TOTAL ABSOLUTE HOROSPHERICAL CURVATURE OF
SUBMANIFOLDS IN HYPERBOLIC SPACE

MARCELO BUOSI, SHYUICHI IZUMIYA, AND MARIA APARECIDA SOARES RUAS

Abstract. We study the horospherical geometry of submanifolds in hyperbolic space. The
main result is a formula for the total absolute horospherical curvature of \( M \), which implies,
for the horospherical geometry, the analogues of classical inequalities of the Euclidean Ge-
ometry. We prove the horospherical Chern-Lashof inequality for surfaces in 3-space and the
horospherical Fenchel and Fary-Milnor’s theorems.

1. Introduction

The hyperbolic Gauss map of a surface in hyperbolic space was introduced by Bryant [1] and
Epstein [5] in the Poincaré ball model. Kobayashi [14, 15] has also independently defined it for a
hypersurface in \( H^n(\mathbb{R}) = SO_0(n, 1)/SO_n \) under a different framework and studied
some basic properties of it from the view point of the theory of Fourier transformations. In
[6], the second named author, D. Pei and T. Sano define the hyperbolic Gauss-Kronecker
curve of hypersurfaces in hyperbolic space. Totally umbilic hypersurfaces in hyperbolic
space are equidistant hypersurfaces, hyperspheres and hyperhorospheres which are the model
hypersurfaces in hyperbolic space. The hyperbolic Gauss-Kronecker curvature is a hyperbolic
invariant which describes the contact of hypersurfaces with such model hypersurfaces. The
notion of horospherical Gauss-Kronecker curvature of hypersurfaces in the model of hyperbolic
space in Minkowski space was introduced in [12]. This curvature is not a hyperbolic invariant
but an \( SO(n) \)-invariant, where we consider the canonical \( SO(n) \)-subgroup in the group of
hyperbolic motions. However it describes the contact of hypersurfaces with hyperhorospheres
and it is independent of the choice of the model of hyperbolic space. We call the geometry
related to this curvature the horospherical geometry of hypersurfaces in hyperbolic space ([6, 7,
9, 10, 13, 12]). Moreover, it has been shown in [12] that a Gauss-Bonnet type theorem holds
when integration of this new curvature is taken over a closed orientable even-dimensional
hypersurface.

On the other hand, the horospherical geometry of higher codimension submanifolds in hy-
perbolic space had been developed in [11]. In this paper we continue this investigation, with
the purpose of studying global properties of such submanifolds. The main result is a for-
formula for the total absolute horospherical curvature of \( M \), which implies the analogues, for the
horospherical geometry, of classical inequalities of the Euclidean geometry. We discuss the

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Chern-Lashof type inequality for surfaces in 3-space

$$\int_M |\tilde{K}_h| \, d\nu_M \geq 2\pi(4 - \chi(M)).$$

This is a special feature of horospherical geometry. In fact, R. Langevin and G. Solanes in [17] contract examples of surfaces in hyperbolic space which do not satisfy the Chern-Lashof type inequality, when the integral is taken with respect to the extrinsic curvature of the surface. In the last section we study curves in $H^2_+(1)$, and obtain the horospherical analogues of the Fenchel, and the Fary-Milnor theorems.

2. Notation and definitions

We outline in this section the local differential geometry of hypersurfaces in the hyperbolic $n$-space developed in the previous papers [6, 12]. We adopt, for this purpose, the model of hyperbolic $n$-space in Minkowski $(n + 1)$-space. Let $\mathbb{R}^{n+1} = \{(x_0, x_1, \ldots, x_n) \mid x_i \in \mathbb{R} \ (i = 0, 1, \ldots, n)\}$ be an $(n + 1)$-dimensional vector space. For any $x = (x_0, x_1, \ldots, x_n)$, $y = (y_0, y_1, \ldots, y_n) \in \mathbb{R}^{n+1}$, the pseudo scalar product of $x$ and $y$ is defined by $\langle x, y \rangle = -x_0y_0 + \sum_{i=1}^n x_iy_i$. We call $(\mathbb{R}^{n+1}, \langle \cdot, \cdot \rangle)$ Minkowski $(n + 1)$-space and denote it by $\mathbb{R}^{n+1}_1$. We say that a non-zero vector $x \in \mathbb{R}^{n+1}$ is spacelike, lightlike or timelike if $\langle x, x \rangle > 0$, $\langle x, x \rangle = 0$ or $\langle x, x \rangle < 0$ respectively. For a vector $v \in \mathbb{R}^{n+1}_1$ and a real number $c$, we define the hyperplane with pseudo normal $v$ by $HP(v,c) = \{x \in \mathbb{R}^{n+1}_1 \mid \langle x, v \rangle = c\}$. We call $HP(v,c)$ a spacelike hyperplane, a timelike hyperplane or a lightlike hyperplane if $v$ is timelike, spacelike or lightlike respectively.

We now define hyperbolic $n$-space by $H^+_n(1) = \{x \in \mathbb{R}^{n+1}_1 \mid \langle x, x \rangle = -1, x_0 \geq 1\}$ and de Sitter $n$-space by $S^+_n = \{x \in \mathbb{R}^{n+1}_1 \mid \langle x, x \rangle = 1\}$. Given $x_1, x_2, \ldots, x_n \in \mathbb{R}^{n+1}_1$, we define a vector $x_1 \wedge x_2 \wedge \cdots \wedge x_n$ by

$$x_1 \wedge x_2 \wedge \cdots \wedge x_n = \begin{vmatrix} -e_0 & e_1 & \cdots & e_n \\ x_0 & x_1 & \cdots & x_n \\ x_0 & x_1 & \cdots & x_n \\ \vdots & \vdots & \cdots & \vdots \\ x_0 & x_1 & \cdots & x_n \end{vmatrix},$$

where $e_0, e_1, \ldots, e_n$ is the canonical basis of $\mathbb{R}^{n+1}$ and $x_i = (x_0^i, x_1^i, \ldots, x_n^i)$. We can easily show that $x_1 \wedge x_2 \wedge \cdots \wedge x_n$ is pseudo orthogonal to any $x_i \ (i = 1, \ldots, n)$. We also define a set $LC_+^n = \{x = (x_0, \ldots, x_n) \in LC_0 \mid x_0 > 0\}$, which is called the future lightcone at the origin.

