26**], and Römer's [**18, 19**].

Throughout this section, S, E denote a polynomial ring $K[x_1, \cdots, x_n]$ over a field K and an exterior algebra $\bigwedge\langle y_1, \cdots, y_n\rangle$, respectively.

Definition 3.1. [**24**] A finitely generated graded module $M \in \mathfrak{mod}_{\mathbb{Z}^n} S$ is said to be *squarefree* if it is $\mathbb{N}^n$-graded, i.e., $M = \bigoplus_{\mathbf{a}\in\mathbb{N}} M_{\mathbf{a}}$ and satisfies that the multiplication map $M_{\mathbf{a}} \ni m \mapsto mx^{\mathbf{b}} \in M_{\mathbf{a}+\mathbf{b}}$ is bijective whenever $\mathrm{supp}(\mathbf{a}+\mathbf{b}) = \mathrm{supp}(\mathbf{a})$.

For example, every face rings and ideals are squarefree.

This definition is equivalent to say that M has a finite presentation of the form: $\bigoplus_{G\subset[n]} S(-G) \to \bigoplus_{F\subset[n]} S(-F) \to M$, whence in particular any squarefree module is finitely generated by its squarefree components.

By $\mathfrak{Sq}(S)$, we denote the category consisting of squarefree S-modules and degree preserving maps in $\mathfrak{mod}_{\mathbb{Z}^n} S$. $\mathfrak{Sq}(S)$ enjoys many nice properties, which enable us to study them and face rings, systematically.

Proposition 3.2. [**24**] *$\mathfrak{Sq}(S)$ is an abelian category, and has enough projectives and enough injectives. An indecomposable projective (resp. injective) module in $\mathfrak{Sq}(S)$ is isomorphic to $S(-F)$ (resp. $S/\mathfrak{p}_F$) for some $F \subset [n]$.*

The followings are immediate consequence of the above Proposition and the fact that $\mathrm{Hom}_S(S(-F), \omega_S) = S(-F^{\mathrm{c}})$.

Corollary 3.3. [**24**] *For $M \in \mathfrak{Sq}(S)$, its syzygy module $\Omega_i(M)$ and $\mathrm{Ext}^i_S(M, \omega_S) \in \mathfrak{Sq}(S)$ is again squarefree for all $i \geq 0$.*

Definition 3.4. [**18**] A finitely generated graded module $N \in \mathfrak{mod}_{\mathbb{Z}^n} E$ is said to be *squarefree* if $N_{\mathbf{a}} = 0$ for all non-squarefree $\mathbf{a} \in \mathbb{Z}^n$, i.e., $N = \bigoplus_{F\in[n]} N_F$.

Exterior face rings $K\langle\Delta\rangle$ and monomial ideals J_Δ are squarefree.

Let us denote, by $\mathfrak{Sq}(E)$, the category consisting of squarefree E-modules and degree preserving maps in $\mathfrak{mod}_{\mathbb{Z}^n} E$. Then it is clear that $\mathfrak{Sq}(E)$ is abelian.

Now we shall construct the functor $\mathbf{E}$ and $\mathbf{S}$ giving a equivalence between $\mathfrak{Sq}(S)$ and $\mathfrak{Sq}(E)$. Before that, The notation below is convenient: For monomials $u, v \in E$ with $\mathrm{supp}(v) \subset \mathrm{supp}(u)$, there is a unique monomial $w \in E$ such that $u = vw$; then we set $v^{-1}u := w$. For monomials $u, v, w \in E$, the following hold whenever the expressions can be defined: $(v^{-1}u)w = v^{-1}(uw)$ and $(w^{-1}v)(v^{-1}u) = w^{-1}u$.

Well, we set about the construction. Let $(P_\bullet, \partial)$ be a free complex in $\mathfrak{Sq}(S)$. Then each free module P_i has a basis B_i such that $\deg(f) \in \mathbb{N}^n$ is squarefree. For $\mathbf{a} \in \mathbb{N}^n$ and $f \in B_i$, we regard $\mathbf{y}^{(\mathbf{a})} f$ as a symbol with $\deg(\mathbf{y}^{\mathbf{a}} f) = \mathbf{a} + \deg(f)$. let $Q_j \in \mathfrak{mod}_{\mathbb{Z}^n} E$ be the $\mathbb{N}^n$-graded free E-module with basis $\mathbf{y}^{(\mathbf{a})} f$ where $\mathbf{a} \in \mathbb{N}^n$, $f \in B_i$, $\operatorname{supp}(\mathbf{a}) \subset \operatorname{supp}(f)$, and $j = \sharp\mathbf{a} + i$. Note that for $f \in B_i$, $\partial(f)$ can be written as $\partial(f) = \bigoplus_{g \in B_{i-1}} \lambda_g \mathbf{x}^{F-G} g$ with $\lambda_g \in K$, $F = \deg(f)$, and $G = \deg(g)$. By using this expression, define two morphisms $\eta_j, \xi_j : Q_j \to Q_{j-1}$ as follows:

$$\eta_j(\mathbf{y}^{\mathbf{a}} f) = (-1)^{\sharp F} \bigoplus_{k \in \operatorname{supp}(\mathbf{a})} \mathbf{y}^{\mathbf{a}-\mathbf{e}_k} f y_k, \quad \text{and} \quad \xi_j(\mathbf{y}^{\mathbf{a}} f) = (-1)^{\sharp \mathbf{a}} \bigoplus_{g \in B_{i-1}} \mathbf{y}^{\mathbf{a}} g \lambda_g (\textstyle\bigwedge^G \mathbf{y})^{-1} \textstyle\bigwedge^F \mathbf{y}.$$

We set $\delta := \eta + \xi$. Then it can be easily verified that $(Q_\bullet, \delta)$ forms a complex of $\mathbb{N}^n$-graded free modules. By the construction, the complex $(Q_\bullet, \delta)$ may depend on the choice of basis B_i, but in fact it is uniquely determined up to isomorphisms of complexes:

Proposition 3.5. [**1**] *Let $(Q'_\bullet, \delta')$ be the complex obtained from basis B'_i. Then we have $(Q_\bullet, \delta) \cong (Q'_\bullet, \delta')$ as complexes of $\mathbb{N}^n$-graded modules.*

Since each component of a minimal graded free resolution of $M \in \mathfrak{Sq}(S)$ is again in $\mathfrak{Sq}(S)$, we can apply the above construction to $P_\bullet$.

Theorem 3.6. [**1, 18**] *Let $M \in \mathfrak{Sq}(S)$ and $(P_\bullet, \partial)$ its minimal graded free resolution. If we apply the above construction to $(P_\bullet, \partial)$, then the resulting complex $(Q_\bullet, \delta)$ is also a minimal $\mathbb{N}^n$-graded free E-resolution of* $\operatorname{coker}(\delta_1 : Q_1 \to Q_0) \in \mathfrak{Sq}(E)$.

Now we can define the functor $\mathbf{E} : \mathfrak{Sq}(S) \to \mathfrak{Sq}(E)$: for $M \in \mathfrak{Sq}(S)$ and its minimal graded free resolution $(P_\bullet, \partial)$, set $\mathbf{E}(M) := \operatorname{coker}(\delta_1)$, where δ is the differential of the complex $(Q_\bullet, \delta)$ obtained from $P_\bullet$.

We shall set about the inverse of $\mathbf{E}$. Let $N \in \mathfrak{Sq}(E)$, $(Q'_\bullet, \delta)$ a minimal graded free resolution of N, B_i the basis of Q_i, and set $\bar{B}_i := \{ f \in B_i \mid \deg(f) \text{ is squarefree} \}$. Then we define the complex $(P_\bullet, \partial)$ of $\mathbb{N}^n$-graded free S-modules as follows: let P_i be a graded free module with basis $\bar{B}_i$, and when $f \in \bar{B}_i$ can be written as $\delta(f) = \bigoplus_{g \in \bar{B}_{i-1}} g \lambda_g (\bigwedge^G \mathbf{y})^{-1} \bigwedge^F \mathbf{y}$ with $F = \deg(f)$, $G = \deg(g)$, and $\lambda_g \in K$, we set

$$\partial_i(f) = \bigoplus_{g \in \bar{B}_{i-1}} g \lambda_g \mathbf{x}^{F-G}.$$

It is a routine to check that $(P_\bullet, \partial)$ is indeed a complex.

Theorem 3.7. [**18**] *Let $N \in \mathfrak{Sq}(E)$, and $(Q'_\bullet, \delta)$ its minimal graded free resolution. Then the complex $(P_\bullet, \partial)$ obtained by the above construction is also a minimal $\mathbb{N}^n$-graded free S-resolution of* $\operatorname{coker}(\partial_1 : P_1 \to P_0) \in \mathfrak{Sq}(S)$.

The functor $\mathbf{S} : \mathfrak{Sq}(E) \to \mathfrak{Sq}(S)$ is now defined by assigning $N \in \mathfrak{Sq}(E)$ to $\mathbf{S}(N) := \operatorname{coker}(\partial_1)$ where ∂ is the differential of the complex $(P_\bullet, \partial)$ obtained from $(Q_\bullet, \delta)$.

Proposition 3.8. [**18**] *The functors* $\mathbf{S}$ *and* $\mathbf{E}$ *are additive covariant exact functors giving an equivalence $\mathfrak{Sq}(S) \cong \mathfrak{Sq}(E)$.*

The following helps us grasp the properties of these functors: For $M \in \mathfrak{Sq}(S)$, $N \in \mathfrak{Sq}(E)$, and simplicial complexes $\Gamma \subset \Delta$, we have

(1) $\mathbf{E}(M)_F = M_F$ for all $F \subset [n]$;
(2) $\mathbf{S}(N)_F = N_F$ for all $F \subset [n]$;
(3) $\mathbf{E}(I_\Gamma/I_\Delta) = J_\Gamma/J_\Delta$;
(4) $\mathbf{S}(J_\Gamma/J_\Delta) = I_\Gamma/I_\Delta$.

We set $\mathbf{D}_E := \mathrm{Hom}_E(-, E)$. Then it is a contravariant functor from $\mathfrak{mod}_{\mathbb{Z}^n} E$ to itself, and moreover since $\mathbf{D}_E(N) \in \mathfrak{Sq}(E)$ for $N \in \mathfrak{Sq}(E)$, it is a contravariant functor from $\mathfrak{Sq}(E)$ to itself. It is known that $\mathbf{D}_E$ is exact (see [**2**]). The following hold: for simplicial complexes $\Gamma \subset \Delta$,

(1) $\mathbf{D}_E(J_\Gamma/J_\Delta) = J_{\Delta^\vee}/J_{\Gamma^\vee}$;
(2) in particular, $\mathbf{D}_E(K\langle\Delta\rangle) = J_{\Delta^\vee}$.

Definition 3.9. [**18**] $\mathbf{D}_E(N)$ is called the *(generalized) Alexander dual* of $N \in \mathfrak{Sq}(E)$.

Composing the functors $\mathbf{E}$, $\mathbf{S}$, and $\mathbf{D}_E$, we obtain the contravariant exact functor $\mathbf{A} := \mathbf{S} \circ \mathbf{D}_E \circ \mathbf{E} : \mathfrak{Sq}(S) \to \mathfrak{Sq}(S)$. Clearly, for simplicial complexes $\Gamma \subset \Delta$, we have

(1) $\mathbf{A}(I_\Gamma/I_\Delta) = I_{\Delta^\vee}/I_{\Gamma^\vee}$;
(2) in particular, $\mathbf{A}(K[\Delta]) = I_{\Delta^\vee}$.

Definition 3.10. [**18**] For $M \in \mathfrak{Sq}(S)$, we call $\mathbf{A}(M)$ the *(generalized) Alexander dual* of M, and $\mathbf{A}$ the *Alexander duality functor*.

There is the following beautiful equation.

Proposition 3.11 ([**20, 23**]). *For $M \in \mathfrak{Sq}(S)$, we have* $\mathrm{reg}_S(M) = \mathrm{proj.\,dim}_S(\mathbf{A}(M))$.

As a corollary, we have the following, which is a generalization of the following result due to Eagon and Reiner.

Corollary 3.12. [**7, 20**] *Let $M \in \mathfrak{Sq}(S)$. Then M is Cohen-Macaulay $\iff$ $\mathbf{A}(M)$ has a linear resolution.*

4. BGG Correspondence and Squarefree Modules

We denote by $\mathfrak{D}^b(\mathfrak{C})$ the derived category of bounded cochain complexes in a category $\mathfrak{C}$. Here we do not mention details of derived categories. We refer the reader to [**8**] for them. In this section, we introduce BGG correspondence, which is an equivalence $\mathfrak{D}^b(\mathfrak{mod}_{\mathbb{Z}}S) \cong \mathfrak{D}^b(\mathfrak{mod}_{\mathbb{Z}}E)$, and its relation to squarefree modules.

