Scattering Theory for the Dirac Equation with a Nonlocal Term

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Abstract

Consider a scattering problem for the Dirac equation with a nonlocal term including the Hartree type. We show the existence of scattering operators for small initial data in the subcritical and critical Sobolev spaces.

1 Introduction

We consider a scattering problem for the Dirac equation with a nonlocal term

$$\partial_t \psi + \alpha \cdot \nabla \psi + i\beta\psi = \lambda[V * |\psi|^{p-1}]\psi, \quad (t,x) \in \mathbf{R} \times \mathbf{R}^n. \tag{1.1}$$

Here $p \geq 3,\ n \geq 3,\ \partial_t = \partial/\partial t,\ \nabla = (\partial_1, \cdots, \partial_n),\ \partial_j = \partial/\partial x_j,\ j = 1, \cdots, n,\ \lambda \in \mathbf{C},\ \alpha \cdot \nabla = \sum_{j=1}^{n} \alpha_j \partial_j,\ \alpha_j$'s, $j = 1, \cdots, n$ and β are $2^{[(n+1)/2]} \times 2^{[(n+1)/2]}$ Hermitian matrices satisfying the usual anticommutation relations [‡]. The unknown function ψ is a $2^{[(n+1)/2]}$-spinor field defined on $\mathbf{R} \times \mathbf{R}^n$. The function $V = V(x)$ satisfies $|V(x)| \leq |x|^{-\gamma}$ with $\gamma > 0$, and $*$ denotes the convolution in space.

As a special case of (1.1), the Dirac equation of Hartree type, say $p = 3$, with the Coulomb potential $V(x) = |x|^{-1}$ is derived from the Maxwell-Dirac equations with zero magnetic field. See Chadam and Glassey [2]. As another case, Dias and Figueira [3] proved that there exist

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‡$\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk}I,\ \alpha_j\beta + \beta\alpha_j = 0,\ \beta^2 = I$ for $1 \leq j,\ k \leq n$.

weak solutions of a special form of the massless Dirac equation with the nonlinear term $\lambda(e^{-|x|}|x|^{-\gamma} * |\psi|^p)\psi$. See also [4].

In this paper we study the scattering problem for (1.1) with small initial data. The scaling argument for the massless Dirac equation, say $\partial_t\psi + \alpha\cdot\nabla\psi = \lambda(V * |\psi|^{p-1})\psi$ with $V(x) = |x|^{-\gamma}$ gives the value of the critical Sobolev exponent $s_c = (\gamma - 1 + n(p-3)/2)/(p-1)$. Our aim of this paper is to prove the existence of scattering operators for (1.1) with small initial data in the subcritical Sobolev space H^s with $s > s_c$ if $p \geq 3$ and in the critical Sobolev space H^{s_c} if $p > 3$ under some conditions.

The present paper is organized as follows. In Section 2, we give some notation and conditions, and then state our main results. In Section 3, we prove the results. Our basic tool is the Strichartz estimate for the Klein-Gordon equation which was shown by Machihara, Nakanishi and Ozawa [7]. In Section 4, we state and prove similar results for the critical case $s = s_c$ and $p > 3$. We use the interpolation inequality shown by Escobedo and Vega [5] for the proof.

2 Notation and Main Results

We use the standard notation. We denote by H^s_r and B^s_r the usual inhomogeneous Sobolev and Besov spaces on $\mathbf{R}^n$, respectively. We write $H^s = H^s_2$. For the definitions of these spaces, see, e.g., [1]. For functions defined on space-time, we write $L^q_t B^s_r = L^q_t(\mathbf{R}; B^s_r)$. We use $A \lesssim B$ to denote the estimate $A \leq CB$, where C is a positive constant.

Let $p \geq 3$. We assume that (γ, s) satisfy the following condition:

$$\text{(H1)} \quad \begin{cases} s > s_c = \dfrac{\gamma - 1}{p-1} + \dfrac{n(p-3)}{2(p-1)}, \\ s > \dfrac{\gamma}{n(p-1)} + \dfrac{1}{2}, \\ \max\left\{ \dfrac{n(p-3)}{2(p-2)},\ 2 - \dfrac{n(p-3)}{2} \right\} < \gamma < n. \end{cases}$$

The set of (γ, s) satisfying (H1) is shown in Figure 1. By choosing $0 \leq \theta \leq 1$ depending on