We now construct the local extrinsic differential geometry on hypersurfaces in $H^+_n(1)$. Let $x : U \rightarrow H^+_n(1)$ be an embedding, where $U \subset \mathbb{R}^{n-1}$ is an open subset. We shall identify $M = x(U)$ and $U$ through the embedding $x$. Since $\langle x, x \rangle = -1$, we have $\langle x_{u_i}(u), x(u) \rangle \equiv 0 \ (i = 1, \ldots, n - 1)$, for any $u = (u_1, \ldots, u_{n-1}) \in U$. Therefore, if we define

$$e(u) = \frac{x(u) \wedge x_{u_1}(u) \wedge \cdots \wedge x_{u_{n-1}}(u)}{\|x(u) \wedge x_{u_1}(u) \wedge \cdots \wedge x_{u_{n-1}}(u)\|},$$

we have $\langle e(u), x_{u_i}(u) \rangle \equiv \langle e(u), x(u) \rangle \equiv 0$, $\langle e(u), e(u) \rangle = 1$, so that $x(u) \pm e(u) \in LC_+^n$ and hence we can define a map

$$L^\pm : U \rightarrow LC_+^n.$$
by \( L^\pm(u) = x(u) \pm e(u) \) which is called the hyperbolic Gauss indicatrix (or the lightcone dual) of \( x \). In order to define the hyperbolic Gauss-Kronecker curvature of the hypersurface \( M = x(U) \), we have shown in [6] that \( D_v L^\pm \in T_p M \). Here, \( D_v \) denotes the covariant derivative with respect to the vector \( v \in T_p M \). This means that \( dL^\pm(u_0) \) is a linear transformation on the tangent space \( T_p M \). We call the linear transformation \( S^\pm_p = -dL^\pm(u_0) : T_p M \to T_p M \) the hyperbolic shape operator of \( M = x(U) \) at \( p = x(u_0) \). We denote the eigenvalues of \( S^\pm_p \) by \( \kappa_i(p) \) \( (i = 1, \ldots, n-1) \) which are called the hyperbolic principal curvatures. The hyperbolic Gauss-Kronecker curvature of \( M = x(U) \) at \( p = x(u_0) \) is defined to be

\[
K_h^\pm(u_0) = \det S^\pm_p = \kappa_1(p) \cdots \kappa_{n-1}(p).
\]

We say that a point \( p = x(u) \) is an umbilic point if \( S^\pm_p = \bar{\kappa}^\pm(p)1_{T_p M} \). In [6] the hyperbolic version of the Weingarten formula was shown. Since \( x_{u_i} \) \( (i = 1, \ldots, n-1) \) are spacelike vectors, we have the Riemannian metric (hyperbolic first fundamental form) given by \( ds^2 = \sum_{i=1}^{n-1} g_{ij}(u_0) du_i du_j \) on \( M = x(U) \), where \( g_{ij}(u) = \langle x_{u_i}(u), x_{u_j}(u) \rangle \) and the hyperbolic second fundamental invariant defined by \( \bar{h}^\pm(u) = \langle -L^\pm_{u_i}(u), x_{u_j}(u) \rangle \) for any \( u \in U \).

**Proposition 2.1.** Under the above notations, we have the following formula:

\[
\mathbb{L}^\pm_{u_i} = -\sum_{j=1}^{n-1} (\bar{h}^\pm)^j \mathbb{L}^\pm_j \mathbb{L}^\pm_{u_j} \quad \text{(The hyperbolic Weingarten formula)},
\]

where \( (\bar{h}^\pm)^j = (\bar{h}^\pm_{ij}) (g^{kj}) \) and \( (g^{kj}) = (g_{kj})^{-1} \). Therefore we have the following formula:

\[
K_h^\pm = \frac{\det (\bar{h}^\pm_{ij})}{\det (g_{ij})}.
\]

In the previous paragraphs we reviewed the properties of hyperbolic Gauss indicatrices and hyperbolic Gauss-Kronecker curvatures. The original definition of the hyperbolic Gauss map introduced by Bryant [1] and Epstein [5] is given in the Poincaré ball model. Here, we introduce the corresponding definition in Minkowski model as follows: If \( x = (x_0, x_1, \ldots, x_n) \) is a lightlike vector, then \( x_0 \neq 0 \). Therefore we have

\[
\hat{x} = \left( 1, \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0} \right) \in S^\pm_{n-1} = \{ x = (x_0, x_1, \ldots, x_n) \mid \langle x, x \rangle = 0, \ x_0 = 1 \}.
\]

We call \( S^\pm_{n-1} \) the lightcone \((n - 1)\)-sphere. We define a map

\[
\hat{L}^\pm : U \to S^\pm_{n-1}
\]

by \( \hat{L}^\pm(u) = \mathbb{L}^\pm(u) \) and call it the hyperbolic Gauss map of \( x \). We remark that for \( n = 3 \), our definition of hyperbolic Gauss map is equivalent to the one introduced in [1, 5]. Let \( N_p M \) be the pseudo-normal space of \( T_p M \) in \( T_p \mathbb{R}^{n+1} \). We have the decomposition \( T_p \mathbb{R}^{n+1} = T_p M \oplus N_p M \), so that we also have the Whitney sum \( T \mathbb{R}^{n+1} = TM \oplus NM \). Therefore we have the canonical projection \( \Pi : T \mathbb{R}^{n+1} \to TM \). It follows that we have a linear transformation \( \Pi_p \circ dL^\pm(u) : T_p M \to T_p M \) for \( p = x(u) \) by the identification of \( U \) and \( x(U) = M \) via \( x \). In [12] the following formula was shown:
Proposition 2.2. Under the above notation we have the following horospherical Weingarten formula:

$$\Pi_p \circ \tilde{L}^\pm_{u_i} = -\sum_{j=1}^{n-1} \frac{1}{\ell^\pm_{j,0}(u)} (\tilde{h}^\pm)^j_i x_{u_j},$$

where $\tilde{L}^\pm(u) = (\ell^\pm_{1,0}(u), \ell^\pm_{2,0}(u), \ldots, \ell^\pm_{n,0}(u))$.

We call the linear transformation $\tilde{S}^\pm_p = -\Pi_p \circ d\tilde{L}^\pm$ the *horospherical shape operator* of $M = x(U)$. We also define the principal horospherical curvature $\tilde{\kappa}^\pm_i(p)$ ($i = 1, \ldots, n-1$) as the eigenvalues of $\tilde{S}^\pm_p$. By the above proposition, we have $\tilde{\kappa}^\pm_i(p) = (1/\ell^\pm_{0,0}(u)) \kappa^\pm_i(p)$. The horospherical Gauss-Kronecker curvature of $x(U) = M$ is defined to be $\tilde{K}^\pm_h(u) = \det \tilde{S}^\pm_p$. It follows that we have the following relation between the horospherical Gauss-Kronecker curvature and the hyperbolic Gauss-Kronecker curvature:

$$\tilde{K}^\pm_h(u) = \left( \frac{1}{\ell^\pm_{0,0}(u)} \right)^{n-1} K^\pm_h(u).$$

We remark that $\tilde{\kappa}^\pm_i(p)$ is not invariant under hyperbolic motions but it is an $SO(n)$-invariant. We also remark that the notion of horospherical curvatures is independent of the choice of the model of hyperbolic space. For the purpose, we introduce a smooth function on the unit tangent sphere bundle of hyperbolic space which plays the principal role of the horospherical geometry. Let $SO_0(n, 1)$ be the identity component of the matrix group

$$SO(n, 1) = \{ g \in GL(n + 1, \mathbb{R}) \mid g I_{n,1} g^t = I_{n,1}, \},$$

where

$$I_{n,1} = \begin{pmatrix} -1 & 0 \\ \mathbb{0} & I_n \end{pmatrix} \in GL(n + 1, \mathbb{R}).$$