Let $M \in \mathfrak{mod}_{\mathbb{Z}}S$. For each i, $\mathrm{Hom}_K(E, M_i)$ has an $\mathbb{Z}$-graded E-module structure whose multiplication is given by $(af)(y) = f(ya)$ for $a, y \in E$ and $f \in \mathrm{Hom}_K(E_j, M_i)$, and whose grading by $[\mathrm{Hom}_K(E, M_i)]_p = \mathrm{Hom}_K(E_{i-p}, M_i)$. We define the functor $\mathbf{R}$ from $\mathfrak{mod}_{\mathbb{Z}}S$ to the category $\mathfrak{Com}(\mathfrak{mod}_{\mathbb{Z}}E)$ of cochain complexes in $\mathfrak{mod}_{\mathbb{Z}}E$ as follows: We set the part of cohomological degree i of $\mathbf{R}(M)$ to be $\mathrm{Hom}_K(E, M_i) \cong E(n-i)^{\dim_K(M_i)}$ and define the differential map $\phi : \mathbf{R}(M)^i \to \mathbf{R}(M)^{i+1}$ by

$$\mathrm{Hom}_K(E, M_i) \ni f \longmapsto (z \mapsto \sum_i x_i f(y_i z)) \in \mathrm{Hom}_K(E, M_{i+1}).$$

We can lift up $\mathscr{R}$ to the functor from the category $\mathfrak{Com}(\mathfrak{mod}_{\mathbb{Z}}S)$ of cochain complexes in $\mathfrak{mod}_{\mathbb{Z}}S$ to $\mathfrak{Com}(\mathfrak{mod}_{\mathbb{Z}}E)$ as follows: For a cochain complex $M^\bullet$ in $\mathfrak{mod}_{\mathbb{Z}}S$ with a differential map $\partial_{M^\bullet}$, we assign $\mathbf{R}(M^\bullet)$ to the total complex of a double complex such that $\mathbf{R}(M^\bullet)^{i,j} := \mathrm{Hom}_K(E, M^i_j)$, its vertical differential ψ is given by $\psi(f) = \partial_{M^\bullet} \circ f$ for $f \in \mathscr{R}(M^\bullet)^{i,j}$, and its horizontal one by ϕ. Thus the differential of $\mathbf{R}(M^\bullet)$ is given by

$$\mathbf{R}(M^\bullet)^i \supset \mathrm{Hom}_K(E, M^i_j) \ni f \longmapsto \phi(f) + (-1)^i \psi(f) \in \mathbf{R}(M^\bullet)^{i+1}.$$

The gradings of $\mathbf{R}(M^\bullet)$ is given by

$$\mathbf{R}(M^\bullet)^p_q := \bigoplus_{p=i+j, q=-l-j} \mathrm{Hom}_K(E_l, M^i_j).$$

Next, we shall construct the functor $\mathbf{L} : \mathfrak{Com}(\mathfrak{mod}_{\mathbb{Z}}E) \to \mathfrak{Com}(\mathfrak{mod}_{\mathbb{Z}}S)$. Let $N^\bullet$ be a complex in $\mathfrak{mod}_{\mathbb{Z}}E$ with a differential map $\partial_{N^\bullet}$. Then we define $\mathbf{L}(N^\bullet)$ to be the total complex of the double complex such that $\mathbf{L}(N^\bullet)^{i,j} := S \otimes_K N^i_j$ and the differential maps $\mathbf{L}(N^\bullet)^i \to \mathbf{L}(N^\bullet)^{i+1}$ are defined by

$$\mathbf{L}(N^\bullet)^i \supset S \otimes_K N^i_j \ni w \otimes z \longmapsto \sum_{k=1}^n (x_k w) \otimes (y_k z) + (-1)^i (w \otimes \partial_{N^\bullet}(z)) \in \mathbf{L}(N^\bullet)^{i+1}.$$

The gradings of $\mathbf{L}(N^\bullet)$ is given by

$$\mathbf{L}(N^\bullet)^p_q := \bigoplus_{p=i+j, q=l-j} S_l \otimes_K N^i_j.$$

We can compute homologies of $\mathbf{R}, \mathbf{L}$.

Proposition 4.1 ([**5, 28**])**.** *For $M \in \mathfrak{mod}_{\mathbb{Z}}S$ and $N \in \mathfrak{mod}_{\mathbb{Z}}E$, we have*

(1) $\mathrm{Tor}^S_i(K, M)_j \cong H^{j-i}(\mathbf{R}(M))_{-j}$;
(2) $\mathrm{Ext}^i_E(K, N)_j \cong H^{j-i}(\mathbf{L}(N))_{-j}$.

We can compute the regularity of given $M \in \mathfrak{mod}_{\mathbb{Z}}S$ with $\mathbf{R}$.

Corollary 4.2. $\mathrm{reg}_S(M) = \max\{\, i \mid H^i(\mathbf{R}(M)) \neq 0 \,\}$.

Proposition 1.23 and Corollary 4.2 implies that for $M \in \mathfrak{mod}_{\mathbb{Z}} S$, $\mathbf{R}(M)$ has finitely many non-vanishing cohomologies. Since $\mathbf{R}$ preserves quasi-isomorpshims, it induces a covariant functor $\mathfrak{D}^b(\mathfrak{mod}_{\mathbb{Z}} S) \to \mathfrak{D}^b(\mathfrak{mod}_{\mathbb{Z}} E)$, which we denote by $\mathscr{R}$. As for $\mathbf{L}$, it is clear that $\mathbf{L}(N)$ has finitely many non-vanishing cohomologies. Since $\mathbf{L}$ also preserves quasi-isomorphisms, it induces a covariant functor $\mathfrak{D}^b(\mathfrak{mod}_{\mathbb{Z}} E) \to \mathfrak{D}^b(\mathfrak{mod}_{\mathbb{Z}} S)$ as well, which we denote by $\mathscr{L}$.

Theorem 4.3 (BGG correspondence [**5**])**.** *The above functors* $\mathscr{R}, \mathscr{L}$ *give an equivalence* $\mathfrak{D}^b(\mathfrak{mod}_{\mathbb{Z}} S) \cong \mathfrak{D}^b(\mathfrak{mod}_{\mathbb{Z}} E)$.

A $\mathbb{Z}^n$-graded module M is regarded as a $\mathbb{Z}$-graded one by $M_i := \bigoplus_{\|\mathbf{a}\|=i} M_{\mathbf{a}}$, where $\|\mathbf{a}\| := a_1 + \cdots + a_n$, and moreover the functors $\mathbf{R}$ (resp. $\mathbf{L}$) takes a complex in $\mathfrak{mod}_{\mathbb{Z}^n} E$ (resp. $\mathfrak{mod}_{\mathbb{Z}^n} S$) to one in $\mathfrak{mod}_{\mathbb{Z}^n} S$ (resp. $\mathfrak{mod}_{\mathbb{Z}^n} S$), where the gradings are given as follows:

$$\mathbf{R}(M^\bullet)^p_{\mathbf{q}} := \bigoplus_{p=i+\|\mathbf{b}\|, \mathbf{q}=-\mathbf{a}-\mathbf{b}} \mathrm{Hom}_K(E_{\mathbf{a}}, M^i_{\mathbf{b}}) \quad \text{and} \quad \mathbf{L}(N^\bullet)^p_{\mathbf{q}} := \bigoplus_{p=i+\|\mathbf{b}\|, \mathbf{q}=\mathbf{a}-\mathbf{b}} S_{\mathbf{a}} \otimes_K N^i_{\mathbf{b}}$$

for $M^\bullet \in \mathfrak{Com}(\mathfrak{mod}_{\mathbb{Z}^n} S)$ and $N^\bullet \in \mathfrak{Com}(\mathfrak{mod}_{\mathbb{Z}^n} E)$. When we thus treat $\mathbf{R}, \mathbf{L}$ as the functors between $\mathfrak{Com}(\mathfrak{mod}_{\mathbb{Z}^n} S)$ and $\mathfrak{Com}(\mathfrak{mod}_{\mathbb{Z}^n} E)$, we write ${}^*\mathbf{R}$ (resp. ${}^*\mathbf{L}$) for $\mathbf{R}$ (resp. $\mathbf{L}$).

${}^*\mathbf{R}, {}^*\mathbf{L}$ induces the functors ${}^*\mathscr{R}, {}^*\mathscr{L}$ giving the $\mathbb{Z}^n$-graded version of BGG correspondence.

Theorem 4.4 ($\mathbb{Z}^n$-graded version of BGG correspondence [**26**])**.** *There is an equivalence* $\mathfrak{D}^b(\mathfrak{mod}_{\mathbb{Z}^n} S) \cong \mathfrak{D}^b(\mathfrak{mod}_{\mathbb{Z}^n} E)$.

Let ${}^*\mathbf{R_1}$ (resp. ${}^*\mathbf{L_{-1}}$) be the functor given by shifting ${}^*\mathbf{R}$ (resp. ${}^*\mathbf{L}$) by $\mathbf{1} = \{1, \cdots, 1\}$ (resp. $-\mathbf{1}$) respectively. That is, we set ${}^*\mathbf{R_1}(M^\bullet)^{i,j} = \mathrm{Hom}_K(E(-\mathbf{1}), M^i_j)$ for $M \in \mathfrak{mod}^n_{\mathbb{Z}} S$ and ${}^*\mathbf{L_{-1}}(M)^{i,j} = S(-\mathbf{1}) \otimes_K N^i_j$. We denote, by ${}^*\mathscr{R}_{\mathbf{1}}$ (resp. ${}^*\mathscr{L}_{\mathbf{-1}}$), the functor from $\mathfrak{D}^b(\mathfrak{mod}_{\mathbb{Z}^n} S)$ (resp. $\mathfrak{D}^b(\mathfrak{mod}_{\mathbb{Z}^n} E)$) to $\mathfrak{D}^b(\mathfrak{mod}_{\mathbb{Z}^n} E)$ (resp. $\mathfrak{D}^b(\mathfrak{mod}_{\mathbb{Z}^n} S)$ which is induced by ${}^*\mathbf{R_1}$ (resp. ${}^*\mathbf{L_{-1}}$).

Let us use R to denote S or E for simplicity. We denote, by $\mathfrak{D}^b_{\mathfrak{Sq}(R)}(\mathfrak{mod}_{\mathbb{Z}^n} R)$, the full subcategory of $\mathfrak{D}^b(\mathfrak{mod}_{\mathbb{Z}^n} R)$ consisting of objects $M^\bullet \in \mathfrak{D}^b(\mathfrak{mod}_{\mathbb{Z}^n} R)$ such that $H_i(M^\bullet) \in \mathfrak{Sq}(R)$ for all i.

Proposition 4.5 ([**26**])**.** *There exist natural equivalence* $\mathfrak{D}^b(\mathfrak{Sq}(S)) \cong \mathfrak{D}^b_{\mathfrak{Sq}(S)}(\mathfrak{mod}_{\mathbb{Z}^n} S)$ *and* $\mathfrak{D}^b(\mathfrak{Sq}(E)) \cong \mathfrak{D}^b_{\mathfrak{Sq}(E)}(\mathfrak{mod}_{\mathbb{Z}^n} E)$.

Thus $\mathfrak{D}^b(\mathfrak{Sq}(S))$ (resp. $\mathfrak{D}^b(\mathfrak{Sq}(E))$ can be regard as a full subcategory of $\mathfrak{D}^b(\mathfrak{mod}_{\mathbb{Z}^n} S)$ (resp. $\mathfrak{D}^b(\mathfrak{mod}_{\mathbb{Z}^n} E)$.

Next, we shall construct a functor from $\mathfrak{D}^b(\mathfrak{Sq}(S))$ to itself. For $M \in \mathfrak{Sq}(S)$, we define $\mathbf{D}(M)$ to be the complex

$$\mathbf{D}(M) : 0 \longrightarrow \mathbf{D}(M)^{-n} \longrightarrow \mathbf{D}(M)^{-n+1} \longrightarrow \cdots \longrightarrow \mathbf{D}(M)^0 \longrightarrow 0,$$

$$\mathbf{D}(M)^i := \bigoplus_{F \subset [n], \sharp F = -i} (M_F)^* \otimes_K (S/\mathfrak{p}_F),$$

where we set $(-)^* := \mathrm{Hom}_K(-, K)$, the degree $(M_F)^*$ is $\mathbf{0} := \{0, \cdots, 0\}$, and the differential map $\varphi : \mathbf{D}(M)^{-i} \to \mathbf{D}(M)^{-i+1}$ is the sum of the maps

$$(-1)^{\alpha(F,i)} \cdot (x_j)^* \otimes_K \mathrm{nat} : (M_F)^* \otimes_K S/\mathfrak{p}_F \longrightarrow (M_{F\setminus\{j\}})^* \otimes_K S/\mathfrak{p}_{F\setminus\{j\}}$$

with $\sharp F = i$, where $(x_j)^*$ denotes the K-dual of the multiplication map $x_j : M_{F\setminus\{j\}} \ni m \mapsto mx_j \in M_F$ and "nat" the natural surjective map $S/\mathfrak{p}_F \to S/\mathfrak{p}_{F\setminus\{j\}}$. Then we lift up $\mathbf{D}$ to a functor from $\mathfrak{Com}^b(\mathfrak{Sq}(S))$ to itself as follows: for $M^\bullet \in \mathfrak{Com}^b(\mathfrak{Sq}(S))$ with a differential map δ, we set $\mathbf{D}(M^\bullet)^t := \bigoplus_{i-j=t} \mathbf{D}(M^j)^i = \bigoplus_{-\sharp F - j = t} (M^j_F) \otimes_K S/\mathfrak{p}_F$, and define the differential map by

$$\mathbf{D}(M^\bullet)^t \supset (M^j_F) \otimes_K S/\mathfrak{p}_F \ni m \otimes x \longmapsto \varphi(m \otimes x) + (-1)^t(\delta^*(m) \otimes x) \in \mathbf{D}(M)^{t+1},$$

where δ^* is the K-dual of δ. $\mathbf{D}$ induces a contravariant functor from $\mathfrak{D}^b(\mathfrak{Sq}(S))$ to itself, which we also denote by $\mathscr{D}$.

The homology of $\mathscr{D}$ is an Ext module.