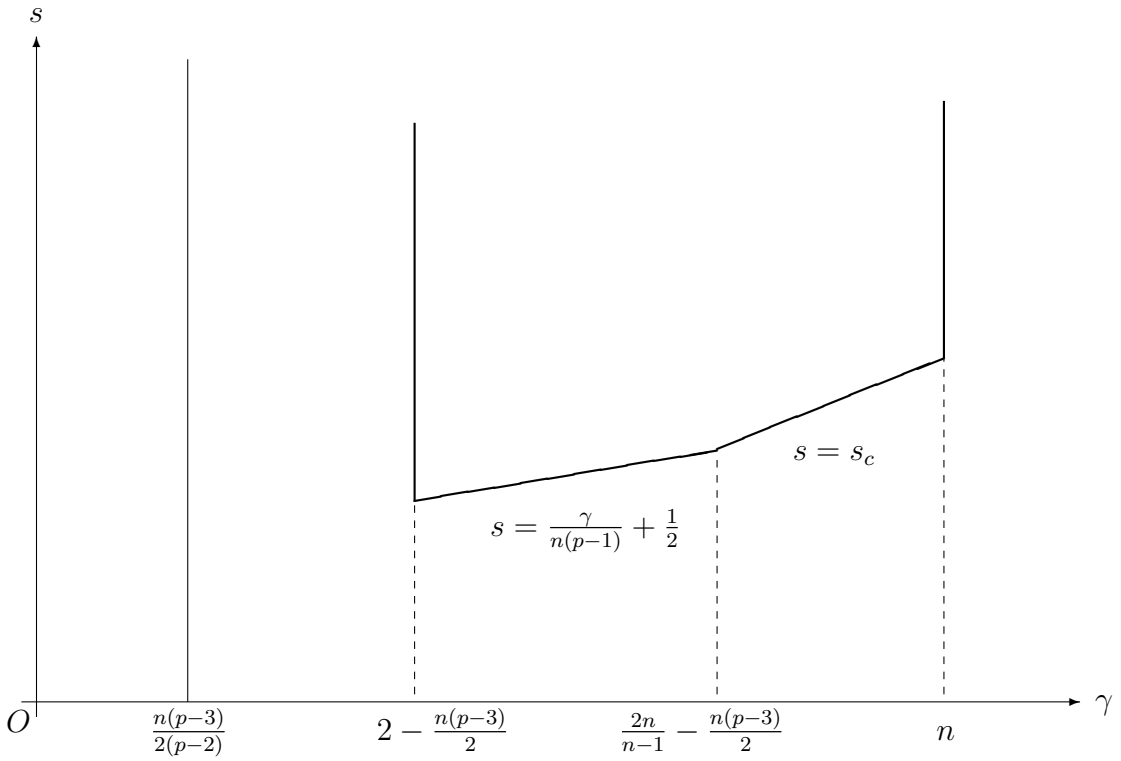

Figure 1: Domain of (γ, s) satisfying (H1)

$s,\ \gamma,\ p$ and n, we put

$$\frac{1}{r}=\frac{1}{2}-\frac{2}{(p-1)(n-1+\theta)},\quad \sigma=\frac{1}{p-1}+\frac{2}{(p-1)(n-1+\theta)}. \tag{2.1}$$

We determine θ in Section 3 so that $s>\sigma$. Note that the pair (r,σ) in (2.1) with $p=n=3$ and $\theta=0$ corresponds to the endpoint where the Strichartz estimates fail. See the remark after Lemma 3.1 below. Thus, in case $p=n=3$, we choose $0<\theta\leq 1$.

For $p,\ r$ and σ above and $s>0$, we define X^s by

$$X^s=L^\infty(\mathbf{R};H^s)\bigcap L^{p-1}(\mathbf{R};B_r^{s-\sigma}). \tag{2.2}$$

We now state our first main result, which is about the Cauchy problem for (1.1) with the initial data $\psi_0(x)$:

$$\begin{cases} \partial_t\psi+\alpha\cdot\nabla\psi+i\beta\psi=\lambda[V*|\psi|^{p-1}]\psi, & (t,x)\in\mathbf{R}\times\mathbf{R}^n,\\ \psi(0,x)=\psi_0(x) & x\in\mathbf{R}^n. \end{cases} \tag{2.3}$$

Theorem 2.1 *(i) Let $p \geq 3$. Assume (H1) and let $s < (p-1)/2$ if p is not an odd integer. Then there exists $0 \leq \theta \leq 1$ ($0 < \theta \leq 1$ if $p = n = 3$) , depending only on s, γ, p and n such that if $\|\psi_0\|_{H^s}$ is sufficiently small, then (2.3) admits a unique global solution $\psi \in X^s$ with $s > \sigma$ defined in (2.2).*

(ii) For the global solution ψ given in (i), there exist unique $\psi_\pm \in H^s$ such that

$$\lim_{t \to \pm\infty} \|\psi(t) - U(t)\psi_\pm\|_{H^s} = 0, \tag{2.4}$$

where $U(t)$ denotes the Dirac group, which solves the free Dirac equation.