It is well-known that $SO_0(n, 1)$ acts transitively on $H^n_+(1)$ and the isotropic group at $p = (1, 0, \ldots, 0)$ is $SO(n)$ which is naturally embedded in $SO_0(n, 1)$. Moreover the action induces isometries on $H^n_+(1)$. On the other hand, we consider a submanifold $\Delta = \{ (v, w) \mid \langle v, w \rangle = 0 \}$ of $H^n_+(1) \times S^1_v$ and the canonical projection $\tilde{\pi} : \Delta \longrightarrow H^n_+(1)$. Let $\pi : S(TH^n_+(1)) \longrightarrow H^n_+(1)$ be the unit tangent sphere bundle over $H^n_+(1)$. For any $v \in H^n_+(1)$, we have the local (global) coordinates $(v_1, \ldots, v_n)$ of $H^n_+(1)$ such that $v = (\sqrt{v_1^2 + \cdots + v_n^2} + 1, v_1, \ldots, v_n)$. We can represent the tangent vector $w = \sum_{i=1}^n w_i \partial/\partial v_i \in T_v H^n_+(1)$ by

$$w = (\frac{1}{v_0} \sum_{i=1}^n w_i v_i, w_1, \ldots, w_n)$$

as a vector in Minkowski $(n + 1)$-space. Then $\langle w, v \rangle = (-\frac{1}{v_0} \sum_{i=1}^n w_i v_i) v_0 + \sum_{i=1}^n w_i v_i = 0$. Therefore $w \in S(Tv H^n_+(1))$ if and only if

$$\langle w, w \rangle = 1 \text{ and } \langle v, w \rangle = 0.$$

The above conditions are equivalent to the condition $(v, w) \in \Delta$. This means that we can canonically identify $\tilde{\pi} : \Delta \longrightarrow H^n_+(1)$. Moreover, the
linear action of $SO(1, 1)$ on $\mathbb{R}^{n+1}$ induces the canonical action on $\Delta$ (i.e., $g(v, w) = (gv, gw)$ for any $g \in SO(1, 1)$). For any $(v, w) \in \Delta$, the first component of $v \pm w$ is given by

$$v_0 \pm w_0 = \sqrt{v_1^2 + \cdots + v_n^2} \pm \frac{1}{\sqrt{v_1^2 + \cdots + v_n^2} + 1} \sum_{i=1}^{n} v_i w_i,$$

so that it can be considered as a function on the unit tangent bundle $S(TH^+_n(-1))$.

We now define a function

$$\mathcal{N}^\pm_h : \Delta \to \mathbb{R} ; \mathcal{N}^\pm_h(v, w) = \frac{1}{v_0 \pm w_0}.$$

We call $\mathcal{N}^\pm_h$ a horospherical normalization function on $H^+_n(-1)$. Since $v^2 + \cdots + v_n^2 + 1$ and $\sum_{i=1}^{n} v_i w_i$ are $SO(n)$-invariant functions, $\mathcal{N}^\pm_h$ is an $SO(n)$-invariant function. Therefore, $\mathcal{N}^\pm_h$ can be considered as a function on the unit tangent sphere bundle over the hyperbolic space $SO_0(1, 1)/SO(n)$ which is independent of the choice of the model space. For any embedding $x : U \to H^+_n(-1)$ ($U \subset \mathbb{R}^{n-1}$), we have the unit normal vector field $e : U \to S^1_1$, so that $(x(u), e(u)) \in \Delta$ for any $u \in U$. It follows that

$$\tilde{K}^\pm_h(u) = \mathcal{N}^\pm_h(x(u), e(u))^{n-1} K^\pm_h(u).$$

The right hand side of the above equality is independent of the choice of the model space.

We now consider the global properties of curvatures. Let $M$ be a closed orientable $(n-1)$-dimensional manifold and $f : M \to H^+_n(-1)$ an embedding. We consider the canonical projection $\pi : \mathbb{R}^{n+1}_1 \to \mathbb{R}^n$ defined by $\pi(x_0, x_1, \ldots, x_n) = (0, x_1, \ldots, x_n)$. Then we have orientation preserving diffeomorphisms $\pi|H^+_n(-1) : H^+_n(-1) \to \mathbb{R}^n$ and $\pi|S^{n-1}_1 : S^{n-1}_1 \to S^{n-1}_1$. Consider the outward unit normal $E$ of $f(M)$ in $H^+_n(-1)$, then we define the hyperbolic Gauss indicatrix in the global

$$\mathbb{L} : M \to LC^+_n$$

by

$$\mathbb{L}(p) = f(p) + E(p).$$

The global hyperbolic Gauss-Kronecker curvature function $\mathcal{K}_h : M \to \mathbb{R}$ is then defined in the usual way in terms of the global hyperbolic Gauss indicatrix $\mathbb{L}$. We also define the hyperbolic Gauss map in the global

$$\tilde{\mathbb{L}} : M \to S^{n-1}_1$$

by

$$\tilde{\mathbb{L}}(p) = \tilde{\mathbb{L}}(p).$$

We now define a global horospherical Gauss-Kronecker curvature function $\tilde{\mathcal{K}}_h : M \to \mathbb{R}$ by

$$\tilde{\mathcal{K}}_h(p) = \mathcal{N}_h(f(p), E(p))^{n-1} K_h(p),$$

where we simply write $\mathcal{N}_h = \mathcal{N}^+_h$. In [12] the following Gauss-Bonnet type theorem for the horospherical Gauss-Kronecker curvature was shown.

**Theorem 2.3.** If $M$ is a closed orientable even-dimensional hypersurface in hyperbolic $n$-space, then

$$\int_M \tilde{\mathcal{K}}_h d\mathcal{W}_M = \frac{1}{2} \gamma_{n-1} \chi(M)$$
where $\chi(M)$ is the Euler characteristic of $M$, $d\nu_M$ is the volume form of $M$ and the constant $\gamma_{n-1}$ is the volume of the unit $(n-1)$-sphere $S^{n-1}$.