Proposition 4.6 ([**26**]). *For $M \in \mathfrak{Sq}(S)$, we have $H^i(\mathscr{D}(M)) = \mathrm{Ext}^{n+i}_S(M, \omega_S)$.*

Moreover, $\mathscr{D}$ is a duality of $\mathfrak{D}^b(\mathfrak{Sq}(S))$.

Proposition 4.7 ([**26**]). *It follows that $\mathscr{D} \circ \mathscr{D} \cong \mathrm{id}_{\mathfrak{D}^b(\mathfrak{Sq}(S))}$*

The functors $\mathbf{E}, \mathbf{S}, \mathbf{A}, \mathbf{D}_E$, introduced in the previous section, also induces a functor between derived categories; for example, $\mathbf{E}$ induces a covariant functor from $\mathfrak{D}^b(\mathfrak{Sq}(S))$ to $\mathfrak{D}^b(\mathfrak{Sq}(E))$. We denote the functors induced likewise by $\mathscr{E}, \mathscr{S}\ \mathscr{A}, \mathscr{D}_E$, respectively.

Remark 4.8. $\mathscr{D}$ has the following strange property [**26**]: we have

$$\mathscr{D} \circ \mathscr{A} \circ \mathscr{D} \circ \mathscr{A} \circ \mathscr{D} \circ \mathscr{A} \cong \mathscr{T}^{2n},$$

where $\mathscr{T}^i$ is a translation functor which assign a complex $M^\bullet \in \mathfrak{Com}^b(\mathfrak{mod}_{\mathbb{Z}} S)$ with the differential δ to the complex whose part of cohomological degree j is M^{i+j} and the differential is given by $(-1)^i \cdot \delta$.

We can describe $\mathscr{L}$ with $\mathscr{D}$ and $\mathscr{A}$.

Proposition 4.9 ([**26**]). *Let $N^\bullet \in \mathfrak{Com}^b(\mathfrak{Sq}(E))$. Then ${}^*\mathscr{L}_{-\mathbf{1}}(N^\bullet) \in \mathfrak{Com}^b(\mathfrak{Sq}(S))$ and hence $\mathscr{L}$ gives a functor from $\mathfrak{D}^b(\mathfrak{Sq}(E))$ to $\mathfrak{D}^b(\mathfrak{Sq}(S))$. Moreover it follows that ${}^*\mathscr{L}_{-\mathbf{1}} \circ \mathscr{E}(M^\bullet) = \mathscr{A} \circ \mathscr{D}(M^\bullet)$ for $M^\bullet \in \mathfrak{Com}^b(\mathfrak{Sq}(S))$.*

Since a free E-module $E(-\mathbf{a})$ is squarefree if and only if $\mathbf{a} = \mathbf{0}$, ${}^*\mathscr{R}_{\mathbf{1}}(M^\bullet)$ is not a complex of squarefree E-module. However the following holds:

Proposition 4.10 ([**26**]). *${}^*\mathscr{R}_{\mathbf{1}}(M^\bullet) \in \mathfrak{D}^b_{\mathfrak{Sq}(E)}(\mathfrak{mod}_{\mathbb{Z}^n} E) \cong \mathfrak{D}^b(\mathfrak{Sq}(E))$ for $M^\bullet \in \mathfrak{D}^b(\mathfrak{Sq}(S))$, and hence ${}^*\mathscr{R}_{\mathbf{1}}$ gives a functor from $\mathfrak{D}^b(\mathfrak{Sq}(S))$ to $\mathfrak{D}^b(\mathfrak{Sq}(E))$. Moreover we have a natural equivalence $\mathscr{S} \circ {}^*\mathscr{R}_{\mathbf{1}} \cong \mathscr{D} \circ \mathscr{A}$.*

5. Linearity Defects

Let $R = \bigoplus_{i \in \mathbb{N}} R_i$ be a positively graded (not necessary commutative) Noetherian K-algebra. (Recall the assumption, in the beginning of Section 1, that all the positively graded K-algebra are connected and generated by their component of degree 1.) We say R is *Koszul* if K has a linear resolution. A polynomial ring and an exterior algebra are typical examples of a Koszul algebra. Henceforth all the positively graded K-algebra are assumed to be Koszul.

Let $M \in \mathfrak{mod}_{\mathbb{Z}} R$ and $P_\bullet$ a minimal graded free resolution of M. Then erasing all terms of degree ≥ 2 from the matrices representing of the differential maps of $P_\bullet$, we obtain the new complex $\mathrm{lin}(P_\bullet)$ such that $\mathrm{lin}(P_\bullet)_i = P_i$.

Definition 5.1 (Eisenbud et.al [**5**])**.** With the above notation, we call $\mathrm{lin}(P_\bullet)$ the *linear part* of $P_\bullet$.

"Linearity defect" is the value indicating when a minimal graded free resolution begins to predominated by its linear part.

Definition 5.2 (Herzog-Iyengar [**11**])**.** Let $M \in \mathfrak{mod}_{\mathbb{Z}} R$ and let $P_\bullet$ be a minimal graded free resolution. We call $\mathrm{ld}_R(M) := \max\{\, i \mid H_i(\mathrm{lin}(P_\bullet)) \neq 0 \,\}$ the *linearity defect* of M.

Example 5.3. Let $S = \mathbb{Q}[x_1, \cdots, x_4]$ and $I := (x_1 - x_4, x_3^2 - 5x_3x_4 + 4x_4^2, x_2^2 - x_3x_4)$ be an ideal of S. We shall compute a minimal graded free resolution $P_\bullet$ of $M := S/I$ and its linear part with the software system Macaulay2 ([**9**]). $P_\bullet$ is as follows (see Appendix for commands):

```
i1 : S := QQ[x_1,x_2,x_3,x_4]

i2 : I = ideal(x_1 - x_4, x_3^2 - 5*x_3*x_4 + 4*x_4^2, x_2^2 - x_3*x_4)

i3 : C = res(S^1/I)

i4 : C.dd
           1                                                  3
o4 = 0 : S  <-------------------------------------------------- S  : 1
               | x_1-x_4 x_2^2-x_3x_4 x_3^2-5x_3x_4+4x_4^2 |

           3                                                                  3
     1 : S  <---------------------------------------------------------------- S  : 2
             {1} | -x_2^2+x_3x_4 -x_3^2+5x_3x_4-4x_4^2 0                     |
             {2} | x_1-x_4       0                     -x_3^2+5x_3x_4-4x_4^2 |
             {2} | 0             x_1-x_4               x_2^2-x_3x_4          |

           3                                  1
     2 : S  <-------------------------------- S  : 3
               {3} | x_3^2-5x_3x_4+4x_4^2 |
               {3} | -x_2^2+x_3x_4        |
               {4} | x_1-x_4              |

           1
     3 : S  <----- 0 : 4
               0
```

Hence the linear part of M is:

```
i5 : linC = linearPart C

i6 : linC.dd

           1                       3
o6 = 0 : S  <------------------- S  : 1
               | x_1-x_4 0 0 |

           3                                 3
     1 : S  <---------------------------- S  : 2
               {1} | 0       0       0 |
               {2} | x_1-x_4 0       0 |
               {2} | 0       x_1-x_4 0 |

           3                       1
     2 : S  <------------------- S  : 3
               {3} | 0       |
               {3} | 0       |
               {4} | x_1-x_4 |

           1
     3 : S  <----- 0 : 4
               0
```

Thus we have $\mathrm{ld}_S(M) = 2$.

It is clear that $\mathrm{ld}_R(M) \le \mathrm{proj.dim}_R(M)$. Hence when $R = S$, there is a uniform bound of linearity defects of $M \in \mathfrak{mod}_{\mathbb{Z}}S$; we have $\sup\{\, \mathrm{ld}_S(M) \mid M \in \mathfrak{mod}_{\mathbb{Z}}S \,\} \le n$.

Now consider the case $R = E := \bigwedge\langle y_1, \cdots, y_n\rangle$, an exterior algebra. Note that it follows that we have $\mathrm{proj.dim}_E(N) = \infty$ for $N \in \mathfrak{mod}_{\mathbb{Z}}E$, unless N is free; in fact assume that $\mathrm{proj.dim}_E(N) = t < \infty$; then by induction on t, there is the exact sequence

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow N \longrightarrow 0,$$

where the P_i are finitely generated free modules for $i = 0, 1$. Since E is injective, so is P_1, and hence P_1 is a direct summand of P_0, so that we deduce that N is free.

Thus in this case we can not easily see whether or not $\mathrm{ld}_E(N), N \in \mathfrak{mod}_{\mathbb{Z}}E$ is finite and there is a uniform bound. Actually, the latter assertion is false when $n \ge 2$; in fact we can construct $N \in \mathfrak{mod}_{\mathbb{Z}}E$ with $\mathrm{ld}_E(N) = i$ for given $i \ge 0$, for we can take ”cosyzygies” of N since E itself is an injective E-module. On the other hand, the following holds from Proposition 1.23 and Theorem 5.11 in the below:

Proposition 5.4 (Eisenbud et.al [**5**]). $\mathrm{ld}_E(N)$ *is finite for all* $N \in \mathfrak{mod}_{\mathbb{Z}}E$.

More precisely, we have:

Proposition 5.5 (Yanagawa [**27**]). *For* $N \in \mathfrak{mod}_{\mathbb{Z}}E$, $\mathrm{ld}_E(N) \le t^{n!}2^{(n-1)!}$ *holds, where* $t := \max\{\, \dim_K N_i \mid i \in \mathbb{Z} \,\}$.

Linearity defects of squarefree modules behaves better than those of finite modules:

Theorem 5.6 (Herzog-Römer [**20**]). *We have* $\mathrm{ld}_E(N) \le \mathrm{proj.dim}_S(\mathscr{S}(N)) \le n$ *for* $N \in \mathfrak{Sq}(E)$.

In particular, we have $\mathrm{ld}_E(K\langle\Delta\rangle) \leq n$ for any simplicial complex Δ on $[n]$. This fact is one of motivation of our study; we want to know "what can we say more about $\mathrm{ld}_E(K\langle\Delta\rangle)$".

It is noteworthy that we can characterize linearity defects by the concept "componentwise linear".

Definition 5.7 (Herzog-Hibi [**10**]). $M \in \mathfrak{mod}_{\mathbb{Z}} R$ is said to be *componentwise linear* if $M_{\langle i\rangle}$ has a linear resolution for all i, where $M_{\langle i\rangle}$ is the submodule of M generated by M_i.

Proposition 5.8 (Römer and Yanagawa). *It follows that*

$$\mathrm{ld}_R(M) = \min\{\, i \mid \textit{The } i^{th} \textit{ syzygy } \Omega_i(M) \textit{ is componentwise linear} \,\}$$

for $M \in \mathfrak{mod}_{\mathbb{Z}} R$ (if R is Koszul).

Linear parts become easier to handle, described by "linear strands".

Definition 5.9. Let R be a positively graded (not necessary commutative) Noetherian Koszul K-algebra, $M \in \mathfrak{mod}_{\mathbb{Z}} R$, and let $P_\bullet$ be a minimal graded free resolution of M. The l^{th}- *linear strand* $\mathrm{lin}_l(P_\bullet)$ of $P_\bullet$ is the complex as follows:

(1) $\mathrm{lin}_l(P)_i = S(-i-l)^{\beta_{i,i+l}(M)}$, which is a direct summand of P_i;
(2) differential maps $\mathrm{lin}_l(P_\bullet)_i \to \mathrm{lin}_l(P_\bullet)_{i-1}$ are corresponding components of differential maps of $P_\bullet$.

Let $R, M, P_\bullet$ be as above. Then we can easily verify that $\mathrm{lin}(P_\bullet) = \bigoplus_{l\in\mathbb{Z}} \mathrm{lin}_l(P_\bullet)$, and hence we have $\mathrm{ld}_R(M) = \max\{\, i \mid H_i(\mathrm{lin}_l(P_\bullet)) \neq 0, l \in \mathbb{Z} \,\}$.

Let $M^\bullet \in \mathfrak{Com}(\mathfrak{mod}_{\mathbb{Z}} R)$. We define $\mathscr{H}$ to be a functor from $\mathfrak{Com}(\mathfrak{mod}_{\mathbb{Z}} R)$ to itself such that $\mathscr{H}^i(M^\bullet) = H^i(M^\bullet)$ and the differential maps are zero maps.

We can consider a chain complex $P_\bullet : \cdots \to P_i \to P_{i-1} \to \cdots$ as the cochain one $P^\bullet : \cdots \to P^{-i} \to P^{-i+1} \to \cdots$ by setting $P^i := P_{-i}$, and henceforth we deal with a chain complex as a cochain one, if necessary.

When R is the polynomial ring $S = K[x_1, \cdots, x_n]$ or the exterior algebra $E = \bigwedge\langle y_1, \cdots, y_n\rangle$, there is the following beautiful description by $\mathscr{H}$ and the functors $\mathscr{R}, \mathscr{L}$ in the previous section.

Theorem 5.10 ([**5, 26**]). *Let $M \in \mathfrak{mod}_{\mathbb{Z}} S$, $N \in \mathfrak{mod}_{\mathbb{Z}} E$, $P_\bullet$ a minimal graded free resolution of M, and $Q_\bullet$ a minimal graded free resolution of N. Then we have*

$$\mathrm{lin}(P_\bullet) \cong \mathscr{L} \circ \mathscr{H} \circ \mathscr{R}(M) \quad \textit{and} \quad \mathrm{lin}(Q_\bullet) \cong \mathscr{D}_E \circ \mathscr{R} \circ \mathscr{H} \circ \mathscr{L} \circ \mathscr{D}_E(N)$$

in $\mathfrak{D}^b(\mathfrak{mod}_{\mathbb{Z}} S)$ and $\mathfrak{D}^b(\mathfrak{mod}_{\mathbb{Z}} E)$, respectively. Here $\mathrm{lin}(P_\bullet)$ and $\mathrm{lin}(Q_\bullet)$ are considered as cochain complexes.