Remark. (i) The set of (γ, s) satisfying (H1) and $s < (p-1)/2$ is not empty since $p \geq 3$ and $n \geq 3$.

(ii) The condition $s < (p-1)/2$ in the theorem is not necessary if p is an odd integer. See the proof of Theorem 2.1 in Section 3.

We next consider the final value problem for (1.1) with data given at $t = -\infty$:

$$\psi(t,x) = U(t)\psi^-(x) + \int_{-\infty}^{t} U(t-t')F(\psi(t'))dt', \tag{2.5}$$

where $F(\psi) = \lambda[V * |\psi|^{p-1}]\psi$. Note that $U(t)\psi^-(x)$ is a solution of the linear Dirac equation $\partial_t\psi + \alpha \cdot \nabla\psi + i\beta\psi = 0$ with the initial data ψ^- at $t = 0$.

Theorem 2.2 *Let s, γ, p and n be as in the preceding theorem. Then there exists $0 \leq \theta \leq 1$ ($0 < \theta \leq 1$ if $p = n = 3$) , depending only on s, γ, p and n such that if $\|\psi^-\|_{H^s}$ is sufficiently small, then the itegral equation (2.5) has a unique solution $\psi \in X^s$ with $s > \sigma$ satisfying*

$$\lim_{t \to -\infty} \|\psi(t) - U(t)\psi^-\|_{H^s} = 0. \tag{2.6}$$

From Theorems 2.1 and 2.2, we can define the scattering operator for small initial data in the case $p \geq 3$ under the assumptions of Theorem 2.1. See, e.g., [8].

3 Proofs of Theorems

To prove Theorem 2.1, we rewrite (2.3) as the following integral equation

$$\psi(t) = U(t)\psi_0 + \int_0^t U(t-t')F(\psi(t'))dt', \tag{3.1}$$

where $F(\psi) = \lambda[V * |\psi|^{p-1}]\psi$, and $U(t)$ is the free propagator defined on $L^2(\mathbf{R}^n; \mathbf{C}^{2^{[(n+1)/2]}})$ given by

$$U(t) = I\cos t(1-\Delta)^{1/2} - (\alpha\cdot\nabla + i\beta)(1-\Delta)^{-1/2}\sin t(1-\Delta)^{1/2}.$$

The following lemma is the Strichartz estimates for $U(t)$. See [7].

Lemma 3.1 *Let $n \geq 2$. Then one has the estimates*

$$\|U(t)u\|_{L_t^{q_1}B_{r_1}^{-s_1}} \lesssim \|u\|_{L^2}, \tag{3.2}$$

$$\left\|\int_{t'<t} U(t-t')F(t')dt'\right\|_{L_t^{q_2}B_{r_2}^{-s_2}} \lesssim \|F\|_{L_t^{q_3'}B_{r_3'}^{s_3}}, \tag{3.3}$$

where $2/q_j = (n-1+\theta)(1/2-1/r_j)$, $2s_j = (n+1+\theta)(1/2-1/r_j)$ for $0 \leq \theta \leq 1$, $2 \leq q_j,\ r_j \leq \infty,\ (q_j, r_j) \neq (2,\infty),\ j = 1,2,3$ when $n = 2,\ 3$, and p' denotes the conjugate exponent to p, i.e., $1/p + 1/p' = 1$.

Remark. When we apply Lemma 3.1 in the case $n = 3$ and $q = 2$, we take $\theta > 0$ since $(q_j, r_j) \neq (2,\infty)$ which is called "end point" where the Strichartz estimates fail.

Proof of Theorem 2.1. We first prove Part (i) of Theorem 2.1. For (s,γ) satisfying (H1), if $n(p-3)/(2(p-2)) < 2n/(n-1) - n(p-3)/2$, we choose $0 \leq \theta \leq 1$ ($0 < \theta \leq 1$ if $p = n = 3$) such that

$$s > \frac{\gamma-1}{p-1} + \frac{n(p-3)}{2(p-1)} + \frac{2\theta}{(p-1)(n-1+\theta)} \tag{3.4}$$

and

$$\frac{n(p-3)}{2(p-2)} \leq \frac{2n}{n-1+\theta} - \frac{n(p-3)}{2} < \gamma < n. \tag{3.5}$$

On the other hand, if $n(p-3)/(2(p-2)) \geq 2n/(n-1) - n(p-3)/2$, we take $\theta = 0$. In either case, we have by (H1)

$$\frac{2n}{n-1+\theta} - \frac{n(p-3)}{2} < \gamma < n. \tag{3.6}$$

Using θ chosen above, we set (2.1).

Let $M > 0$. We define the complete metric space X_M^s by

$$\begin{aligned} X_M^s &= \{\psi \in X^s \,:\, \|\psi\|_{X^s} \le M\}, \\ \|\psi\|_{X^s} &= \|\psi\|_{L_t^\infty H^s} + \|\psi\|_{L_t^{p-1} B_r^{s-\sigma}}, \end{aligned}$$

where r and σ are defined in (2.1).