In [11] the horospherical geometry on general submanifolds in $H^n_+(−1)$ was studied. Let $x : U → H^n_+(−1)$ be an embedding of codimension $(r + 1)$, where $U ⊂ \mathbb{R}^r$ is an open subset $(s + r + 1 = n)$. We also write that $M = x(U)$ and identify $M$ and $U$ through the embedding $x$. For any $p = x(u) ∈ M ⊂ H^n_+(−1)$, we have $(x(u), x(u)) = −1$, so that $(x_{u_i}(u), x(u)) = 0$, where $u = (u_1, u_2, \ldots, u_s)$. The tangent space of $M$ at $p = x(u)$ is

$$T_p M = \langle x_{u_1}(u), x_{u_2}(u), \ldots, x_{u_s}(u) \rangle_\mathbb{R}.$$ 

Let $N_p(M)$ be the normal space of $M$ at $p = x(u)$ in $\mathbb{R}^{n+1}$ and we define $N^h_p(M) = N_p(M) \cap T_p H^n_+(−1)$. Since the normal bundle $N(M)$ is trivial, we can arbitrarily choose a unit normal section $n(u) ∈ S'(N^h_p(M))$. We can consider the differential geometry of general submanifolds in hyperbolic space which generalizes the differential geometry of hypersurfaces in hyperbolic space in [6]. Since $(n(u), n(u)) = 1$ and $(x(u), n(u)) = 0$, $n_{u_i}(u)$ $(i = 1, \ldots, s)$ are orthogonal to both of $n(u)$ and $x(u)$. Therefore we have $n_{u_i}(u) ∈ T_p M ⊕ N^h_p(M)$.

We now consider the orthogonal projections $\pi^T : T_p M ⊕ N^h_p(M) → T_p M$ and $\pi^N : T_p M ⊕ N^h_p(M) → N^h_p(M)$. Let $dn_u : T_u U → T_p M ⊕ N^h_p(M)$ be the derivative of $n$. We define that $d\nu^T_u = \pi^T \circ dn_u$ and $d\nu^N_u = \pi^N \circ dn_u$. We call the linear transformation $A_{p_0}(n) = −d\nu^T_{u_0} : T_{p_0} M → T_{p_0} M$ the $n$-shape operator of $M = x(U)$ at $p_0 = x(u_0)$. Under the identification of $U$ and $M$, the derivative $dx_u$ can be identified with the identity mapping $id_{T_u M}$. We also call the linear transformation $S_{p_0}(n) = −(id_{T_{p_0} M} + d\nu^T_{u_0}) : T_{p_0} M → T_{p_0} M$ the horospherical (or, hyperbolic) $n$-shape operator of $M = x(U)$ at $p_0 = x(u_0)$. We denote the eigenvalue of $A_{p_0}(n)$ by $κ_{p_0}(n)$ and the eigenvalue of $S_{p_0}(n)$ by $κ_{p_0}(n)$. By the relation of $A_{p_0}(n)$ and $S_{p_0}(n)$ we have a relation $κ_{p_0}(n) = κ_{p_0}(n) − 1$. We call $κ_{p_0}(n)$ the horospherical principal curvature at $p_0$ with respect to $n$.

We now define the notion of curvature as follows. The horospherical (or hyperbolic) curvature with respect to $n$ at $p_0 = x(u_0)$ is defined to be

$$K_h(n)(x(u_0)) = K_h(n)_{p_0} = \det S_{p_0}(n).$$

We give the following generalized hyperbolic Weingarten formula. Since $x_{u_i}$ $(i = 1, \ldots, s)$ are spacelike vectors, we induce the Riemannian metric (the hyperbolic first fundamental form ) $ds^2 = \sum_{i=1}^s g_{ij} du_i du_j$ on $M = x(U)$, where $g_{ij}(u) = \langle x_{u_i}(u), x_{u_j}(u) \rangle$ for any $u ∈ U$. We also define the horospherical (or, hyperbolic) second fundamental invariant with respect to the unit normal vector field $n$ by $h_{ij}(n)(u) = \langle -(x + n)_{u_i}(u), x_{u_j}(u) \rangle$ for any $u ∈ U$. If we define the second fundamental invariant with respect to the normal vector field $n$ by $\tilde{h}_{ij}(n)(u) = \langle n_{u_i}(u), x_{u_j}(u) \rangle$, then we have the following relation:

$$\tilde{h}_{ij}(n)(u) = −g_{ij}(u) + h_{ij}(n)(u), \ (i, j = 1, \ldots, s).$$

**Proposition 2.4.** Under the above notations, we have the following horospherical (or, hyperbolic) Weingarten formula with respect to $n$ :

$$\pi^T \circ (x + n)_{u_i} = −\sum_{j=1}^s \tilde{h}_{ij}(n) x_{u_j},$$
with respect to geometry. Let \( f \) and \( \Pi(L) \) where \( T \) write \( K \). We might also say that a point \( K \) is defined by the following commutative diagram

\[
\begin{array}{ccc}
N_p(M) = \langle x(u), n_1(u), \ldots, n_{r+1}(u) \rangle & \rightarrow & \mathbb{R} \\
\sum \mu_k n_k + \mu_{r+2} x & \mapsto & (n_j)_{u_i} \in \langle x(u), n_1(u), \ldots, n_{r+1}(u) \rangle \in \mathbb{R},
\end{array}
\]

Since \( \langle - (x + n)(u), x_u(u) \rangle = 0 \), we have \( h_{ij}(n)(u) = \langle x(u) + n(u), x_{u}(u) \rangle \). Therefore the horospherical second fundamental invariant at a point \( p_0 = x(u_0) \) depends only on \( x(u_0) + n(u_0) \) and \( x_u(u_0) \). By the above corollary, the horospherical curvature also depends only on \( x(u_0) + n(u_0) \) and \( x_u(u_0) \). It is independent on the choice of the normal vector field \( n \). We write \( K(h)(n)(u_0) \) as the horospherical curvature at \( p_0 = x(u_0) \) with respect to \( n_0 = n(u_0) \).

We now arbitrarily choose unit orthonormal sections \( n_j(u) \in S^n(N^h(M)) \) \( (j = 1, \ldots, r + 1) \). Therefore we have

\[
N_p(M) = \langle x(u), n_1(u), \ldots, n_{r+1}(u) \rangle \mathbb{R}.
\]

Since \( \{x, x_u, \ldots, x_{u_1}, n_1, \ldots, n_{r+1} \} \) is a frame of \( T\mathbb{R}^{n+1} \) along \( M \), we have \( n_j(u) = \sum \lambda_i x_{u_i} + \sum \mu_k n_k + \mu_{r+2} x \) for some \( \lambda_i, \mu_k \in \mathbb{R}, i = 1, 2, \ldots, s, j = 1, \ldots, r + 2 \), where we denote \( (n_j)_{u_i} = (\partial n_{j}/\partial u_i)(u) \). It follows from the fact \( \langle n_j, n_j \rangle = 1 \) that \( \langle (n_j)_{u_i}, n_j \rangle = 0 \). Thus we have \( \mu_j = 0 \). By the relation \( \langle x, n_j \rangle = 0 \), we have \( \langle x, (n_j)_{u_i} \rangle = -\langle x_u, n_j \rangle = 0 \). Hence \( \mu_{r+2} = 0 \). Therefore, we have a relation

\[
(n_j)_{u_i}(u) \in \langle x_u(u), \ldots, x_{u_1}(u), n_1(u), \ldots, n_{j-1}(u), n_{j+1}(u), \ldots, n_{r+1}(u) \rangle \mathbb{R}.
\]