More precisely, it follows that

$$\mathrm{lin}_l(P_\bullet) \cong \mathscr{L}(H^l(\mathscr{R}(M)))[-l] \quad \textit{and} \quad \mathrm{lin}_l(Q_\bullet) \cong \mathscr{D}_E \circ \mathscr{R}(H^{n-l}(\mathscr{L} \circ \mathscr{D}_E(N)))[n-l].$$

Theorem 5.11 ([**5, 26**]). *We have*

$$\mathrm{ld}_E(N) = \max\{\, \mathrm{reg}_S(H^i(\mathscr{L} \circ \mathscr{D}_E(N))) + i \mid i \in \mathbb{Z} \,\}$$

for $N \in \mathfrak{mod}_{\mathbb{Z}} E$.

Proof. Let $Q_\bullet$ be a minimal graded free resolution of N. It suffices to show that $\max\{\, i \mid H_i(\mathrm{lin}_l(Q_\bullet)) \neq 0 \,\} = \mathrm{reg}_S(H^{n-l}(\mathscr{L} \circ \mathscr{D}_E(N))) + n - l$. By Theorem 5.10, we have

$$\begin{aligned} H_i(\mathrm{lin}_l(Q_\bullet)) = H^{-i}(\mathrm{lin}_l(Q_\bullet)) &\cong H^{-i}(\mathscr{D}_E \circ \mathscr{R}(H^{n-l}(\mathscr{L} \circ \mathscr{D}_E(N)))[n-l]) \\ &= H^{-i+n-l}(\mathscr{D}_E \circ \mathscr{R}(H^{n-l}(\mathscr{L} \circ \mathscr{D}_E(N)))) \\ &\cong \mathscr{D}_E(H^{i-n+l}(\mathscr{R}(H^{n-l}(\mathscr{L} \circ \mathscr{D}_E(N))))), \end{aligned}$$

where the last isomorphism follows from the exactness of the contravariant functor $\mathscr{D}_E$. Therefore by Corollary 4.2 we conclude that $\max\{\, i \mid H_i(\mathrm{lin}_l(Q_\bullet)) \neq 0 \,\} = \max\{\, i \mid H^{i-n+l}(\mathscr{R}(H^{n-l}(\mathscr{L} \circ \mathscr{D}_E(N)))) \neq 0 \,\} = \mathrm{reg}_S(H^{n-l}(\mathscr{L} \circ \mathscr{D}_E(N))) + n - l$. □

Furthermore, using Proposition 4.9, we have the following:

Corollary 5.12 ([**26**]). *For $N \in \mathfrak{Sq}(E)$, it follows that*

$$\mathrm{ld}_E(N) = \max\{\, n - l - \mathrm{depth}_S(\mathrm{Ext}^l_S(\mathscr{S} \circ \mathscr{D}_E(N), \omega_S)) \mid 0 \leq l \leq n \,\},$$

where we set the depth of the zero module to be ∞.

Proof. By Theorem 5.11, it suffices to show that

$$\mathrm{reg}_S(H^{n-l}(\mathscr{L} \circ \mathscr{D}_E(N))) = -\,\mathrm{depth}_S(\mathrm{Ext}^l_S(\mathscr{S} \circ \mathscr{D}_E(N), \omega_S)).$$

By Proposition 4.9 we have ${}^*\mathscr{L}_{-\mathbf{1}} \cong \mathscr{A} \circ \mathscr{D} \circ \mathscr{S}$, which implies that ${}^*\mathscr{L} \circ \mathscr{D}_E(N) \cong \mathscr{A} \circ \mathscr{D} \circ \mathscr{S} \circ \mathscr{D}_E(N)(\mathbf{1})$. Hence it follows from Proposition 4.6 and the exactness of the contravariant functor $\mathscr{A}$ that $H^{n-l}({}^*\mathscr{L} \circ \mathscr{D}_E(N)) \cong H^{n-l}(\mathscr{A} \circ \mathscr{D} \circ \mathscr{S} \circ \mathscr{D}_E(N)(\mathbf{1})) \cong \mathscr{A}(H^{-n+l}(\mathscr{D} \circ \mathscr{S} \circ \mathscr{D}_E(N)))(\mathbf{1})$. Since $H^{n-l}({}^*\mathscr{L} \circ \mathscr{D}_E(N)) \cong H^{n-l}(\mathscr{L} \circ \mathscr{D}_E(N))$ as $\mathbb{Z}$-graded modules, by Proposition 3.11 we have

$$\begin{aligned} \mathrm{reg}_S(H^{n-l}(\mathscr{L} \circ \mathscr{D}_E(N))) &= \mathrm{reg}_S(\mathscr{A}(\mathrm{Ext}^l_S(\mathscr{S} \circ \mathscr{D}_E(N), \omega_S))) - n \\ &= \mathrm{proj.\,dim}_S(\mathrm{Ext}^l_S(\mathscr{S} \circ \mathscr{D}_E(N), \omega_S)) - n \\ &= -\,\mathrm{depth}_S(\mathrm{Ext}^l_S(\mathscr{S} \circ \mathscr{D}_E(N), \omega_S)) \end{aligned}$$

as required, where the last equality follows from Auslander-Buchsbaum formula. □

We shall introduce some important results in [**17**] in the following. Since these results are shown by Yanagawa, we do not give their proof here, except for Theorem 5.16.

Let $M \in \mathfrak{Sq}(S)$. $\mathrm{ld}_S(M)$ can be computed with the depth of $\mathrm{Ext}^\bullet_S(M, \omega_S)$, which will be made use of in Appendix A as a method for commuting linearity defects.

Proposition 5.13 ([**17**]). *Let $M \in \mathfrak{Sq}(S)$, and $P_\bullet$ its minimal graded free resolution. Then*

$$\max\{\, i \mid H_i(\mathrm{lin}_l(P_\bullet)) \neq 0 \,\} = n - l - \mathrm{depth}_S(\mathrm{Ext}^l_S(\mathscr{A}(M), \omega_S)),$$

and hence

$$\mathrm{ld}_S(M) = \max\{\, i - \mathrm{depth}_S(\mathrm{Ext}^{n-i}_S(\mathscr{A}(M), \omega_S)) \mid 0 \leq i \leq n \,\}.$$

Let us recall the definition of the concept “sequentially Cohen-Macaulay”.

Definition 5.14. Let R be a positively graded K-algebra and let $M \in \mathrm{mod}_{\mathbb{Z}} R$. M is said to be *sequentially Cohen-Macaulay* if M has a (unique) filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_t = M$ with $\dim(M_i/M_{i-1}) < \dim(M_{i+1}/M_i)$ for $i = 1, \cdots, n-1$ such that M_i/M_{i-1} is Cohen-Macaulay for $i = 1, \cdots, n$.

The following characterization of sequentially Cohen-Macaulayness is well known.

Proposition 5.15. *M is sequentially Cohen-Macaulay if and only if $\operatorname{Ext}^i_S(M, \omega_S)$ is Cohen-Macaulay of dimension $n - i$ for each i.*

By Proposition 5.15, we see that Proposition 5.13 is a generation of the following well-known result [**10, 19**]: for $M \in \mathfrak{Sq}(S)$,

$$M \text{ is sequentially Cohen-Macaulay} \iff \mathscr{A}(M) \text{ is componentwise linear.}$$

Of course, the latter assertion is equivalent to say that $\operatorname{ld}_S(\mathscr{A}(M)) = 0$.

Proposition 5.13 and Corollary 5.12 yields the following surprising result.

Theorem 5.16 ([**17**])**.** *For $N \in \mathfrak{Sq}(E)$, we have $\operatorname{ld}_E(N) = \operatorname{ld}_S(\mathscr{S}(N)) \le n - 1$. In particular, for a simplicial complex $\Delta \subset 2^{[n]}$, we have $\operatorname{ld}_E(K\langle\Delta\rangle) = \operatorname{ld}_S(K[\Delta])$.*

Proof. Since $\mathscr{S} \circ \mathscr{D}_E \cong \mathscr{S} \circ \mathscr{D}_E \circ \mathscr{E} \circ \mathscr{S} = \mathscr{A} \circ \mathscr{S}$, Proposition 5.12 and Proposition 5.13 yield that

$$\begin{aligned} \operatorname{ld}_E(N) &= \max\{\, n - l - \operatorname{depth}_S(\operatorname{Ext}^l_S(\mathscr{S} \circ \mathscr{D}_E(N), \omega_S)) \mid 0 \le l \le n \,\} \\ &= \max\{\, n - l - \operatorname{depth}_S(\operatorname{Ext}^l_S(\mathscr{A} \circ \mathscr{S}(N), \omega_S)) \mid 0 \le l \le n \,\} \\ &= \operatorname{ld}_S(\mathscr{S}(N)) \end{aligned}$$

as required. □

By Theorem 5.16, we may set $\operatorname{ld}(\Delta) := \operatorname{ld}_S(K[\Delta]) = \operatorname{ld}_E(K\langle\Delta\rangle)$.

As was shown by Yanagawa, the Alexander dual $\Delta^\vee$ of Δ is essential for $\operatorname{ld}(\Delta)$.

Theorem 5.17 ([**17**])**.** *If $I_\Delta \neq (0)$ (equivalently, $\Delta \neq 2^{[n]}$), then $\operatorname{ld}_S(I_\Delta)$ is a topological invariant of the geometric realization $|\Delta^\vee|$ of the Alexander dual $\Delta^\vee$ of Δ. If $\Delta \neq 2^T$ for any $T \subset [n]$, then $\operatorname{ld}(\Delta)$ is also a topological invariant of $|\Delta^\vee|$ (also independent from the number $n = \dim S$).*

Remark 5.18. $\operatorname{ld}(\Delta)$ depends on the characteristic of K. Let Δ be the Alexander dual of a triangulation of $P^2\mathbb{R}$. Since $P^2\mathbb{R}$ is a manifold, $K[\Delta^\vee]$ is Buchsbaum. Hence we have $H^2_\mathfrak{m}(K[\Delta^\vee]) \cong \tilde{H}_1(\Delta^\vee; K)$ by Theorem 2.15. So, if $\operatorname{char} K = 2$, then we have $\operatorname{depth}_S(\operatorname{Ext}^4_S(K[\Delta^\vee], \omega_S)) = 0$ by Theorem 1.17. Thus by Proposition 5.13, $\operatorname{ld}(\Delta) \ge 3 - \operatorname{depth}(\operatorname{Ext}^3_S(I_{\Delta^\vee}, \omega_S)) = 3$. On the other hand, according to Corollary 6.3 in the next section, we have $\operatorname{ld}(\Delta) \le \dim \Delta^\vee + 1 = 3$. Therefore $\operatorname{ld}(\Delta) = 3$ when $\operatorname{char} K = 2$. In the case $\operatorname{char} K \neq 2$, $\Delta^\vee$ is Cohen-Macauly, and hence I_Δ has a linear resolution by Corollary 3.12. Summing up, we have

$$\operatorname{ld}(\Delta) = \begin{cases} 3 & \text{if } \operatorname{char} K = 2; \\ 1 & \text{otherwise.} \end{cases}$$

Finally, we give a conjecture due to Herzog which is another motivation of the study.

Conjecture 5.19 (Herzog)**.** *Let Δ be a simplicial complex Δ on $[n]$. Then*

$$\operatorname{ld}(\Delta) + \operatorname{ld}(\Delta^\vee) \le n.$$

Since the simplicial complex Δ in Example 2.20 satisfies $\Delta = \Delta^\vee$, the above remark and Theorem 5.17 implies that $\operatorname{ld}(\Delta) + \operatorname{ld}(\Delta^\vee) = 6 = n$ if $\operatorname{char} K = 2$, and thus Δ is an example which satisfies the equality of the conjecture. No counterexample has not been found yet.

6. An Upper Bound of Linearity Defects

In the previous section, we stated that $\operatorname{ld}_E(N) = \operatorname{ld}_S(\mathscr{S}(N))$ for $N \in \mathfrak{Sq}(E)$, in particular $\operatorname{ld}_E(K\langle\Delta\rangle) = \operatorname{ld}_S(K[\Delta])$ for a simplicial complex Δ. In this section, we will give an upper bound of them, and see that the bound is sharp.

Let $0 \neq N \in \operatorname{mod}_{\mathbb{Z}}E$ and $0 \neq M \in \operatorname{mod}_{\mathbb{Z}}S$. We set $\operatorname{indeg}_E(N) := \min\{\, i \mid N_i \neq 0 \,\}$ and $\operatorname{indeg}_S(M) := \min\{\, i \mid M_i \neq 0 \,\}$, which we call the *initial degree* of N, M, respectively. Note that for a simplicial complex Δ on $[n]$, unless $\Delta \neq 2^{[n]}$ (or equivalently, $I_\Delta \neq 0$ or $J_\Delta \neq 0$), we have $\operatorname{indeg}_S(I_\Delta) = \operatorname{indeg}_E(J_\Delta) = \min\{\, \sharp F \mid F \subset [n], F \notin \Delta \,\}$, where $\sharp F$ denotes the cardinal number of F. So we set $\operatorname{indeg}(\Delta) := \operatorname{indeg}_S(I_\Delta) = \operatorname{indeg}_E(J_\Delta)$. It is clear that $\operatorname{indeg}(\Delta) = n - \dim \Delta^\vee - 1$.