Let A be the map defined by

$$A\psi(t) = U(t)\psi_0 + \int_0^t U(t-t')F(\psi(t'))dt' \quad \text{for } \psi \in X_M^s. \tag{3.7}$$

We show that A is a contraction of X_M^s into itself. From (3.2), (3.3) and (3.7), we have

$$\|A\psi\|_{X^s} \lesssim \|\psi_0\|_{H^s} + \|F(\psi)\|_{L_t^1 H^s}. \tag{3.8}$$

Take

$$\frac{1}{p_2} = \frac{1}{\widetilde{q_2}} = \frac{1}{p-1}\left(1 - \frac{\gamma}{n}\right) \tag{3.9}$$

and set

$$\frac{1}{p_1} = \frac{1}{2} - \frac{1}{p_2} \quad \text{and} \quad \frac{1}{\widetilde{q_1}} = \frac{p-2}{\widetilde{q_2}} + \frac{1}{2}. \tag{3.10}$$

Then (3.9) and (3.10) imply

$$\frac{1}{p_1} = \frac{\gamma}{n} + \frac{1}{\widetilde{q_1}} - 1. \tag{3.11}$$

We note that $0 < 1/p_1 < 1/2$ and $1/2 < 1/\widetilde{q_1} < 1$ since $p \ge 3$ and $\gamma > n(p-3)/(2(p-2))$ by assumption.

We estimate the norm of $F(\psi) = \lambda(V * |\psi|^{p-1})\psi$ in (3.8). The following inequality holds:

$$\begin{aligned} \|F(\psi)\|_{L_t^1 H^s} &\lesssim \|V * |\psi|^{p-1}\|_{L_t^{(p-1)/(p-2)} H_{p_1}^s} \|\psi\|_{L_t^{p-1} L^{p_2}} + \|V * |\psi|^{p-1}\|_{L_t^1 L^\infty} \|\psi\|_{L_t^\infty H^s} \\ &\equiv I_1 + I_2, \end{aligned} \tag{3.12}$$

where p_1 and p_2 are given in (3.9) and (3.10).

We first estimate I_1. By (3.10) and (3.11), the Hardy-Littlewood-Sobolev and Hölder inequalities yield

$$\begin{aligned} \|V * |\psi|^{p-1}\|_{H_{p_1}^s} &\lesssim \||\psi|^{p-1}\|_{H_{\widetilde{q_1}}^s} \\ &\lesssim \|\psi\|_{L^{\widetilde{q_2}}}^{p-2} \|\psi\|_{H^s}, \end{aligned} \tag{3.13}$$

where $\widetilde{q_1}$ and $\widetilde{q_2}$ are given in (3.9) and (3.10). We note by (2.1), (3.4) and (3.6) that

$$\frac{1}{r} - \frac{s-\sigma}{n} < \frac{1}{p-1}\left(1 - \frac{\gamma}{n}\right) = \frac{1}{p_2} = \frac{1}{\widetilde{q_2}} < \frac{1}{r}.$$

Using the embedding $B_r^{s-\sigma} \hookrightarrow L^{p_2}$ and $B_r^{s-\sigma} \hookrightarrow L^{\widetilde{q_2}}$, we obtain

$$I_1 \lesssim \|\psi\|^{p-1}_{L_t^{p-1}B_r^{s-\sigma}} \|\psi\|_{L_t^\infty H^s}. \tag{3.14}$$

We next estimate I_2. We use the following proposition:

Proposition 3.2 *Let $p \geq 3$. Let γ and s satisfy (H1) and let r and σ be given in (2.1). Then there exists $p_5 \geq 1$ such that*

$$\begin{aligned} &-\frac{\gamma}{n} - \frac{1}{r} + 1 < \frac{p-2}{p_5} < 1 - \frac{1}{r}, \\ &0 < \frac{p-2}{p_5} < \frac{s-\sigma}{n} - \frac{\gamma}{n} - \frac{1}{r} + 1, \\ \textit{and} \quad &\frac{1}{r} - \frac{s-\sigma}{n} \leq \frac{1}{p_5} \leq \frac{1}{r}. \end{aligned}$$

Proof. It is sufficient to show that

$$\frac{s-\sigma}{n} - \frac{\gamma}{n} - \frac{1}{r} + 1 > 0, \tag{3.15}$$

$$m > -\frac{\gamma}{n} - \frac{1}{r} + 1, \tag{3.16}$$

$$m > (p-2)\left(\frac{1}{r} - \frac{s-\sigma}{n}\right), \tag{3.17}$$

$$(p-2)\left(\frac{1}{r} - \frac{s-\sigma}{n}\right) < \frac{s-\sigma}{n} - \frac{\gamma}{n} - \frac{1}{r} + 1, \tag{3.18}$$

where $m = \min\{1 - 1/r,\ (p-2)/r\}$. Recall that (3.4) and (3.6) are satisfied. We notice (3.16) is equivalent to (3.6) when $m = (p-2)/r$. If $m = 1 - 1/r$, (3.16) holds true since $\gamma > 0$. We can also easily verify that (3.18) is equivalent to (3.4) by substituting (2.1) into the both sides of (3.18). Moreover, we see that (3.18) yields (3.15) by $\gamma < n$.