3. Total Absolute Horospherical Curvature

In this section \( M \) denotes a compact \( s \)-dimensional manifold and \( f : M^s \rightarrow H^n_+(-1) \) denotes an immersion of codimension \( (r + 1) \). Let \( \nu^1 M \) denote the unitary normal bundle of the immersion \( f \), i.e.:

\[
\nu^1 M = \{(p, \xi) ; \xi \in N^h_p(M) \text{ and } \langle \xi, \xi \rangle = 1 \}.
\]

**Definition 3.1.** The horospherical Gauss map \( \tilde{\mathbb{L}} : \nu^1 M \rightarrow S^{n-1}_+ \) of the immersion \( f : M^s \rightarrow H^n_+(-1) \) is defined by the following commutative diagram

\[
\begin{array}{ccc}
\nu^1 M & \xrightarrow{\tilde{\mathbb{L}}} & LC^*_+ \\
\downarrow \Pi & & \downarrow \Pi \\
S^{n-1}_+ & \xrightarrow{\Pi} & S^{n-1}_+
\end{array}
\]

where \( \mathbb{L} : \nu^1 M \rightarrow LC^*_+ \); \( \mathbb{L}(p, \xi) = f(p) + \xi \) is the hyperbolic Gauss indicatrix of the immersion \( f \) and \( \Pi(v) = \tilde{\nu} \).

The horospherical Gauss map lead us to a curvature in the framework of horospherical geometry. Let \( T_{(x, n)} \nu^1 M \) be the tangent space of \( \nu^1 M \) at \( (x, n) \). We have the canonical identification \( T_{(x, n)} \nu^1 M = T_x M \oplus T_n S^r \subset T_x M \oplus N_x M = T_x \mathbb{R}^{n+1} \), where \( N_x M \) is the normal
vector space of $M$ at $x$ in $\mathbb{R}^{n+1}$. Let $P : \tilde{\mathbb{L}}^* T\mathbb{R}^{n+1} = T\nu^1 M \oplus \mathbb{R}^2 \to T\nu^1 M$ be the canonical projection. It follows that we have a linear transformation

$$P_{\tilde{\mathbb{L}}(x,n)} \circ d\tilde{\mathbb{L}} : T(x,n)\nu^1 M \to T(x,n)\nu^1 M.$$ 

**Definition 3.2.** The horospherical curvature with respect to $n$ at $x$ is defined to be

$$K_h(x,n) = \det \left( P_{\tilde{\mathbb{L}}(x,n)} \circ (-d\tilde{\mathbb{L}}) \right).$$

Let $x : (U, 0) \to (H^+_1(-1), x(0))$ be an embedding and let $\{n_1(u), \ldots, n_r(u), n_{r+1}(u)\}$ be orthonormal normal vector fields on $x(U)$ such that $n_{r+1} = n$. Since $\{\ell_0, \ell_1, \ldots, \ell_{n+1}\}$ is a basis of $T(x,n)\nu^1 M$ and writing $(\ell_0, \ell_1, \ldots, \ell_{n+1})$ as the coordinates of $\mathbb{L}$,

$$K_h(n)(x(u)) = \det \left( \left( \pi^T \circ (x + n)_{u_i}, x_{u_j} \right) (g^{ij}) \right) \quad \text{(3.1)}$$

$$= \det \left( \left( \begin{array}{cc} \langle x + n \rangle_{u_i}, x_{u_j} & \langle x + n \rangle_{u_i}, n_j \end{array} \right) \left( \begin{array}{cc} g^{ij} & 0 \\ 0 & I_r \end{array} \right) \right)$$

Since $\ell_0\tilde{\mathbb{L}} = \mathbb{L}$, we have $(\ell_0)\mathbb{L}_{u_i} = \mathbb{L}_{u_i} - (\ell_0)_{u_i}\tilde{\mathbb{L}}$. It follows that $P_{\tilde{\mathbb{L}}}(\ell_0)\mathbb{L}_{u_i} = \mathbb{L}_{u_i}$, so that we have

$$\tilde{K}_h(x,n)(u) = \det \left( \begin{array}{cc} \frac{1}{\ell_0} \langle x + n \rangle_{u_i}, x_{u_j} & \frac{1}{\ell_0} \langle x + n \rangle_{u_i}, n_j \\ 0 \end{array} \right) \left( \begin{array}{cc} g^{ij} & 0 \\ 0 & I_r \end{array} \right)$$

$$= \frac{1}{\ell_0} \det \left( \langle x + n \rangle_{u_i}, x_{u_j} \right) \quad \text{(3.2)}$$

$$= \frac{1}{\ell_0} \det \left( \pi^T \circ (x + n)_{u_i}, x_{u_j} \right)$$

$$= \frac{1}{\ell_0} K_h(n)(x(u)) = N_{h}^{s+r}(x,n)K_h(n)(x(u)).$$

**Lemma 3.3.** If $\bar{\delta}$ denotes the canonical volume density of $\nu^1 M$ and $\sigma$ denotes the canonical volume density of the lightcone sphere $S^n_{\mathbb{R}}$ then

$$(\tilde{\mathbb{L}}^* \sigma)(x,n) = |\tilde{K}_h(x,n)| \bar{\delta}(x,n).$$

for all $(x,n) \in \nu^1 M$.

**Proof.** Under the above notation it suffices to show that if $(x,n)$ is a nonsingular point of $\tilde{\mathbb{L}}$ then $\det((\partial_i\tilde{\mathbb{L}}, \partial_j\tilde{\mathbb{L}})) = \left(\tilde{K}_h(x,n)\right)^2 \det((\partial_i x, \partial_j x))$. Locally we may write

$$\mathbb{L}(u,t) = x(u) + t_1 n_1(u) + \cdots + t_r n_r(u) + \sqrt{1 - \sum_{i=1}^r t_i^2} n(u).$$
Since $\ell_0\tilde{L} = L$, we calculate that $(\ell_0)\partial t\tilde{L}_{(0,0)} = (\partial_t L - (\partial_t\ell_0)\tilde{L})_{(0,0)}$, where $L = (\ell_0, \ell_1, \ldots, \ell_{n+1})$. It follows that

$$\tilde{g} = \left(\begin{array}{cccc}
(\tilde{L}_{u_1}, \tilde{L}_{u_2})_{1 \leq i, j \leq s} & (\tilde{L}_{u_1}, \tilde{L}_{u_2})_{1 \leq i \leq s, 1 \leq j \leq r} & (\tilde{L}_{u_1}, \tilde{L}_{u_2})_{1 \leq j \leq s, 1 \leq i \leq r} \\
(\tilde{L}_{u_2}, \tilde{L}_{u_3}) & (\tilde{L}_{u_2}, \tilde{L}_{u_3})_{1 \leq i \leq s, 1 \leq j \leq r} & (\tilde{L}_{u_2}, \tilde{L}_{u_3})_{1 \leq j \leq s, 1 \leq i \leq r} \\
(\tilde{L}_{u_3}, \tilde{L}_{u_4}) & (\tilde{L}_{u_3}, \tilde{L}_{u_4})_{1 \leq i \leq s, 1 \leq j \leq r} & (\tilde{L}_{u_3}, \tilde{L}_{u_4})_{1 \leq j \leq s, 1 \leq i \leq r}
\end{array}\right)$$

$$= \left(\frac{1}{\ell_0}\right)^2 \left(\begin{array}{ccc}
\langle (x + n)_{u_1}, (x + n)_{u_2} \rangle_{1 \leq i, j \leq s} & \langle (x + n)_{u_1}, n_j \rangle_{1 \leq i \leq s, 1 \leq j \leq r} & \langle (x + n)_{u_1}, n_j \rangle_{1 \leq j \leq s, 1 \leq i \leq r} \\
\langle (x + n)_{u_2}, n_i \rangle_{1 \leq i \leq s} & \langle n_i, n_j \rangle_{1 \leq i, j \leq r} \\
\langle n_i, n_j \rangle_{1 \leq i \leq s, 1 \leq j \leq r} & I_r 
\end{array}\right).$$

