ELLIPTIC ESTIMATES INDEPENDENT OF DOMAIN EXPANSION

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ABSTRACT. In this paper, we consider elliptic estimates for a system with smooth variable coefficients on a domain $\Omega \subset \mathbb{R}^n, n \geq 2$ containing the origin. We first show the invariance of the estimates under a domain expansion defined by the scale that $y = Rx$, $x, y \in \mathbb{R}^n$ with parameter $R > 1$, provided that the coefficients are in a homogeneous Sobolev space. Then we apply these invariant estimates to the global existence of unique strong solutions to a parabolic system defined on an unbounded domain.

1. INTRODUCTION

We consider the linear elliptic system $Lu = f$ defined by

$$(Lu)^\alpha \equiv -\sum_{i,j,\beta} \partial_i \left(A_{ij}^{\alpha\beta} \partial_j u^\beta \right) + mu^\alpha = f^\alpha \qquad \text{in} \quad \Omega_R. \tag{1.1}$$

Here u is an N dimensional vector field defined on $\mathbb{R}^n$ for $N \geq 1, n \geq 2$ and $1 \leq i,j \leq n$, $1 \leq \alpha, \beta \leq N$. m is a nonnegative bounded scalar function. Ω_R is the domain scaled with parameter R and is defined by

$$\Omega_R = \{y : y = Rx, \quad x \in \Omega\}, \quad R > 1. \tag{1.2}$$

The domain Ω is a bounded domain with $C^{1,1}$ boundary containing the origin in $\mathbb{R}^n$. We assume that the coefficients $A_{ij}^{\alpha\beta}$ and m are functions of $x \in \Omega_R$ satisfying the following assumptions:

- $A_{ij}^{\alpha\beta}$ are uniformly continuous on $\overline{\Omega_R}$, m is bounded and for some fixed positive constant Λ

$$\sum_{i,j,\alpha,\beta} \sup_{\overline{\Omega_R}} |A_{ij}^{\alpha\beta}| + \sup_{\overline{\Omega_R}} m \leq \Lambda, \tag{1.3}$$

- For some fixed positive constant λ

$$\sum_{i,j,\alpha,\beta} A_{ij}^{\alpha\beta}(x)\xi_i^\alpha \xi_j^\beta \geq \lambda |\xi|^2 \quad \text{for all} \quad x \in \Omega_R, \ \xi \in \mathbb{R}^{nN}. \tag{1.4}$$

The gradient ∇ is defined by $\nabla = (\partial_1, \cdots, \partial_n)$, where $\partial_i = \frac{\partial}{\partial x_i}$.

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Our main concerns are to find some conditions for elliptic estimates of the operator L independent of domain expansion, namely, independent of the parameter $R > 1$.

The elliptic estimate in this paper is the following: if Ω is a bounded $C^{1,1}$ domain, $\nabla A \in L^\infty(\Omega)$, $f \in L^q(\Omega)$ and u is a solution of (1.1) in $W_0^{1,q} \cap W^{2,q}(\Omega)$ for some $1 < q < \infty$, then u satisfies that

$$\|\nabla^2 u\|_{L^q(\Omega)} \le C(\|f\|_{L^q(\Omega)} + \|u\|_{L^q(\Omega)}), \tag{1.5}$$

where the constant C depends on $\Lambda, \lambda, n, q, \Omega, \partial\Omega, \|\nabla A\|_{L^\infty(\Omega)}$, the modulus of continuity of A and so on. Furthermore, if the uniqueness of solutions to the system (1.1) is guaranteed, then we can say that there exists a constant C independent of u and f such that

$$\|\nabla^2 u\|_{L^q(\Omega)} \le C\|f\|_{L^q(\Omega)}. \tag{1.6}$$

However, we do not know on which parameters the constant C depends exactly (especially, on the scale parameter R). For details of the estimates (1.5) and (1.6), we refer the readers to the papers and books of [3, 7, 10, 11, 15, 16, 17].

Throughout this paper, we use the notation for Sobolev space $W^{k,q}(\Omega)$, $k \ge 0, 1 \le q \le \infty$ and $H^k(\Omega) = W^{k,2}(\Omega)$. The space $W_0^{k,q}(\Omega)(1 \le q < \infty)$ is the closure of $C_0^k(\Omega)$ functions in $W^{k,q}(\Omega)$.