Since $\operatorname{ld}(2^{[n]}) = \operatorname{ld}_S(S) = \operatorname{ld}_E(E) = 0$, we henceforth exclude this trivial case; we assume that $\Delta \neq 2^{[n]}$.

We often make use of the following facts:

Lemma 6.1. *Let $0 \neq M \in \operatorname{mod}_{\mathbb{Z}}S$ and let $P_\bullet$ be a minimal graded free resolution of M. Then*

(1) $\operatorname{lin}_l(P_\bullet) = 0$ *for all $l < \operatorname{indeg}_S(M)$ or $l > \operatorname{reg}_S(M)$, i.e., there are only l-linear strands with $\operatorname{indeg}_S(M) \leq l \leq \operatorname{reg}_S(M)$ in $P_\bullet$;*
(2) $\operatorname{lin}_{\operatorname{indeg}_S(M)}(P_\bullet)$ *is a subcomplex of $P_\bullet$;*
(3) *if $M \in \mathfrak{Sq}(S)$, then $\operatorname{lin}(P_\bullet) = \bigoplus_{0 \leq l \leq n} \operatorname{lin}_l(P_\bullet)$, and $\operatorname{lin}_l(P_\bullet)_i = 0$ for $i > n - l$ and all $0 \leq l \leq n$, where the subscript i is a homological degree.*

Proof. (1) and (2) are clear. (3) follows since $P_i \cong \bigoplus_{F \subset [n]} S(-F)^{\beta_{i,F}(M)}$. □

Theorem 6.2. *For $N \in \mathfrak{Sq}(E)$, it follows that*

$$\operatorname{ld}_E(N) \leq \max\{\, 0, n - \operatorname{indeg}_E(N) - 1 \,\}.$$

By Theorem 5.16. this is equivalent to say that for $M \in \mathfrak{Sq}(S)$,

$$\operatorname{ld}_S(M) \leq \max\{\, 0, n - \operatorname{indeg}_S(M) - 1 \,\}.$$

Proof. It suffices to show the assertion for $M \in \mathfrak{Sq}(S)$. Set $\operatorname{indeg}_S(M) = d$ and let $P_\bullet$ be a minimal graded free resolution of M. The case $d = n$ is trivial by Lemma 6.1 (1), (3). Assume that $d \leq n - 1$. Then the last few steps of $P_\bullet$ is of the form

$$0 \longrightarrow S(-n)^{\beta_{n-d,n}(M)} \longrightarrow S(-n)^{\beta_{n-d-1,n}(M)} \oplus S(-n+1)^{\beta_{n-d-1,n-1}(M)}$$

by Lemma 6.1 (1), (3), again. Hence $\operatorname{lin}_d(P_\bullet)_{n-d} = S(-n)^{\beta_{n-d,n}(M)} = P_{n-d}$. Since $\operatorname{lin}_d(P_\bullet)$ is a subcomplex of the acyclic complex $P_\bullet$ by Lemma 6.1 (2), it follows $H_{n-d}(\operatorname{lin}_d(P_\bullet)) = 0$, so that $\operatorname{ld}_S(M) \leq n - d - 1$. □

Note that $J_\Delta \in \mathfrak{Sq}(E)$ (resp. $I_\Delta \in \mathfrak{Sq}(S)$). Since $\operatorname{ld}(\Delta) \leq \operatorname{ld}_E(J_\Delta) + 1$ (resp. $\operatorname{ld}(\Delta) \leq \operatorname{ld}_S(I_\Delta) + 1$), we have a bound for $\operatorname{ld}(\Delta)$, applying Theorem 6.2 to J_Δ (resp. I_Δ).

Corollary 6.3. *For a simplicial complex Δ on $[n]$, we have*

$$\operatorname{ld}(\Delta) \leq \max\{\, 1, n - \operatorname{indeg}\Delta \,\} = \max\{\, 1, \dim \Delta^\vee + 1 \,\}.$$

Let Δ be a simplicial complex on $[n]$. For our convenience, we set $\operatorname{ver}(\Delta) := \{\, v \in [n] \mid \{v\} \in \Delta \,\}$. If $\operatorname{ver}(\Delta) \neq [n]$, then Δ can be regarded as a simplicial complex on a set V with $\operatorname{ver}(\Delta) \subset V \subset [n]$. To avoid confusion, when we treat Δ in this view, we write Δ^V, instead of Δ. Note that $K[\Delta^V] = S^V/I_{\Delta^V} \cong K[\Delta]$, where $S^V := K[x_i \mid i \in V] \cong K[V]$ and $I_{\Delta^V} = (\mathbf{x}^F \in S^V \mid F \subset V, F \notin \Delta) = I_\Delta \cap S^V$ ($K\langle\Delta^V\rangle$, E^V, and J_{Δ^V} are also similar). Thus $\mathrm{ld}(\Delta^V) = \mathrm{ld}_{S^V}(K[\Delta^V]) = \mathrm{ld}_{E^V}(K\langle\Delta^V\rangle)$, and $\mathrm{indeg}(\Delta^V) = \mathrm{indeg}_{S^V}(K[\Delta^V]) = \mathrm{indeg}_{E^V}(K[\Delta^V])$.

For two simplicial complexes Δ and Γ, we denote, by $\Delta * \Gamma$, the *join* $\{F \cup G \mid F \in \Delta, G \in \Gamma\}$ of Δ and Γ.

Proposition 6.4. *Let* Δ *be a simplicial complex on* $[n]$*. Assume that* $\mathrm{indeg}(\Delta) = 1$, *or equivalently* $\mathrm{ver}(\Delta) \neq [n]$*. Then we have* $\mathrm{ld}(\Delta) = \mathrm{ld}(\Delta * 2^{[n]\setminus V}) = \mathrm{ld}(\Delta^V)$ *for any* $\mathrm{ver}(\Delta) \subset V \subset [n]$.

Proof. We have $(\Delta^{V''})^{V'} = \Delta^{V'}$ for two sets V', V'' with $\mathrm{ver}(\Delta) \subset V' \subset V'' \subset [n]$, and

$$\begin{aligned}\Delta * 2^{[n]\setminus V} = \Delta * 2^{\{v_1,\cdots,v_r\}} &= (\Delta * 2^{\{v_1,\cdots,v_{r-1}\}}) * \{v_r\} \\ &= \cdots = (\cdots((\Delta * \{v_1\}) * \{v_2\}) \cdots) * \{v_r\} \\ &= (\cdots((\Delta * 2^{[n]\setminus V_1}) * 2^{[n]\setminus V_2}) \cdots) * 2^{[n]\setminus V_r},\end{aligned}$$

where we set $[n] \setminus V = \{v_1, \cdots, v_r\}$ and $V_i := [n] \setminus \{v_i\}$ (which hence implies that $\mathrm{ver}(\Delta) \subset V_i \subset [n]$, for $1 \leq i \leq r$), and where we denote, by $\{v_i\}$, the simplex $2^{\{v_i\}}$ with the only vertex v_1. Without loss of generality, we may thus assume $\sharp([n] \setminus V) = 1$ and hence that $[n] \setminus V = \{1\}$.

Now, let $P_\bullet$ be a minimal graded free resolution of $K[\Delta * \{1\}]$ and $\mathcal{K}(x_1)$ the Koszul complex with respect to x_1. Consider the complex $P_\bullet \otimes_S \mathcal{K}(x_1)$. Then by Proposition 1.10, we have $H_i(P_\bullet \otimes_S \mathcal{K}(x_1)) \cong H_i(P_\bullet/x_1P_\bullet)$. Since x_1 is S-regular and $K[\Delta * \{1\}]$-regular, $P_\bullet/xP_\bullet$ is, as is well known, acyclic (cf. [**3**, Proposition 1.1.5]). Thus $P_\bullet \otimes_S \mathcal{K}(x_1)$ is acyclic and hence a minimal graded free S-resolution of $K[\Delta]$. On the other hand, since $H_0(P_\bullet/x_1P_\bullet) = K[\Delta * \{1\}] \otimes_S S/(x_1) = K[\Delta^V]$, the complex $P_\bullet/x_1P_\bullet$ is a minimal graded free S^V-resolution of $K[\Delta^V]$. Note that $\mathrm{lin}(P_\bullet) \otimes_S S/(x_1) = \mathrm{lin}(P_\bullet/x_1P_\bullet)$ and $\mathrm{lin}(P_\bullet \otimes_S \mathcal{K}(x_1)) = \mathrm{lin}(P_\bullet) \otimes_S \mathcal{K}(x_1)$; the first equality is easy to verify, and as for the second, we have

$$\begin{aligned}\mathrm{lin}_l(P_\bullet \otimes_S \mathcal{K}(x_1))_i &= \mathrm{lin}_l(P_\bullet \otimes_S S)_i \oplus \mathrm{lin}_l(P_\bullet[-1] \otimes_S S(-1))_i \\ &= (\mathrm{lin}_l(P_\bullet)_i \otimes_S S) \oplus (\mathrm{lin}_l(P_\bullet)_{i-1} \otimes_S S(-1)) \\ &= (\mathrm{lin}_l(P_\bullet) \otimes_S \mathcal{K}(x_1))_i,\end{aligned}$$

where the subscripts i denote homological degrees, and the differential map $\mathrm{lin}_l(P_\bullet \otimes_S \mathcal{K}(x_1))_i \longrightarrow \mathrm{lin}_l(P_\bullet \otimes_S \mathcal{K}(x_1))_{i-1}$ is composed by $\partial_i^{\langle l\rangle}$, $-\partial_{i-1}^{\langle l\rangle}$, and the multiplication map by x_1, where $\partial_i^{\langle l\rangle}$ (resp. $\partial_{i-1}^{\langle l\rangle}$) is the i^{th}(resp. $(i-1)^{\mathrm{st}}$) differential map of the l-linear strand of $P_\bullet$. Hence the isomorphism $H_i(\mathrm{lin}(P_\bullet \otimes_S \mathcal{K}(x_1))) \cong H_i(\mathrm{lin}(P_\bullet/x_1P_\bullet))$ for all i, which follows from again Proposition 1.10, yields the second equality of the required assertion. For the first one, we notice that by Proposition 1.10 there is the exact sequence

$$0 \longrightarrow \mathrm{lin}(P_\bullet) \longrightarrow \mathrm{lin}(P_\bullet \otimes_S \mathcal{K}(x_1)) \longrightarrow \mathrm{lin}(P_\bullet)(-1)[-1] \longrightarrow 0.$$

From the long exact sequence of homology induced by this sequence, we can deduce that $H_i(\mathrm{lin}(P_\bullet \otimes_S \mathcal{K}(x_1))) = 0$ for all $i \geq \mathrm{ld}(\Delta * \{1\}) + 2$ and that there is the exact sequence

$$0 \longrightarrow H_{\mathrm{ld}(\Delta*\{1\})+1}(\mathrm{lin}(P_\bullet \otimes_S \mathcal{K}(x_1))) \longrightarrow H_{\mathrm{ld}(\Delta*\{1\})}(\mathrm{lin}(P_\bullet)(-1)) \\ \xrightarrow{x_1} H_{\mathrm{ld}(\Delta*\{1\})}(\mathrm{lin}(P_\bullet)) \longrightarrow H_{\mathrm{ld}(\Delta*\{1\})}(\mathrm{lin}(P_\bullet \otimes_S \mathcal{K}(x_1))).$$

Since x_1 does not appear in any entry of the matrices representing the differentials of $\mathrm{lin}(P_\bullet)$, it is regular on $H_\bullet(\mathrm{lin}(P_\bullet))$, hence we have $H_{\mathrm{ld}(\Delta*\{1\})+1}(\mathrm{lin}(P_\bullet \otimes_S \mathcal{K}(x_1))) = 0$, and Nakayama's lemma implies that $H_{\mathrm{ld}(\Delta*\{1\})}(\mathrm{lin}(P_\bullet \otimes_S \mathcal{K}(x_1))) \neq 0$, since $H_{\mathrm{ld}(\Delta*\{1\})}(\mathrm{lin}(P_\bullet)) \neq 0$. Therefore the required assertion holds. □

By the above proposition, we can deduce the following:

Corollary 6.5. *Let* Δ *be a simplicial complex on* $[n]$*. Then we have*

$$\mathrm{ld}(\Delta) \leq \max\{1, \sharp\,\mathrm{ver}(\Delta) - \mathrm{indeg}(\Delta^{\mathrm{ver}(\Delta)})\} = \max\{1, \dim{(\Delta^{\mathrm{ver}(\Delta)})^\vee} + 1\}.$$

In particular, if $\mathrm{indeg}\,\Delta = 1$*,*

$$\mathrm{ld}(\Delta) \leq \max\{1, n-3\}.$$

Hence as is already shown in [**28**, Proposition 4.15], we have

Corollary 6.6. *For any simplicial complex* Δ *on* $[n]$*,*

$$\mathrm{ld}(\Delta) \leq \{1, n-2\}.$$

Example 6.7. According to [**28**, Proposition 4.14], we can construct a squarefree module $N \in \mathfrak{Sq}(E)$ with $\mathrm{ld}_E(N) = \mathrm{proj.dim}_S(\mathscr{S}(N)) = n-1$. By its construction it is easy to check that $\mathrm{indeg}_E(N) = 0$, and hence by Theorem 5.16 $M := \mathscr{S}(N)$ satisfies that $\mathrm{indeg}_S(M) = 0$ and $\mathrm{ld}_S(M) = n-1$. For $0 \leq i \leq n-1$, let $\Omega_i(M)$ be the i^{th} syzygy of M. Then $\Omega_i(M)$ is squarefree, and we have that $\mathrm{ld}_S(\Omega_i(M)) = \mathrm{ld}_S(M) - i = n - i - 1$ and $\mathrm{indeg}_S(\Omega_i(M)) \geq \mathrm{indeg}_S(M) + i = i$. Thus by Theorem 6.2, we know that $\mathrm{indeg}_S(\Omega_i(M)) = i$ and $\mathrm{ld}_S(\Omega_i(M)) = n - \mathrm{indeg}_S(\Omega_i(M)) - 1$. So the bound in Theorem 6.2 is optimal.