For $j = 1, \ldots, s$ the j-th columns of the two above matrixes satisfy the equation

$$\tilde{g}^j = \tilde{g}_0 + \sum_{l=1}^{r} \langle n_{aj}, n_i \rangle \tilde{g}^{s+l}.$$ 

It follows that

$$\det(\tilde{g}) = \det(\tilde{g}_0) = \left(\frac{1}{\ell_0}\right)^{2(s+r)} \det \left(\begin{array}{c}
\langle \pi^T \circ (x + n)_{u_1}, \pi^T \circ (x + n)_{u_2} \rangle_{1 \leq i, j \leq s} \\
\langle n_i, n_j \rangle_{1 \leq i \leq s, 1 \leq j \leq r} \\
0_{r \times s} \\
I_r
\end{array}\right).$$

By [11, Proposition 3.2], we have $\pi^T \circ (x + n)_{u_1} = -\sum_{\alpha=1}^{s} \tilde{h}^\alpha_i(n)x_{u_\alpha}$, thus

$$\langle \pi^T \circ (x + n)_{u_1}, \pi^T \circ (x + n)_{u_2} \rangle = \left\langle \sum_{\alpha=1}^{s} \tilde{h}^\alpha_i(n)x_{u_\alpha}, \sum_{\beta=1}^{s} \tilde{h}^\beta_j(n)x_{u_\beta} \right\rangle$$

$$= \sum_{\alpha, \beta} \tilde{h}^\alpha_i(n)\tilde{h}^\beta_j(n)\langle x_{u_\alpha}, x_{u_\beta} \rangle$$

$$= \sum_{\alpha, \beta} \tilde{h}^\alpha_i(n)\tilde{h}^\beta_j(n)g_{\alpha\beta}.$$ 

Finally, by [11, Corollary 3.3], (3.3) and (3.1), we obtain

$$\det(\tilde{g}) = \left(\frac{1}{\ell_0^{s+r}}\right)^2 (K_h(x, n))^2 \det(g_{\alpha\beta}) = (\tilde{K}_h(x, n))^2 \det(g_{\alpha\beta}).$$

$\square$
Definition 3.4. The total absolute horospherical curvature of the immersion \( f \) is defined by

\[
\tau_h(f) = \frac{1}{\text{vol}(S^n_{+})} \int_{\nu^1\mathcal{M}} \hat{\mathcal{L}}^* \sigma.
\]

From Lemma 3.3 we obtain the following formula for \( \tau_h(f) \):

\[
(3.4) \quad \tau_h(f) = \frac{1}{\text{vol}(S^n_{+})} \int_{\nu^1\mathcal{M}} |\tilde{K}_h(x, n)| d\mu_\delta(x, n),
\]

where \( \delta \) denotes the canonical volume density of \( \nu^1\mathcal{M} \). Now we will relate the total absolute horospherical curvature to the critical point of certain height functions.

Definition 3.5. Let \( v \in S^n_{+} \) be a fixed vector. We define the horospherical height function in the \( v \) direction as:

\[
h_v : \mathcal{M} \rightarrow \mathbb{R}, \quad h_v(x) = \langle f(x), v \rangle.
\]

Theorem 3.6. A point \( x \in \mathcal{M} \) is critical for \( h_v \) if and only if there is a unit normal \( n \) to \( \mathcal{M} \) at \( x \) such that \( \tilde{L}(x, n) = v \). Moreover, \( h_v \) is a Morse function if and only if \( v \) is a regular value of the Horospherical Gauss map.

Let \( D \subset S^n_{+} \) denote the set of regular values of \( \tilde{L} \). Since \( \mathcal{M} \) is compact, \( D \) is open and, by Sard’s theorem, the complement of \( D \) in \( S^n_{+} \) has null measure. We define the integer valued map \( \kappa : D \rightarrow \mathbb{N} \) by setting

\[
\kappa(v) = \text{number of elements of } \tilde{L}^{-1}(v),
\]

which turns out to be continuous.

Lemma 3.7. Let \( \phi : \nu^1\mathcal{M} \rightarrow \mathbb{R} \) be a \( \mu_\delta \)-integrable function. Then the function \( D \ni n \mapsto \sum_{x \in \tilde{L}^{-1}(n)} \phi(x, n) \in \mathbb{R} \) is \( \mu_\sigma \)-integrable and the following identity holds:

\[
\int_{\nu^1\mathcal{M}} |\tilde{K}_h(x, n)| \phi(x, n) d\mu_\delta(x, n) = \int_{D} \left( \sum_{x \in \tilde{L}^{-1}(n)} \phi(x, n) \right) d\mu_\delta(x, n).
\]

Proof. Since \( \tilde{K}_h(x, n) \) vanishes when \( (x, n) \) is a critical point of \( \tilde{L} \) and since the set of regular points of \( \tilde{L} \) outside \( \tilde{L}^{-1}(D) \) has null measure we have:

\[
\int_{\nu^1\mathcal{M}} |\tilde{K}_h(x, n)| \phi(x, n) d\mu_\delta(x, n) = \int_{\tilde{L}^{-1}(D)} |\tilde{K}_h(x, n)| \phi(x, n) d\mu_\delta(x, n).
\]

Observe that the restriction of the horospherical Gauss map to \( \tilde{L}^{-1}(D) \) is a smooth covering map onto \( D \). Hence the conclusion follows by applying Fubini’s theorem for covering maps to compute the righthand side of the equality above.

Taking \( \phi \) constant and equal to 1 in the above lemma we obtain

Theorem 3.8. Let \( f : M^s \rightarrow H^n_+(-1) \) be an immersion of the compact manifold \( M \). Then

\[
\tau_h(f) = \frac{1}{\text{vol}(S^n_{+})} \int_{D} \kappa(v) d\mu_\sigma(v).
\]

Recall that the Morse number of a compact manifold \( M \), \( \gamma(M) \), is defined as the minimum number of critical points which any Morse function \( \varphi : M \rightarrow \mathbb{R} \) can posses.
Corollary 3.9. Let \( f : M^s \to H^s_+(−1) \) be an immersion of the compact manifold \( M \). Then

1. \( \tau_h(f) \geq \gamma(M) \geq 2; \)
2. if \( \tau_h(f) < 3 \) then \( M \) is homeomorphic to the sphere \( S^s \).