Finally, we show (3.17) dividing into two cases. If $n(p-3)/(2(p-2)) < 2n/(n-1) - n(p-3)/2$, then $m = (p-2)/r$. In fact, from (3.5) we have

$$\frac{(p-3)(p-1)}{4(p-2)} \leq \frac{1}{n-1+\theta},$$

which yields

$$p \le 3 + \frac{(p-3)(p-1)}{p-2} \le 3 + \frac{4}{n-1+\theta}.$$

We have used $p \ge 3$ for the first inequality. Hence, $m = (p-2)/r$. Clearly, if $m = (p-2)/r$, (3.17) holds true since $s > \sigma$ and $p \ge 3$.

If $n(p-3)/(2(p-2)) \ge 2n/(n-1) - n(p-3)/2$, recall $\theta = 0$. We have only to deal with the case $m = 1 - 1/r$ since in the other case (3.17) is clearly true as above. From (H1), we have

$$\begin{aligned} s > s_c &= \frac{\gamma - 1}{p-1} + \frac{n(p-3)}{2(p-1)} \\ &> \frac{1}{p-1}\left(\frac{n(p-3)}{2(p-2)} - 1\right) + \frac{n(p-3)}{2(p-1)} \\ &> \frac{1}{p-1} + \frac{2}{(n-1)(p-1)} + \frac{n(p-1)}{p-2}\left(\frac{1}{2} - \frac{2}{(n-1)(p-1)}\right) - \frac{n}{p-2} \\ &= \sigma + \frac{n}{p-2}\left(\frac{p-1}{r} - 1\right) \end{aligned}$$

since $\theta = 0$. By simple calculation we see that the last inequality is equivalent to (3.17) with $m = 1 - 1/r$. This completes the proof of the proposition. ∎

We now return to the proof of Theorem 2.1. Let p_5 be as in Proposition 3.2. We set

$$\frac{1}{p_3} = \frac{\gamma}{n} + \frac{1}{p_4} - 1, \quad \text{and} \quad \frac{1}{p_4} = \frac{1}{r} + \frac{p-2}{p_5}$$

for $p \ge 3$. Then by $\gamma < n$ and Proposition 3.2, we see that $0 < 1/p_3,\ 1/p_4 < 1$ and

$$s - \sigma > \frac{n}{p_3}.$$

By the Sobolev and Hardy-Littlewood-Sobolev inequalities, we have

$$\|V * |\psi|^{p-1}(t)\|_{L^\infty} \lesssim \|V * |\psi|^{p-1}(t)\|_{H^{s-\sigma}_{p_3}} \lesssim \||\psi|^{p-1}\|_{H^{s-\sigma}_{p_4}}.$$

By the Hölder and Sobolev inequalities and Proposition 3.2, we obtain

$$\||\psi|^{p-1}\|_{H^{s-\sigma}_{p_4}} \lesssim \|\psi\|_{B^{s-\sigma}_r} \|\psi\|^{p-2}_{L^{p_5}} \lesssim \|\psi\|^{p-1}_{B^{s-\sigma}_r},$$

and hence,

$$I_2 \lesssim \|\psi\|^{p-1}_{L^{p-1}_t B^{s-\sigma}_r} \|\psi\|_{L^\infty_t H^s}. \tag{3.19}$$

Combining (3.8), (3.12), (3.14) and (3.19) together, we obtain

$$\|A\psi\|_{X^s} \le C_1\|\psi_0\|_{H^s} + C_2\|\psi\|_{X^s}^p \le C_1\|\psi_0\|_{H^s} + C_2M^p.$$