Taking the above examples into consideration, it is natural to ask whether the bound given in Corollary 6.3 is optimal or not. Indeed, we can construct simplicial complexes satisfying the equality $\mathrm{ld}(\Delta) = n - \mathrm{indeg}\,\Delta$, when $1 \leq \dim\Delta \leq n-3$.

Example 6.8. Set $\Sigma := 2^{[n]}$, and let Γ be a simplicial complex on $[n]$ whose geometric realization $|\Gamma|$ is homeomorphic to the $(d-1)$-dimensional sphere $\mathbb{S}^{d-1}$ with $2 \leq d < n-1$. Consider the simplicial complex $\Delta := \Gamma \cup \Sigma^{(d-2)}$. We will verify that Δ is a desired complex, that is, $\mathrm{ld}(\Delta) = n - \mathrm{indeg}\,\Delta$. For brief notation, we put $t := \mathrm{indeg}\,\Delta$ and $l := \mathrm{ld}(\Delta)$.

First, from our definition, it is clear that $t \geq d$. Thus it is enough to show that $n-d \leq l$; in fact we have that $l \leq n-t \leq n-d \leq l$ by Corollary 6.3, and hence that $t = d$ and $l = n-d$. Our aim is to prove that

$$\beta_{n-d,n}(K[\Delta]) \neq 0 \quad \text{and} \quad \beta_{n-d-1,n-1}(K[\Delta]) = 0,$$

since, in this case, we have $H_{n-d}(\mathrm{lin}_d(P_\bullet)) \neq 0$, and hence $n-d \leq l$.

Now, let $F \subset [n]$, and $\tilde{C}_\bullet(\Delta_F; K)$, $\tilde{C}_\bullet(\Gamma_F; K)$ be the augmented chain complexes of

Δ_F and Γ_F, respectively. Since $\Sigma^{(d-2)}$ have no faces of dimension $\geq d-1$, we have $\tilde{C}_{d-1}(\Delta_F; K) = \tilde{C}_{d-1}(\Gamma_F; K)$ and hence $\tilde{H}_{d-1}(\Delta_F; K) = \tilde{H}_{d-1}(\Gamma_F; K)$. On the other hand, our assumption that $|\Gamma| \approx \mathbb{S}^{d-1}$ implies that Γ is Gorenstein, and hence that

$$\tilde{H}_{d-1}(\Gamma_F; K) = \begin{cases} K & \text{if } F = [n]; \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, by Proposition 2.13, we have that

$$\beta_{n-d,n}(K[\Delta]) = \dim_K \tilde{H}_{d-1}(\Gamma; K) = 1 \neq 0;$$
$$\beta_{n-d-1,n-1}(K[\Delta]) = \sum_{F \subset [n], \sharp F = n-1} \dim_K \tilde{H}_{d-1}(\Gamma_F; K) = 0.$$

These examples implies the bound in Corollary 6.3 is optimal if $\mathrm{indeg}(\Delta) \geq 2$, that is, $\mathrm{ver}(\Delta) = [n]$.

7. A Simplicial Complex with The Maximal Linearity Defect

By Corollary 6.6, we have $\mathrm{ld}(\Delta) \le n-2$ when $n \ge 3$, and so it is natural to ask which simplicial complex attains the equality $\mathrm{ld}(\Delta) = n-2$. According to Example 6.8, there exists indeed a simplicial complex Δ with $\mathrm{ld}(\Delta) = n-2$. (In fact we have only to set $d = 2$ in Example 6.8). We call this complex an *n-gon*. Thus, a simplicial complex on $[n]$ is an n-gon if its facets are $\{1,2\},\{2,3\},\cdots,\{n-1,n\}$, and $\{n,1\}$ after a suitable permutation of vertices. One of our goals in this section is to give the answer to the above question, that is, to show the following;

Theorem 7.1. *Let Δ be a simplicial complex on $[n]$ with $n \ge 4$. Then $\mathrm{ld}(\Delta) = n-2$ if and only if Δ is an n-gon.*

As a corollary, we see that Herzog's conjecture 5.19 is affirmative for a simplicial complex Δ with $\dim \Delta \le 2$.

Corollary 7.2. *For a simplicial complex Δ on $[n]$, we have $\mathrm{ld}(\Delta)+\mathrm{ld}(\Delta^\vee) \le n$, if $\dim \Delta \le 2$.*

Proof. Since we have $\mathrm{ld}(\Delta) = \mathrm{ld}(\Delta^\vee) = 0$ when $n = 1$, we may assume that $n \ge 2$. The assertion follows from Corollary 6.3 and Corollary 6.6 when $\dim \Delta < 2$, expect for the case $n = 2$ and $\dim \Delta = 1$. In the case that $n = 2$ and $\dim \Delta = 1$, or that $n = 3$ and $\dim \Delta = 2$, Δ is the simplex $2^{[n]}$ and $\Delta^\vee$ is the void $\varnothing$, and hence the assertion also follows in this case. What remains to be shown is the case that $\dim \Delta = 2$ and $n \ge 4$; in this case, the above theorem and Corollary 6.6 implies $\mathrm{ld}(\Delta) \le n-3$. Since by Corollary 6.3 again, $\mathrm{ld}(\Delta^\vee) \le 3$ holds, we have the required. □

The below lemma serve the reduction of the proof of Theorem 7.1.

Lemma 7.3. *Let Δ be a simplicial complex on $[n]$, and $P_\bullet$ a minimal graded free resolution of $K[\Delta]$. We denote $Q_\bullet$ for the subcomplex of $P_\bullet$ such that $Q_i := \mathrm{lin}_1(P_\bullet)_i \oplus \mathrm{lin}_0(P_\bullet)_i$. Assume $n \ge 4$. Then the following are equivalent.*

(1) $\mathrm{ld}(\Delta) = n-2$;
(2) $H_{n-2}(\mathrm{lin}_2(P_\bullet)) \ne 0$;
(3) $H_{n-3}(Q_\bullet) \ne 0$.

It is noteworthy that the condition (3) is equivalent to $H_{n-3}(\mathrm{lin}_1(P_\bullet)) \ne 0$ in the case $n \ge 5$ (see the proof below).

Proof. First of all, we shall show $H_i(Q_\bullet) = H_i(\mathrm{lin}_1(P_\bullet))$ for $i \ge 2$. Note that $\mathrm{lin}_0(P_\bullet)$ is always acyclic; in fact the case $\mathrm{indeg}(\Delta) \ge 2$ is clear and in the case $\mathrm{indeg}(\Delta) = 1$, $\mathrm{lin}_0(P_\bullet)$ coincides the Koszul complex with respect to the sequence $x_{i_1},\cdots,x_{i_t}$ with $[n] \setminus \mathrm{ver}(\Delta) = \{i_1,\cdots,i_t\}$, and hence is acyclic by Proposition 1.7. Now since $\mathrm{lin}_0(P_\bullet)$ is a subcomplex of $Q_\bullet$, there is the exact sequence of complexes;

$$0 \longrightarrow \mathrm{lin}_0(P_\bullet) \longrightarrow Q_\bullet \longrightarrow \mathrm{lin}_1(P_\bullet) \longrightarrow 0.$$

The exact sequence of homologies induced by this and the acyclicity of $\mathrm{lin}_0(P_\bullet)$ implies the required assertion.

Well, observing that for $i \geq n-2$ and $l \geq 3$ we have $\mathrm{lin}_l(P_\bullet)_i = 0$ by Lemma 6.1 and that $\mathrm{ld}(\Delta) \leq n-2$ by Corollary 6.3, we see that it suffices to show the following.

$$H_{n-2}(\mathrm{lin}_2(P_\bullet)) \cong H_{n-3}(Q_\bullet) \quad \text{and} \quad H_i(Q_\bullet) = 0 \quad \text{for } i \geq n-2\,(\geq 2). \tag{7.1}$$

Since $Q_\bullet$ is a subcomplex of $P_\bullet$, there exists the following short exact sequence of complexes.

$$0 \longrightarrow Q_\bullet \longrightarrow P_\bullet \longrightarrow \tilde{P}_\bullet := P_\bullet/Q_\bullet \longrightarrow 0,$$

which induces the exact sequence of homology groups

$$H_i(P_\bullet) \longrightarrow H_i(\tilde{P}_\bullet) \longrightarrow H_{i-1}(Q_\bullet) \longrightarrow H_{i-1}(P_\bullet).$$

Hence the acyclicity of $P_\bullet$ implies that $H_i(\tilde{P}_\bullet) \cong H_{i-1}(Q_\bullet)$ for all $i \geq 2$. Now $H_i(\tilde{P}_\bullet) = 0$ for $i \geq n-1$ by Lemma 6.1 and the fact that $\tilde{P}_i = \oplus_{l \geq 2} \mathrm{lin}_l(P_\bullet)_i$. So the latter assertion of (7.1) holds, since $n-2 \geq 2$. The former follows from the equality $H_{n-2}(\tilde{P}_\bullet) = H_{n-2}(\mathrm{lin}_2(P_\bullet))$, which is a direct consequence of the fact that $\mathrm{lin}_2(P_\bullet)$ is a subcomplex of $\tilde{P}_\bullet$ and $\tilde{P}_{n-2} = \mathrm{lin}_2(P_\bullet)_{n-2}$. □

Let Δ be a 1-dimensional simplicial complex on $[n]$. A *cycle* C in Δ of length t (≥ 3) is a subcomplex of Δ consisting of a sequence of distinct edges of the form (v_1, v_2), $(v_2, v_3), \ldots, (v_t, v_1)$ joining distinct vertices $v_1, \ldots v_t$. Let C be such a cycle. We say C has a *chord* if there exists an edge (v_i, v_j) of G such that $j \not\equiv i+1 \pmod t$. A cycle without cords is said to be *minimal*. It is easy to see that the 1^{st} homology of Δ is generated by those of minimal cycles contained in Δ, that is, we have the surjective map:

$$\bigoplus_{\substack{C \subset \Delta \\ C:\text{minimal cycle}}} \tilde{H}_1(C; K) \longrightarrow \tilde{H}_1(\Delta; K). \tag{7.2}$$

where the first map composed by the natural ones $\tilde{H}_1(C; K) \longrightarrow \tilde{H}_1(\Delta; K)$.

Now we are ready for the proof of Theorem 7.1.

Proof of Theorem 7.1. The implication "⇐" has been already done in the beginning of this section. So we shall show the inverse. By Proposition 6.5, we may assume that $\mathrm{indeg}(\Delta) \geq 2$. Let $P_\bullet$ be a minimal graded free resolution of $K[\Delta]$ and $Q_\bullet$ as in Lemma 7.3. Note that $Q_\bullet$ is determined only by $[I_\Delta]_2$ and that it follows $[I_\Delta]_2 = [I_{\Delta^{(1)}}]_2$. If the 1-skeleton $\Delta^{(1)}$ of Δ is an n-gon, then so is Δ itself. Thus by Lemma 7.3, we may assume that $\dim \Delta = 1$. Since $\mathrm{ld}(\Delta) = n-2$, we have by Lemma 7.3

$$\tilde{H}_1(\Delta; K) \cong \tilde{H}^1(\Delta; K) \cong [\mathrm{Tor}_S^{n-2}(K[\Delta], K)]_{[n]} \neq 0,$$

and hence Δ contains at least one cycle as a subcomplex. So it suffices to show that Δ has no cycles of length $\leq n-1$. Suppose not, i.e., Δ has some cycles of length $\leq n-1$. To give a contradiction, we shall show

$$0 \longrightarrow \mathrm{lin}_2(P_\bullet)_{n-2} \longrightarrow \mathrm{lin}_2(P_\bullet)_{n-3} \tag{7.3}$$

is exact; in fact it follows $H_{n-2}(\mathrm{lin}_2(P_\bullet)) = 0$, which contradicts to Lemma 7.3. For that, we need some observations (this is a similar argument to that done in Theorem 4.1 of

[**24**]). Consider the chain complex $K[\Delta] \otimes_K \bigwedge S_1 \otimes_K S$. We can define two differential map ϑ, ∂ on it as follows:

$$\vartheta(f \otimes \textstyle\bigwedge^G \mathbf{x} \otimes g) = \displaystyle\sum_{i \in G} (-1)^{\alpha(i,G)} (x_i f \otimes \textstyle\bigwedge^{G \setminus \{i\}} \mathbf{x} \otimes g);$$

$$\partial(f \otimes \textstyle\bigwedge^G \mathbf{x} \otimes g) = \displaystyle\sum_{i \in G} (-1)^{\alpha(i,G)} (f \otimes \textstyle\bigwedge^{G \setminus \{i\}} \mathbf{x} \otimes x_i g).$$