Proof. Since \( M \) is compact, for every \( v \in D \), the horospherical height function \( h_v \) has at least \( \gamma(M) \geq 2 \) critical points, so that \( \kappa(v) \geq \gamma(M) \geq 2 \). The proof of (1) follows then from Theorem 3.8. If \( \tau_h(f) < 3 \) then \( \kappa(v) = 2 \) for some \( v \in D \) and therefore \( h_v \) is a Morse function with exactly two critical points. The proof of the second statement then follows from the well known Reeb’s theorem. \( \square \)

When \( \tau_h(f) = \gamma(M) \) it follows that every non degenerate horospherical height function \( h_v \) has the minimum number of critical points allowed by the Morse inequalities, ie \( f \) is a horo-tight immersion. It follows that \( \tau_h(f) \) is a hyperbolic invariant when \( f \) is a horo-tight immersion.

The relationship between horo-tightness and total absolute horospherical curvature of manifolds deserves further study. For example, we have the following questions:

Question 3.10. How is the geometry of \( f(M) \subset H^s_+(−1) \) if \( \tau_h(f) = 2? \) Is \( \gamma(M) \) the greatest lower bound of \( \tau_h(f) \) for all smooth immersions \( f \) of \( M \) into all hyperbolic spaces?

In the following sections we specialize to surfaces and curves in \( H^3_+(−1) \).

4. Surfaces in \( H^3_+(−1) \)

For an immersion of a closed surface \( M \) in Euclidean space, let \( K \) denote its Gaussian curvature. Chern and Lashof proved in [4] that

\[
(4.1) \quad \int_M |K| \geq 2\pi (4 - \chi(M)),
\]

where \( \chi(M) \) is the Euler characteristic of \( M \). For immersed surfaces in hyperbolic spaces, the inequality (4.1) was expected to hold, according to [16]. However, R. Langevin and G. Solanes in [17] construct examples of surfaces in hyperbolic space which do not satisfy the Chern-Lashof inequality, when the integral is taken with respect to the extrinsic curvature of the surface. In this section we prove that the Chern-Lashof type inequality with respect to the horospherical Gauss-Kronecker curvature holds for surfaces in hyperbolic 3-space.

Theorem 4.1. Let \( M^2 \) be an embedded compact surface in \( H^3_+(−1) \), then

\[
\int_M |\tilde{K}_h| \, dv_M \geq 2\pi (4 - \chi(M))
\]

where \( \chi(M) \) is the Euler characteristic of \( M \) and \( dv_M \) is the volume form of \( M \).

Proof. Let \( \tilde{L} : M^2 \to S^2_+ \) denote the horospherical Gauss map of \( M^2 \subset H^3 \) and let \( M^- = \{ p \in M; \tilde{K}_h(p) < 0 \} \). We can write

\[
\int_M |\tilde{K}_h| \, dv_M = \int_{M^+} \tilde{K}_h \, dv_M - \int_{M^-} \tilde{K}_h \, dv_M,
\]

and

\[
\int_M \tilde{K}_h \, dv_M = \int_{M^+} \tilde{K}_h \, dv_M + \int_{M^-} \tilde{K}_h \, dv_M.
\]
From the horospherical Gauss-Bonnet Theorem and the above equations it follows that
\[ \int_M |\tilde{K}_h| \, d\nu_M = 2 \int_{M^+} \tilde{K}_h \, d\nu_M - 2\pi \chi(M) \]
The result now follows from Lemma 4.2 below. \qed

**Lemma 4.2.** Let \( M \subset H_+^3(-1) \) be an embedded compact surface and \( M^+ = \{ p \in M; \tilde{K}_h(p) > 0 \} \) then
\[ \int_{M^+} \tilde{K}_h \, d\nu_M \geq 4\pi. \]

**Proof.** Let \( M_0, M_1, M_2, M_2^+ \) be the subsets of \( M \) defined by \( M_0 = \tilde{K}_h^{-1}(0) \), \( M_1 = \{ p \in M \setminus M_0; \) there exists some point \( q \in M_0 \) with \( \tilde{L}(q) = \tilde{L}(p) \} \), \( M_2 = M \setminus (M_0 \cup M_1) \) and \( M_2^+ = M^+ \cap M_2 \).

Of course \( M_0 \) is the singular set of \( \tilde{L} \) so \( \tilde{L}(M_0) \) has measure zero by Sard’s Theorem and also \( \tilde{L}(M_2) \cup \tilde{L}(M_1) \) is a measure zero set. On the other hand, for any \( \nu \in S^2 \setminus (\tilde{L}(M_0) \cup \tilde{L}(M_1)) \), the horospherical height function \( h_\nu \) has at least two critical points: a maximum and a minimum. At such points the horospherical Gauss-Kronecker curvature is positive. The result is then proved since \( \tilde{L} \) is surjective. \qed

**Remark 4.3.** If we consider \( \mathbb{L}^{-}(p) = f(p) - \mathbb{E}(p) \), we can show that the similar inequality
\[ \int_M |\tilde{K}_h^-| \, d\nu_M \geq 2\pi(4 - \chi(M)) \]
holds for another horospherical Gauss-Kronecker curvature \( \tilde{K}_h^-(p) = \det \tilde{S}_p^- \) instead of \( \tilde{K}_h(p) = \tilde{K}_h^+(p) = \det \tilde{S}_p^+ \). This fact describes the situation in hyperbolic space is quite different from the Euclidean case. The original Chern-Lashof inequality in Euclidean space was given as a corollary of the general codimensional inequality [4]. However, since the unit normal bundle \( \nu^1M \) has two connected components for a hypersurface, we have the following inequality as a direct consequence of Corollary 3.9:
\[ \frac{1}{4\pi} \left( \int_M |\tilde{K}_h^+| \, d\nu_M + \int_M |\tilde{K}_h^-| \, d\nu_M \right) \geq \gamma(M). \]
It follows from the Morse relation that \( \gamma(M) = 4 - \chi(M) \) for a closed surface \( M \). Therefore we have an inequality
\[ \int_M |\tilde{K}_h^+| \, d\nu_M + \int_M |\tilde{K}_h^-| \, d\nu_M \geq 4\pi(4 - \chi(M)). \]
Although \( \mathcal{K}_h^+ \neq \mathcal{K}_h^- \) in general, we have the Chern-Lashof type inequality in Theorem 4.1.