Similarly, we have

$$\begin{aligned}\|F(\psi_1) - F(\psi_2)\|_{L_t^1H^s} &\lesssim (\|\psi_1\|_{X^s}^{p-1} + \|\psi_2\|_{X^s}^{p-1})\|\psi_1 - \psi_2\|_{X^s} \\ &\lesssim M^{p-1}\|\psi_1 - \psi_2\|_{X^s},\end{aligned}$$

and hence

$$\|A\psi_1 - A\psi_2\|_{X^s} \le C_3M^{p-1}\|\psi_1 - \psi_2\|_{X^s}.$$

Thus, if we choose the $\|\psi_0\|_{H^s}$ sufficiently small and $M > 0$ so that,

$$\|\psi_0\|_{H^s} \le \frac{M}{2C_1} \quad \text{and} \quad M^{p-1} \le \min\left\{\frac{1}{2C_2}, \frac{1}{2C_3}\right\},$$

then we obtain

$$\|A\psi\|_{X^s} \le M \quad \text{and} \quad \|A\psi_1 - A\psi_2\|_{X^s} \le \frac{1}{2}\|\psi_1 - \psi_2\|_{X^s},$$

which shows that A is a contraction of X_M^s into itself. Therefore, A admits a unique fixed point $\psi \in X^s$ which satisfies (3.1).

(ii) Using the global solution ψ given in (i), we define the functions $\psi_\pm$ by

$$\psi_\pm(x) = \psi_0(x) + \int_0^{\pm\infty} U(-t')F(\psi(t'))dt'. \tag{3.20}$$

Then $U(t)\psi_\pm$ are solutions of the linear Dirac equation $\partial_t\psi + \alpha\cdot\nabla\psi + i\beta\psi = 0$. Moreover, we see that $\psi_\pm \in H^s$ and (2.4) hold since $\|F(\psi)\|_{H^s} \in L_t^1$ by (3.12), (3.14) and (3.19).

This completes the proof of Theorem 2.1. ■

Proof of Theorem 2.2 We can prove Theorem 2.2 in the same way as the proof of Theorem 2.1. We omit the details. ■

4 Critical Case

In this section, we consider the same problem as in the previous section for the Sobolev exponent including the critical one. Let $p > 3$. We make some preparations to state the

result. We assume that (γ, s) satisfy the following conditions:

$$\text{(H2)} \qquad \begin{cases} s \geq s_c = \dfrac{\gamma - 1}{p-1} + \dfrac{n(p-3)}{2(p-1)}, \\ \\ \max\left\{ \dfrac{n(p-3)}{2(p-2)},\ \dfrac{2n}{n-1} - \dfrac{n(p-3)}{2} \right\} < \gamma < n. \end{cases}$$

We define some function spaces. We put

$$\begin{cases} \dfrac{1}{r_0} = \dfrac{1}{2} - \dfrac{2}{(n-1)(p-1)}, \\ \\ \sigma_0 = \dfrac{n+1}{(n-1)(p-1)}. \end{cases} \tag{4.1}$$

Note that $0 < 1/r_0 < 1/2$. For any (γ, s) satisfying (H2), we have $s > \sigma_0$, hence we can choose σ_1 and σ_2 such that

$$0 < \sigma_2 < \sigma_0 < \sigma_1 < \min\{(s+1)\sigma_0/(1+\sigma_0),\ (n+1-\gamma)\sigma_0,\ (n+1)/(2(n-1))\}. \tag{4.2}$$

For each $\sigma_j,\ j = 1, 2$, we define q_j and $r_j,\ j = 1, 2$ by

$$\frac{1}{q_j} = \frac{(n-1)\sigma_j}{n+1} = \frac{n-1}{2}\left(\frac{1}{2} - \frac{1}{r_j}\right),\ j = 1, 2. \tag{4.3}$$

Then we see that

$$2 < q_1 < p - 1 < q_2. \tag{4.4}$$

For $p,\ r_i,\ \sigma_i,\ i = 0, 1, 2,\ q_j,\ j = 1, 2$ above and $s > 0$, we set

$$Y^s = L^\infty(\mathbf{R}; H^s) \bigcap L^{p-1}(\mathbf{R}; B^{s-\sigma_0}_{r_0}) \bigcap L^{q_1}(\mathbf{R}; B^{s-\sigma_1}_{r_1}) \bigcap L^{q_2}(\mathbf{R}; B^{s-\sigma_2}_{r_2}). \tag{4.5}$$

Theorem 4.1 *Let $p > 3$ and let (γ, s) satisfy (H2). Assume that $s < (p-1)/2$ if p is not an odd integer. If $\|\psi_0\|_{H^s}$ is sufficiently small, then there exists a unique global solution ψ of (2.3) such that $\psi \in Y^s$ defined in (4.5).*

Moreover, for the global solution ψ given in (i), there exist unique $\psi_\pm \in H^s$ satisfying (2.4).

Theorem 4.2 *Let $s,\ \gamma,\ p$ and n be as in the preceding theorem. If $\|\psi^-\|_{H^s}$ is sufficiently small, then the integral equation (2.5) has a unique solution $\psi \in Y^s$ satisfying (2.6).*

From Theorems 4.1 and 4.2, we can also define the scattering operator for small initial data in the case $p > 3$ under the assumptions of Theorem 4.1.