By a routine, we have that $\partial\vartheta + \vartheta\partial = 0$, and easily we can check that the i^{th} homology group of the chain complex $(K[\Delta] \otimes_K \bigwedge S_1 \otimes_K S, \vartheta)$ is isomorphic to the i^{th} graded free module of a minimal free resolution $P_\bullet$ of $K[\Delta]$. Since, moreover, the differential maps of $\operatorname{lin}(P_\bullet)$ is induced by ∂ due to Eisenbud-Goto [**6**] and Herzog-Simis-Vasconcelos [**12**], $\operatorname{lin}_l(P_\bullet)_i \longrightarrow \operatorname{lin}_l(P_\bullet)_{i-1}$ can be identified with

$$\bigoplus_{\substack{F \subset [n], \\ \sharp F = i+l}} [\operatorname{Tor}^i_S(K[\Delta], K)]_F \otimes_K S \xrightarrow{\bar{\partial}} \bigoplus_{\substack{F \subset [n], \\ \sharp F = i-1+l}} [\operatorname{Tor}^{i-1}_S(K[\Delta], K)]_F \otimes_K S, \tag{7.4}$$

where $\bar{\partial}$ is induced by ∂. We set $-\{i\} := [n] \setminus \{i\}$. Then we may identify the sequence (7.3) with $\bar{\partial} : [\operatorname{Tor}^{n-2}_S(K[\Delta], K)]_{[n]} \otimes_K S \longrightarrow \bigoplus_{i \in [n]} [\operatorname{Tor}^{n-3}_S(K[\Delta], K)]_{-\{i\}} \otimes_K S$, and hence by the isomorphism (2.1), with $\bar{\varepsilon} : \tilde{H}^1(\Delta; K) \otimes_K S \longrightarrow \bigoplus_{i \in [n]} \tilde{H}^1(\Delta_{-\{i\}}; K) \otimes_K S$. Here $\bar{\varepsilon}$ is composed by $\bar{\varepsilon}_i : \tilde{H}^1(\Delta; K) \otimes_K S \longrightarrow \tilde{H}^1(\Delta_{-\{i\}}; K) \otimes_K S$, which is induced by the chain map $\varepsilon_i : \tilde{C}^\bullet(\Delta; K) \otimes_K S \longrightarrow \tilde{C}^\bullet(\Delta_F; K) \otimes_K S$ with

$$\varepsilon_i(e^*_G \otimes 1) = \begin{cases} (-1)^{\alpha(i,G)} e^*_G \otimes x_i & \text{if } i \notin G; \\ 0 & \text{otherwise.} \end{cases}$$

Now observing that Δ contains a cycle of length $\leq n-1$ (that is, Δ itself is not a minimal cycle), the surjective map (7.2) yields the surjective one $\bar{\eta} : \bigoplus_{i \in [n]} \tilde{H}_1(\Delta_{-\{i\}}; K) \to \tilde{H}_1(\Delta; K)$ induced by the chain map $\eta : \bigoplus_{i \in [n]} \tilde{C}_\bullet(\Delta_{-\{i\}}; K) \longrightarrow \tilde{C}_\bullet(\Delta; K)$, which is composed by $\eta_i : \tilde{C}_\bullet(\Delta_{-\{i\}}; K) \ni e_G \mapsto (-1)^{\alpha(i,G)} e_G \in \tilde{C}_\bullet(\Delta; K)$. (We need the sign $(-1)^{\alpha(i,G)}$ for the latter convenience). Taking the K-dual of this sequence, we have the injective map $\bar{\eta}^* : \tilde{H}^1(\Delta; K) \longrightarrow \bigoplus_{i \in [n]} \tilde{H}^1(\Delta_{-\{i\}}; K)$, where $\bar{\eta}^*$ is the dual map of $\bar{\eta}$. Then $\bar{\eta}^*$ is composed by the K-dual $(\bar{\eta}_i)^* : \tilde{H}^1(\Delta; K) \to \tilde{H}^1(\Delta_{-\{i\}}; K)$ of $\bar{\eta}_i$, and hence for all $0 \neq z \in \tilde{H}_1(\Delta; K)$, we have $(\bar{\eta}_i)^*(z) \neq 0$ for some i. Recalling the map $\bar{\varepsilon}$ in (7.4) and its construction, we see $\bar{\varepsilon}(z \otimes 1) = \sum_{i=1}^n (\bar{\eta}_i)^*(z) \otimes x_i$ for $z \in \tilde{H}_1(\Delta; K)$; therefore (7.3) is exact. □

Remark 7.4. (1) If Δ is an n-gon, then $\Delta^\vee$ is an $(n-3)$-dimensional Buchsbaum complex with $\tilde{H}_{n-4}(\Delta^\vee; K) \cong \tilde{H}_1(\Delta; K) = K$ by Alexander's duality. If $n = 5$, then $\Delta^\vee$ is a triangulation of the Möbius band. But, for $n \geq 6$, $\Delta^\vee$ is not a homology manifold. In fact, let $\{1,2\}, \{2,3\}, \cdots, \{n-1,n\}, \{n,1\}$ be the facets of Δ, then if $F = [n] \setminus \{1,3,5\}$, easy computation shows that $\operatorname{lk}_{\Delta^\vee} F$ is a 0-dimensional complex with 3 vertices $\{1\}, \{3\}, \{5\}$, and hence $\tilde{H}_0(\operatorname{lk}_{\Delta^\vee} F; K) = K^2$.

(2) Theorem 7.1 implies that when $\operatorname{indeg} \Delta = 2$, the simplicial complexes given in Example 6.8 are the only examples which attain the equality $\operatorname{ld}(\Delta) = n - \operatorname{indeg}(\Delta)$, and so it is natural to ask if the same holds when $\operatorname{indeg} \Delta \geq 3$. But unfortunately, it is false. Let us give two examples.

Let Δ be the triangulation of $P^2\mathbb{R}$ with 6 vertices which is given in Example 2.20. As was seen in Remark 5.18, we have $\mathrm{ld}(\Delta) = \mathrm{ld}(\Delta^\vee) = 3 = 6 - \mathrm{indeg}(\Delta)$ whenever $\mathrm{char}\, K = 2$.

Next, as is well known, there is a triangulation of the torus with 7 vertices. Let Δ be the triangulation. Since $\dim \Delta = 2$, it follows that $\mathrm{indeg}(\Delta^\vee) = 7 - \dim \Delta - 1 = 4$. Observing that $K[\Delta]$ is Buchsbaum, we have, by easy computation, that $\mathrm{ld}(\Delta^\vee) = 3 = 7 - 4 = 7 - \mathrm{indeg}(\Delta^\vee)$. Thus $\Delta^\vee$ attains the equality, but is not a simplicial complex given in Example 6.8, since it follows, from Alexander's duality, that

$$\dim_K \tilde{H}_i(\Delta^\vee; K) = \dim_K \tilde{H}_{4-i}(\Delta; K) = \begin{cases} 2 \neq 1 & \text{for } i = 3; \\ 0 & \text{for } i \geq 4. \end{cases}$$

More generally, the dual complexes of d-dimensional Buchsbaum complexes Δ with $\tilde{H}_{d-1}(\Delta; K) \neq 0$ satisfy the equality $\mathrm{ld}(\Delta^\vee) = n - \mathrm{indeg}(\Delta^\vee)$, but many of them differ from the examples in Example 6.8, and we can construct such complexes more easily as $\mathrm{indeg}(\Delta^\vee)$ is larger.

Using the argument in the proof of Theorem 7.1, we obtain a lower bound of $\mathrm{ld}(\Delta)$, which is shown by the following lemma:

Lemma 7.5. *Let Δ be a $(d-1)$-dimensional Gorenstein* complex on $[n]$ with $\Delta \neq \emptyset, \{\emptyset\}$. Then we have $\tilde{H}^{d-1}(\Delta; K) \neq 0$, and $\tilde{H}^{d-1}(\Delta_F; K) = 0$ for all $F \subset [n]$ with $F \subsetneq \mathrm{ver}(\Delta)$.*

Proof. The former follows from Theorem 2.17. We shall show the latter. We may assume that $\mathrm{ver}(\Delta) = [n]$. Then what we have to show is $\tilde{H}^{d-1}(\Delta_F; K) = 0$ for all $F \subsetneq [n]$. Let $P_\bullet : 0 \to P_t \to \cdots \to P_1 \to P_0 \to 0$ be a minimal graded free resolution of $K[\Delta]$. Then $t = n - d$ by Auslander-Buchsbaum formula. According to the argument in the proof of Theorem 7.1, we have $\mathrm{lin}_l(P_\bullet)_i = \bigoplus_{F \subset [n], \sharp F = i+l} \tilde{H}^{l-1}(\Delta_F; K) \otimes_K S$, and hence in particular if we write $P_t = \bigoplus_{i=1}^r S(-a_i)$, $\max\{\, a_i \mid 1 \leq i \leq r \,\} = n$ holds since $\tilde{H}^{d-1}(\Delta; K) \neq 0$. Therefore by 2.17 again, we have $\beta_{i,i+d}(K[\Delta]) = \beta_{t-i,n-i-d}(K[\Delta]) = \beta_{t-i,t-i}(K[\Delta])$. Now by our assumption $\mathrm{ver}(\Delta) = [n]$ we have $\beta_{i,i}(K[\Delta]) = 0$ for $i \geq 2$, so that $\beta_{i,i+d}(K[\Delta]) = 0$ for $i < t$. Hence $\tilde{H}^{d-1}(\Delta_F; K) \otimes_K S \subset \mathrm{lin}_d(P_\bullet)_{\sharp F - d} = S(-\sharp F)^{\beta_{\sharp F - d, (\sharp F - d) + d}(K[\Delta])} = 0$ holds for $F \subsetneq [n]$ since $\sharp F - d < n - d = t$, and therefore we have the required. □

Proposition 7.6. *For a simplicial complex on $[n]$, we have*

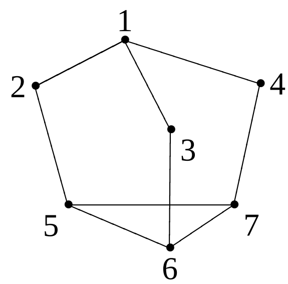
$$\mathrm{ld}(\Delta) \geq \max\{\, \sharp F - \dim \Delta_F - 1 \mid F \subset [n], \Delta_F \text{ is Gorenstein with } \Delta \neq \emptyset, \{\emptyset\} \,\},$$

where we set the value of the right hand of the inequality to be 0 if there is no $F \subset [n]$ such that Δ_F is Gorenstein.

Proof. First, recall the description of $\mathrm{lin}_l(P_\bullet)$ by using reduced cohomologies, which is given in the proof of Theorem 7.1. According to this description, we have $\mathrm{lin}_l(P_\bullet)_i = \bigoplus_{F \subset [n], \sharp F = i+l} \tilde{H}^{l-1}(\Delta_F; K) \otimes_K S$. We denote the differential maps of $P_\bullet$ and $\mathrm{lin}_l(P_\bullet)$ by ∂ and $\partial^{\langle l \rangle}$, respectively.

Assume Δ_F is Gorenstein. Note that Δ_F is of the form $\Delta_F = \Delta_{\mathrm{core}\, F} * 2^{F \setminus \mathrm{core}\, F}$, where we set $\mathrm{core}\, F := \{\, v \in F \mid \mathrm{st}_{\Delta_F} v \neq \Delta \,\}$. Then it follows that $\sharp F - \dim \Delta_F - 1 = \sharp F - (\dim \Delta_{\mathrm{core}\, F} + \sharp F - \sharp\, \mathrm{core}\, F) - 1 = \sharp\, \mathrm{core}\, F - \dim \Delta_{\mathrm{core}\, F} - 1$, and so we may assume that Δ_F is a Gorenstein* complex on F, since $\Delta_{\mathrm{core}\, F}$ is so on $\mathrm{core}\, F$. Set $d := \dim \Delta_F + 1$. Then we have $\tilde{H}^{d-1}(\Delta_F; K) \neq 0$ by Lemma 7.5, whence $\mathrm{lin}_d(P_\bullet)_{\sharp F - d} \supset \tilde{H}^{d-1}(\Delta_F; K) \otimes_K S \neq 0$. Now take a base z of the free S-module $\tilde{H}^{d-1}(\Delta_F; K)\otimes_K S$. Then by minimality of $P_\bullet$, z is not in $\partial^{\langle d\rangle}(\mathrm{lin}_d(P_\bullet)_{\sharp F-d+1})$. On the other hand, though $\partial^{\langle d\rangle}(z) \in \bigoplus_{G\subset F, \sharp G=\sharp F-1} \tilde{H}^{d-1}(\Delta_G; K) \otimes_K S$ holds, we have $\tilde{H}^{d-1}(\Delta_G; K) = 0$ for all such G by Lemma 7.5 again, and so $\partial^{\langle d\rangle}(z) = 0$. Therefore we conclude $H_{\sharp F-d}(\mathrm{lin}_d(P_\bullet)) \neq 0$. □

Remark 7.7. Unfortunately, in the above proposition, the equality does not necessarily hold; in fact, let Δ be the simplicial complex as follows:



Then $\max\{ \sharp F - 2 \mid \Delta_F$ is Gorenstein $\} = 3$, while we see $\mathrm{ld}(\Delta) = 4$, by computation with the software system Macaulay 2 ([**9**]) as follows (see Appendix A in the end):

```
i1 : S = QQ[x_1..x_7]

i2 : D = {{1,2},{2,5},{1,3},{3,6},{1,4},{4,7},{5,6},{6,7},{5,7}}

i3 : linearityDefectByDepth(D,S)

o3 = 4
```

Thus we see that $\mathrm{ld}(\Delta) = 4$.