5. **Total absolute horospherical Gaussian curvature of curves**

In this section we consider curves in \( H_+^3(-1) \). Let \( \gamma : I \rightarrow H_+^3(-1) \) be a regular curve. Since \( H_+^3(-1) \) is a Riemannian manifold, we can reparametrise \( \gamma \) by the arc-length. Hence, we may assume that \( \gamma(s) \) is a unit speed curve. So we have the tangent vector \( t(s) = \gamma'(s) \) with \( ||t(s)|| = 1 \). In the case when \( \langle t'(s), t'(s) \rangle \neq -1 \), we have a unit vector \( n(s) = \frac{t'(s) - \gamma(s)}{||t'(s) - \gamma(s)||} \). Moreover, define \( e(s) = \gamma(s) \wedge t(s) \wedge n(s) \), then we have a pseudo orthonormal frame
We define \( N(s, \theta) = \cos \theta n(s) + \sin \theta e(s) \) as a normal section along \( \gamma \). Therefore we have \( L(s, \theta) = \gamma(s) + N(s, \theta) \). By a straightforward calculation, we have

\[
\frac{\partial L}{\partial \theta}(s, \theta) = (1 - \kappa_h(s)) t(s) - \sin \theta \tau_h(s) n(s) + \cos \theta \tau_h(s) e(s),
\]

and

\[
\frac{\partial L}{\partial s}(s, \theta) = -\sin \theta n(s) + \cos \theta e(s).
\]

It follows that we have

\[
K_h(N_\theta)(\gamma(s)) = \left\langle \frac{\partial L}{\partial \theta}(s, \theta), t(s) \right\rangle = 1 - \kappa_h(s),
\]

so that

\[
\tilde{K}_h(N_\theta)(\gamma(s)) = N_h(\gamma(s), \cos \theta n(s) + \sin \theta e(s))^2(1 - \kappa_h(s)).
\]

By definition, the total absolute horospherical curvature of \( \gamma \) at \( s \) is

\[
\tilde{\kappa}_h(s) = \int_0^{2\pi} N_h(\gamma(s), \cos \theta n(s) + \sin \theta e(s))^2|1 - \kappa_h(s)|d\theta.
\]

By Corollary 2.9, we have the following horospherical Fenchel type theorem:

**Theorem 5.1.** Let \( \gamma : S^1 \rightarrow H^+_1(-1) \) be an embedding. Then we have the following inequality:

\[
\int_{S^1} \tilde{\kappa}_h ds \geq 8\pi.
\]

**Remark 5.2.** We describe here why we say that the above theorem is a Fenchel type theorem. For an embedded curve \( \gamma : S^1 \rightarrow \mathbb{E}^3 \) in Euclidean space, Fenchel’s theorem asserts that

\[
\int_{S^1} \kappa(s) ds \geq 2\pi.
\]

In this case if we consider the canal surface \( CM_{\gamma_r} \) of \( \gamma \) with sufficiently small positive real number \( r \), we have a relation

\[
4 \int_{S^1} \kappa ds = \int_{CM_{\gamma_r}} |K| d\mathbf{v}_{CM_{\gamma_r}}.
\]
So the Fenchel’s theorem follows from the Euclidean Chern-Lashof inequality. However for the horospherical curvature, we have no such a relation.

We can also show the following horospherical Fary-Milnor type theorem.

**Theorem 5.3.** Let \( \gamma : S^1 \to H^3_+(-1) \) be an embedding. If \( \gamma \) is knotted, then 
\[
\int_{S^1} \kappa_h ds \geq 16\pi.
\]

Before we start to give the proof, we need another interpretation of the total absolute horospherical curvature of a surface \( f : M \to H^3_+(-1) \).

We now consider an integer valued function \( \eta : S^2_+ \setminus C \to \mathbb{N} \) defined by
\[
\eta(v) = \text{the number of } \tilde{L}^{-1}(v),
\]
where \( C = \tilde{L}(\tilde{K}_h^{-1}(0)) \) (i.e., the critical value set of \( \tilde{L} \)). By Sard’s Theorem, \( C \) has measure zero, so that \( \eta \) is a well-defined continuous function almost everywhere in \( S^2_+ \).

Any point \( v \in S^2_+ \) has a neighbourhood \( U \) on which \( \eta \) has a constant value \( m \) and such that \( \tilde{L}^{-1}(U) \subset M \) is the disjoint union of \( m \) open sets \( V_1, \ldots, V_m \) on each of which \( \tilde{L} : V_\alpha \to U \) is a diffeomorphism.

Since \( \tilde{L}^*d\nu_{S^2_+} = \tilde{K}_h d\nu_M \), we have
\[
\int_{V_\alpha} |\tilde{K}_h| d\nu_M = \int_U d\nu_{S^2_+}.
\]

From this we see that
\[
\int_M |\tilde{K}_h| d\nu_M = \int_{S^2_+ \setminus C} \eta d\nu_{S^2_+}.
\]

We also need some definitions. If the horosphere is defined by \( H^3_+(-1) \cap HP(v, c) \) for a lightlike vector \( v \), we say that the center direction of the horosphere is \( v \). Moreover if we have two points \( x_1, x_2 \in H^3_+(-1) \cap HP(v, c) \), we have a lightlike affine plane
\[
LP(x_1, x_2; v) = \{ \lambda x_1 + (1 - \lambda)x_2 + \mu v \ | \lambda, \mu \in \mathbb{R} \} \subset HP(v, c).
\]

Then we have a horocycle defined by \( H^3_+(-1) \cap LP(x_1, x_2; v) \). We call it the horocycle joining \( x_1, x_2 \) with the center direction \( v \). Under these preparations, we can prove Theorem 5.3.

**Proof.** Suppose that
\[
\int_{S^1} \kappa_h ds < 16\pi.
\]

Then we have
\[
\int_{S^2_+} \eta d\nu_{S^2_+} < 16\pi.
\]

This means that some point \( v \in S^2_+ \) is the image of \( \tilde{L} \) at most 3 points of \( M = \nu_1(\gamma(S^1)) \). Since the hyperbolic Gauss indicatrix of \( M \) is
\[
L(s, \theta) = \gamma(s) + N(s, \theta) = \gamma(s) + (\cos \theta n(s) + \sin \theta e(s)),
\]
the above condition is equivalent to saying that $t(s) = \gamma'(s)$ is pseudo-perpendicular to $v$ for at most 3 values of $s$. Since

$$\langle v, \gamma'(s) \rangle = \frac{d}{ds} \langle v, \gamma(s) \rangle,$$

the hyperbolic height function from $h_v(s) = \langle v, \gamma(s) \rangle$ of $\gamma$ has at most 3 critical points. Since the number of relative maxima or minima of the hyperbolic height function is even, there must be just one of each. Therefore the curve $\gamma$ must consist of two arcs joining the lowest and highest points, each arc having monotonically increasing height relative to $v$ direction. Since the level surface of the hyperbolic height function is a horosphere, each horosphere $H^3_+(-1) \cap HP(v, c)$ with the center direction $v$ between these lowest and highest points intersects the curve in 2 points. Joining each such pair by the horocycle with the center direction $v$, we obtain an embedded disc whose boundary is the curve $\gamma$. Therefore $\gamma$ is unknotted. This completes the proof. □

References