Proof of Theorem 4.1. For $M > 0$, we define the complete metric space Y_M^s by

$$\begin{aligned} Y_M^s &= \{\psi \in Y^s \ : \ \|\psi\|_{Y^s} \le M\}, \\ \|\psi\|_{Y^s} &= \|\psi\|_{L_t^\infty H^s} + \|\psi\|_{L_t^{p-1} B_{r_0}^{s-\sigma_0}} + \|\psi\|_{L_t^{q_1} B_{r_1}^{s-\sigma_1}} + \|\psi\|_{L_t^{q_2} B_{r_2}^{s-\sigma_2}}, \end{aligned}$$

where $r_i,\ \sigma_i,\ i = 0, 1, 2$ and $q_j,\ j = 1, 2$ are defined in (4.1)-(4.3).

We proceed in a similar way to the proof of Theorem 2.1 to show that the map A defined in (3.7) is a contraction of Y_M^s into itself. As before, we have

$$\|A\psi\|_{Y^s} \lesssim \|\psi_0\|_{H^s} + \|F(\psi)\|_{L_t^1 H^s}. \tag{4.6}$$

We choose $p_1, p_2, \widetilde{q_1}$ and $\widetilde{q_2}$ as in (3.9) and (3.10). Then we have (3.12). We can estimate I_1 in the same manner as before. However, as the Sobolev embedding for $\|\psi\|_{L^{\widetilde{q_2}}}$ in (3.13) and $\|\psi\|_{L^{p_2}}$ in I_1, we use

$$B_{r_0}^{s-\sigma_0} \hookrightarrow B_{r_0}^{s_c-\sigma_0} \hookrightarrow L^{p_2},\ L^{\widetilde{q_2}}.$$

In fact, (3.9) and (4.1) imply

$$\frac{1}{r_0} - \frac{s_c - \sigma_0}{n} = \frac{1}{p-1}\left(1 - \frac{\gamma}{n}\right) = \frac{1}{p_2} = \frac{1}{\widetilde{q_2}},$$

and $s \ge s_c$ by assumption. Thus, we obtain

$$I_1 \lesssim \|\psi\|_{L_t^{p-1} B_{r_0}^{s-\sigma_0}}^{p-1} \|\psi\|_{L_t^\infty H^s}. \tag{4.7}$$

We next estimate I_2 following Escobedo and Vega [5]. Since q_1 and q_2 given in (4.3) satisfy (4.4), there exists $0 < \delta < 1$ such that

$$\delta\left(1 - \frac{p-1}{q_1}\right) + (1-\delta)\left(1 - \frac{p-1}{q_2}\right) = 0. \tag{4.8}$$

We show that

$$\|V * |\psi|^{p-1}\|_{L_t^1 L^\infty} \lesssim \|\psi\|_{L_t^{q_1} B_{r_1}^{s-\sigma_1}}^{(p-1)\delta} \|\psi\|_{L_t^{q_2} B_{r_2}^{s-\sigma_2}}^{(p-1)(1-\delta)}, \tag{4.9}$$

where r_j and $\sigma_j,\ j = 1, 2$ satisfy (4.2) and (4.3). We use the following lemma:

Lemma 4.3 *Let $1 < a, b < \infty$, $0 < \alpha$, $\beta < n$ and $0 < \delta < 1$ satisfy*

$$\delta\left(\frac{1}{a} - \frac{\alpha}{n}\right) + (1-\delta)\left(\frac{1}{b} - \frac{\beta}{n}\right) = 0, \tag{4.10}$$
$$\frac{1}{a} - \frac{\alpha}{n} \neq 0 \quad \text{and} \quad \frac{1}{b} - \frac{\beta}{n} \neq 0.$$

Then we have

$$\|u\|_{L^\infty(\mathbf{R}^n)} \lesssim \|u\|_{H^\alpha_a}^{\delta} \|u\|_{H^\beta_b}^{1-\delta}.$$

See [5] for its proof.

By (4.2), we have

$$(\sigma_1 <)\sigma_1\left(\frac{1}{\sigma_0} + 1\right) - 1 < s,$$

which is equivalent to

$$\frac{p-1}{q_1} - 1 < s - \sigma_1$$

since $\sigma_0 = (n+1)/((n-1)(p-1))$ and $1/q_1 = (n-1)\sigma_1/(n+1)$ by (4.3). Hence, we can take $\alpha > 0$ such that

$$0 < \frac{p-1}{q_1} - 1 < \alpha < \min\left\{s - \sigma_1, n + \frac{p-1}{q_1} - 1\right\}. \tag{4.11}$$