To verify $\max\{ \sharp F - 2 \mid \Delta_F$ is Gorenstein $\} = 3$, recall the concept of "hilbert series"; for a $\mathbb{Z}$-graded K-algebra R and $M \in \mathfrak{mod}_{\mathbb{Z}} R$, the hilbert series of M is defined by $H_M(t) := \sum \dim_K M_i t^i$. We have only to verify that there is no $F \subset \{1, \cdots, 7\}$ with $\sharp F = 6$ such that Δ_F is Gorenstein. Suppose not. Then by the proof of Proposition 7.6 the free module of homological degree 4 in a minimal graded free resolution $P_\bullet$ of $K[\Delta]$ has a base which belongs to kernels but does not to boundaries. Thus $(H_4(\mathrm{lin}(P_\bullet)))_6 \neq 0$ holds, and hence the 6$^{\text{th}}$ term of $H_{H_4(\mathrm{lin}(P_\bullet))}(t)$ does not vanish. On the other hand, according Macaulay 2, we have:

```
i4 : M = faceRing(D,S)

i5 : C = linearPart M

i6 : h = HH_4 C

i7 : hilbertSeries h

          7
        $T
o7 = ---------
            7
     (1 - $T)

o7 : Divide
```

Hence the minimal degree i such that the i^{th}-term of $H_{H_4(\text{lin}(P_\bullet))}(t)$ does not vanish, is 7, which is a contradiction. Therefore we have the required.

Appendix A. Computing Linearity Defects with Macaulay 2

As is seen so far, we know 3 methods of calculating linearity defects: i) the direct one — construct linear parts, and then search for the maximal homological degree where the homology does not vanish —; ii) the method by Proposition 5.13; iii) that by Theorem 5.11. In the following, we shall give 3 methods of computation of linearity defects with Macaulay 2 (henceforth, abbreviated to M2), the software system [**9**]. For undefined commands, see the document in http://www.math.uiuc.edu/Macaulay2/.

Method by "Linear Part". Let $\mathfrak{m}$ be the maximal ideal of S. In M2, we can define the non-well-defined map $f : S/\mathfrak{m}^2 \to S$. This is, rather than a map, the assignment with the following rule: (i) $f(\overline{x_i}) = x_i$ for each i, $f(\overline{0}) = 0$, and $f(\overline{1})$, where for $x \in S$, $\overline{x}$ denotes the image of x in $S/\mathfrak{m}^2$; (ii) for a polynomial p of S, $f(\overline{p})$ is done after simplification of p and reduction of their degrees. For instance, $f(\overline{x_1^2} + \overline{x_2}) = x_2$, $f(\overline{x_1} - \overline{x_1}) = 0$, and so on. Making use of this, we can construct the function which returns the matrix given by erasing terms of degree ≥ 2 from a matrix given as input.

```
i1 : linearPart = method()

i2 : linearPart Matrix := M -> (
              S := ring M; --the base ring of M
              m := ideal vars S; --the maximal ideal of S
              R := S/m^2;
              f := map(R,S,vars R);
              g := map(S,R,vars S);
              g f M);
```

For example,

```
i3 : S = QQ[x_1..x_5]

i4 : M = matrix({{x_1*x_2,x_2 + x_3,x_4*x_5},
             {x_2^2 + x_1*x_5,x_1^2 + x_2 + x_4*x_5,x_1 + x_2*x_3 - x_4^3 + x_5}})

o4 = | x_1x_2       x_2+x_3          x_4x_5                 |
     | x_2^2+x_1x_5 x_1^2+x_4x_5+x_2 -x_4^3+x_2x_3+x_1+x_5 |

              2       3
o4 : Matrix S  <--- S

i5 : linearPart M

o5 = | 0 x_2+x_3 0       |
     | 0 x_2     x_1+x_5 |

              2       3
o5 : Matrix S  <--- S
```

Since M2 has the function "chainComplex" which returns the chain complex with given differential maps, we can make the function "linearPart" which returns the linear part of a given chain complex,

```
i6 : linearPart(ChainComplex) = C -> (
          ListOfdiffs := {};
          ini := min C + 1; --1 + minimal homological degree of C
          ends := max C; --maximal homological degree
```

```
for i from ini to ends do (
     d := C.dd_i; --i-th differential map of C
     lind := map(target d, source d, linearPart d);
     ListOfdiffs = append(ListOfdiffs, lind););
chainComplex(ListOfdiffs))
```

hence the linear part of a minimal graded free resolution of a given module, since M2 can be compute free resolutions of modules.

```
i7 : linearPart(Module) = M ->(
          C := res M;
          complete C; --for complete computation of res M
          linearPart C)
```

For example, the linear part of the resolution of A/I where A is the polynomial ring $Q[x_1, \cdots, x_5]$ over the field Q of rational numbers and $I := (x_3x_5, x_2x_5, x_2x_4, x_1x_4, x_1x_3)$ is an ideal of A, is:

```
i8 : m = matrix({{x_3*x_5,x_2*x_5,x_2*x_4,x_1*x_4,x_1*x_3}})

o8 = | x_3x_5 x_2x_5 x_2x_4 x_1x_4 x_1x_3 |

             1       5
o8 : Matrix S  <--- S

i9 : C = res cokernel m

      1      5      5      1
o9 = S  <-- S  <-- S  <-- S  <-- 0

     0      1      2      3      4

o9 : ChainComplex

i10 : C.dd --describe C with all the differential maps

          1                                               5
o10 = 0 : S  <------------------------------------------ S  : 1
                | x_1x_3 x_1x_4 x_2x_4 x_2x_5 x_3x_5 |

          5                                         5
      1 : S  <------------------------------------ S  : 2
               {2} | 0     -x_4 -x_5 0    0    |
               {2} | -x_2 x_3  0    0    0    |
               {2} | x_1  0    0    0    -x_5 |
               {2} | 0    0    0    -x_3 x_4  |
               {2} | 0    0    x_1  x_2  0    |

          5                        1
      2 : S  <------------------- S  : 3
               {3} | -x_3x_5 |
               {3} | -x_2x_5 |
               {3} | x_2x_4  |
               {3} | -x_1x_4 |
               {3} | -x_1x_3 |

          1
```

```
      3 : S  <----- 0 : 4
                0

o10 : ChainComplexMap

i11 : linC = linearPart cokernel m

       1      5      5      1
o11 = S  <-- S  <-- S  <-- S  <-- 0

      0      1      2      3      4

o11 : ChainComplex

i11 : linC.dd

           1         5
o11 = 0 : S  <----- S  : 1
            0

           5                                       5
      1 : S  <----------------------------------- S  : 2
                {2} | 0    -x_4 -x_5 0    0    |
                {2} | -x_2 x_3  0    0    0    |
                {2} | x_1  0    0    0    -x_5 |
                {2} | 0    0    0    -x_3 x_4  |
                {2} | 0    0    x_1  x_2  0    |

           5         1
      2 : S  <----- S  : 3
                0

           1
      3 : S  <----- 0 : 4
                0

o11 : ChainComplexMap
```

Now we can define the function "linearityDefectByLinearPart" which returns the linearity defect of a given module:

```
i12 : linearityDefectByLinearPart = M -> (
          S := ring M;
          C := linearPart M;
          ldNum := pdim M; --projective dimension of M
          h := HH_ldNum C; --ldNum-th homology
          while h == 0 and ldNum >= 1 do (
               ldNum = ldNum - 1;
               h = HH_ldNum C);
          ldNum)
```

For example, the linearity defect of A/I in the above, is:

```
i13 : linearityDefectByLinearPart cokernel m

o13 = 3
```

Since A/I is a face ring of a 5-gon, ld(Δ) is indeed 3 by theorem 7.1.

Method by "Depth". The methods we introduce from now, comes from Proposition 5.13. Since M2 can compute Ext modules, the rest of problems in realizing this method is the following; "how can we get the Alexander dual of a given squarefree module?". This problem is solved by the command "dual". The command "dual" makes use of Miller's definition of Alexander dual [**16**], which is done for any monomial ideals (of course, for squarefree ideals, his definition and ours coincide). It takes as input a monomial ideal and then returns the "ideal of Alexander dual". Before giving an example, we shall introduce the function "faceIdeal" and "faceRing" which can respectively compute the face ideal/ring of a given simplicial complex. Here we use the list of its facets to describe the simplicial complex.

```
i14 : faceIdeal = (D,R) -> (
          n := numgens R;
          Vars := first entries vars R; --list of variables of R
          V := apply(Vars, i -> index i + 1); --vertex set
          Ass := {};
          assignment := i -> Vars#(i-1);
          for i from 0 to #D - 1 do (
               F := D #i;
               Fc := toList(set(V) - set(F));
               if #Fc == 0 then PF := ideal(0*Vars#0) --0 ideal of R
               --Note that 'ideal 0' is regarded as an ideal of ZZ
               else (
                    xFc := apply(Fc, assignment);
                    PF = ideal(xFc));
               Ass = append(Ass,PF););
          intersect(Ass))

i15 : faceRing = (D,R) -> (
           I := faceIdeal(D,R);
           R^1/I)
```

We can compute the face ideal/ring of the simplicial complex in Example 2.6:

```
i16 : D = {{1,2,3},{1,3,4},{2,3,4},{1,2,4},{3,4,5}}

i17 : I = faceIdeal(D,S)

o17 = ideal (x x , x x , x x x x )
              2 5   1 5   1 2 3 4

o17 : Ideal of S

i18 : faceRing(D,S)

o18 = cokernel | x_2x_5 x_1x_5 x_1x_2x_3x_4 |

                            1
o18 : S-module, quotient of S
```

Then its ideal of Alexander dual is:

```
i19 : I' = monomialIdeal I

o19 = monomialIdeal (x x x x , x x , x x )
                      1 2 3 4   1 5   2 5
```

```
o19 : MonomialIdeal of S

i19 : dual I'

o19 = monomialIdeal (x x , x x , x x , x x , x x )
                      1 2   1 5   2 5   3 5   4 5

o19 : MonomialIdeal of S
```

In fact, by using Example 2.8, we have:

```
i20 : faceIdeal({{1,3,4},{2,3,4},{5}},S)

o20 = ideal (x x , x x , x x , x x , x x )
              4 5   3 5   2 5   1 5   1 2

o20 : Ideal of S
```

M2 has the command "pdim" which returns the projective dimension of a given module, and so by Auslander-Buchsbaum formula (Theorem 1.9) the depth of a given module M is computed by the command n - (pdim M), where n is the depth of the base ring S of M, hence the number of variables if S is a polynomial ring. Now we can construct the "linearityDefectByDepth" function:

```
i21 : linearityDefectByDepth = (D,R) -> (
           M := module dual monomialIdeal faceIdeal(D,R);
           n := numgens R;
           DepthList := {};
           for i from n - pdim M to dim M do (
                H := Ext^(n - i)(M,R^1);
                if H != 0 then DepthList = append(DepthList,i - n + (pdim H)));
           max DepthList)
```

For example,

```
i22 : linearityDefectByDepth(D,S)

o22 = 1

i23 : linearityDefectByDepth({{1,2},{2,3},{3,4},{4,5},{1,5}},S)

o23 = 3
```

Method by "BGG". The last method is the one by BGG correspondence, that is, by Theorem 5.11. Since M2 can compute the regularity of a given module, the realization of the functor $\mathscr{L}$ is only the problem. But due to Decker and Eisenbud [**4**] there is already the command "bgg" which takes as input a number i and a module $M \in \mathfrak{mod}_{\mathbb{Z}}E$ and then returns the differential map $\mathscr{L}(M)_i \to \mathscr{L}(M)_{i+1}$ as output.

```
i24 : bgg = (i,M,B) -> (
           A :=ring M;
           numvarsB := rank source vars B;
           ev:=map(B,A,vars B);
           f0:=basis(i,M);
           f1:=basis(i+1,M);
           g :=((vars A)**f0)//f1;
           b:=(ev g)*((transpose vars B)**(ev source f0));
           --correct the degrees (which are otherwise
```

```
        --wrong in the transpose)
        map(B^{(rank target b):i+1},B^{(rank source b):i}, b))
```

Remark A.1. Actually in [**4**] bgg is introduced the command which returns $\mathcal{R}(M)_i \to \mathcal{R}(M)_{i+1}$ where $\mathcal{R}$ is an adjoint functor of $\mathscr{L}$ (see for details [**4, 5**]). But careful observation of its program tells us that it also works as a tool returning the differential map of $\mathscr{L}(M)$.

The function “linearityDefectByBgg” can be constructed as follows:

```
i25 : linearityDefectByBgg = N -> (
        E := ring N;
        n := numgens E;
        S := (coefficientRing E)[x_1..x_n];
        N' := Hom(N,E);
        ListOfRegs := {};
        D := degrees N'; --all the degrees of a minimal system of generators of N'
        indeg := first min D; --minimal deg in D
        endeg := first max D; --maximal deg in D
        for i from indeg to endeg + n do(
             Z := kernel bgg(i,N',S);
             B := image bgg(i-1,N',S);
             H := Z/B;
             if H == 0 then r = -i
             else r = regularity H;
             ListOfRegs = prepend(r+i,ListOfRegs););
        max ListOfRegs)
```

For example,

```
i26 : E = QQ[y_1..y_5, SkewCommutative => true]

i27 : N = faceRing(D,E)

o27 = cokernel | y_2y_5 y_1y_5 y_1y_2y_3y_4 |

                                   1
o27 : E-module, quotient of E

i28 : linearityDefectByBgg N

o28 = 1

i29 : linearityDefectByBgg faceRing({{1,2},{2,3},{3,4},{4,5},{1,5}},E)

o29 = 3
```

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