We can also take $\beta > 0$ such that

$$0 < \beta < \min\left\{s - \sigma_2, n + \frac{p-1}{q_2} - 1\right\} \tag{4.12}$$

by (4.2) and (4.4). Set

$$\frac{1}{a} = \frac{\alpha}{n} + \frac{1}{n}\left(1 - \frac{p-1}{q_1}\right) \quad \text{and} \quad \frac{1}{b} = \frac{\beta}{n} + \frac{1}{n}\left(1 - \frac{p-1}{q_2}\right).$$

We see by (4.11) and (4.12) that $0 < 1/a < 1$ and $0 < 1/b < 1$. Note that (4.10) is satisfied by (4.8). Applying Lemma 4.3 for $u = V * |\psi|^{p-1}$, we have

$$\|V * |\psi|^{p-1}(t)\|_{L^\infty} \lesssim \|V * |\psi|^{p-1}(t)\|_{H^\alpha_a}^{\delta} \|V * |\psi|^{p-1}(t)\|_{H^\beta_b}^{1-\delta}. \tag{4.13}$$

To estimate the H^α_a - norm in (4.13), we write

$$\frac{1}{a} = \frac{\gamma}{n} + \frac{p-2}{a_1} + \frac{1}{a_2} - 1, \tag{4.14}$$

where

$$\frac{1}{a_1} = \frac{1}{r_1} - \frac{s_c - \sigma_1}{n}, \qquad \frac{1}{a_2} = \frac{\alpha}{n} + \frac{1}{r_1} - \frac{s_c - \sigma_1}{n} \tag{4.15}$$

with $s_c = (\gamma - 1 + n(p-3)/2)/(p-1)$. We note that

$$0 < \frac{1}{a_1} < \frac{1}{a_2} < \frac{1}{2}.$$

In fact, we can verify $0 < 1/a_1 < 1/a_2$ using (4.15),

$$\frac{1}{r_1} = \frac{1}{2} - \frac{2\sigma_1}{n+1}, \quad \sigma_0 = \frac{n+1}{(n-1)(p-1)} \text{ and } \sigma_1 < (n+1-\gamma)\sigma_0$$

in (4.1)-(4.3). Moreover, we have $1/a_1 < 1/a_2 < 1/r_1 \le 1/2$ by (4.11) and (4.15). Note that $s_c - \sigma_1 > 0$ holds by assumption and (4.2).

By (4.14), the Hardy-Littlewood-Sobolev and Hölder inequalities give

$$\|V * |\psi|^{p-1}(t)\|_{H_a^\alpha}^\delta \lesssim \|\psi(t)\|_{L^{a_1}}^{(p-2)\delta} \|\psi(t)\|_{H_{a_2}^\alpha}^\delta.$$

It follows from (4.15), $s \ge s_c$ and the embedding $B_{r_1}^{s-\sigma_1} \hookrightarrow L^{a_1}$ and $B_{r_1}^{s-\sigma_1} \hookrightarrow H_{a_2}^\alpha$ that

$$\|V * |\psi|^{p-1}(t)\|_{H_a^\alpha}^\delta \lesssim \|\psi(t)\|_{B_{r_1}^{s-\sigma_1}}^{(p-1)\delta}.$$

Similarly, the H_b^β - norm in (4.13) is estimated by

$$\|V * |\psi|^{p-1}(t)\|_{H_b^\beta}^{1-\delta} \lesssim \|\psi(t)\|_{B_{r_2}^{s-\sigma_2}}^{(p-1)(1-\delta)},$$

where r_2 and σ_2 are given by (4.2) and (4.3) with $j = 2$. Hence, from (4.13) we have

$$\|V * |\psi|^{p-1}(t)\|_{L^\infty} \lesssim \|\psi(t)\|_{B_{r_1}^{s-\sigma_1}}^{(p-1)\delta} \|\psi(t)\|_{B_{r_2}^{s-\sigma_2}}^{(p-1)(1-\delta)}.$$

Thus, we obtain (4.9) by (4.8) and the Hölder inequality. Therefore,

$$I_2 \lesssim \|\psi\|_{L_t^{q_1} B_{r_1}^{s-\sigma_1}}^{(p-1)\delta} \|\psi\|_{L_t^{q_2} B_{r_2}^{s-\sigma_2}}^{(p-1)(1-\delta)} \|\psi\|_{L_t^\infty H^s} \tag{4.16}$$

holds. Combining (3.12), (4.6), (4.7) and (4.16) together, we obtain

$$\|A\psi\|_{Y^s} \le C_1 \|\psi_0\|_{H^s} + C_2 \|\psi\|_{Y^s}^p \le C_1 \|\psi_0\|_{H^s} + C_2 M^p.$$

Therefore, proceeding as in the proof of Theorem 2.1, we obtain the desired result. ■

Proof of Theorem 4.2. The proof is the same as that of Theorem 4.1. We omit the details. ■

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