Now let us introduce a new question: "Can we get the estimate

$$\|\nabla^2 u\|_{L^q(\Omega_R)} \le C(\|f\|_{L^q(\Omega_R)} + \|u\|_{L^q(\Omega_R)}) \quad \text{or} \quad \|\nabla^2 u\|_{L^q(\Omega_R)} \le C\|f\|_{L^q(\Omega_R)}$$

with the constant C independent of R ?" If A and m are constants, then the answer is positive. In fact, let $\widetilde{u}(x) = u(Rx)$ for a solution u to $Lu = f$ in Ω_R. Then $\widetilde{u}$ is also a solution to the system $L\widetilde{u} = \widetilde{f}$ in Ω, where $\widetilde{f}(x) = R^2 f(Rx)$. The operator L is not changed under the scaling $x \mapsto Rx$. Hence we get from (1.5) and (1.6) the estimates

$$\|\nabla^2 \widetilde{u}\|_{L^q(\Omega)} \le C(\|\widetilde{f}\|_{L^q(\Omega)} + \|\widetilde{u}\|_{L^q(\Omega)}) \ \text{and} \ \|\nabla^2 \widetilde{u}\|_{L^q(\Omega)} \le C\|\widetilde{f}\|_{L^q(\Omega)}$$

with C independent of R and by rescaling

$$\|\nabla^2 \widetilde{u}\|_{L^q(\Omega_R)} \le C(\|\widetilde{f}\|_{L^q(\Omega_R)} + \|\widetilde{u}\|_{L^q(\Omega_R)})$$
$$\|\nabla^2 u\|_{L^q(\Omega_R)} \le C\|f\|_{L^q(\Omega_R)} \tag{1.7}$$

for $R > 1$, respectively. However, if A is not constant, then the situation is quite different. For an invariant estimate under domain expansion, we need to scrutinize on which factors the constant C depends and to find some conditions for the invariance.

In this connection, let us introduce an invariant condition under the domain expansion as follows

- (Scaling condition)

$$\nabla A_{ij}^{\alpha\beta} \in L^r(\Omega_R) \tag{1.8}$$

for some r such that $n < r < \infty$ if $q \leq n$ and $q \leq r < \infty$ if $q > n$.

Using the conditions (1.3), (1.4), (1.8), we have

Theorem 1.1. *Let Ω be a bounded domain containing the origin of $\mathbb{R}^n, n \geq 2$, with $C^{1,1}$ boundary $\partial\Omega$. Assume that $f \in L^q(\Omega_R)$ for some $2 \leq q < \infty$, A and m satisfy the conditions (1.3), (1.4), (1.8) for some fixed r on the scaled domain Ω_R. Then there exists a unique solution $u \in (W_0^{1,q} \cap W^{2,q})(\Omega_R)$ to the boundary value problem $Lu = f$ in $\Omega_R, R > 1$, $u = 0$ on $\partial\Omega_R$ satisfying*

$$\|u\|_{W^{2,q}(\Omega_R)} \leq C(\|f\|_{L^q(\Omega_R)} + (1 + \|\nabla A\|_{L^r(\Omega_R)})^{\frac{2r}{r-n}} \|u\|_{L^q(\Omega_R)}). \tag{1.9}$$

The constant C may depend on $\Lambda, \lambda, N, n, q, r, \Omega$ and the modulus continuity of A but not on R.

For the proof of Theorem 1.1, we use the classical method of freezing coefficient. Before freezing, we scale the space variables with the factor R. Then in freezing coefficients we partition the domain Ω into $\frac{\rho}{R}$ scaled balls, where ρ is the modulus of continuity of A. We revisit the Calderón-Zygmund theories for the whole and half spaces.

If $1 < p < 2$ and $N \geq 2$, then contrary to the scalar case we cannot assert the uniqueness of solution due to the absence of mean value property nor even the existence of solution to the boundary value problem. However, once $u \in (W_0^{1,q} \cap W^{2,q})(\Omega_R)$ is a solution of (1.1) for $1 < q < 2$, the same arguments as above show that u satisfies the estimate (1.9).

Strictly speaking, the right hand side of the estimate (1.9) is not completely uniform on the parameter R because we still do not know how to control the norm $\|u\|_{L^q(\Omega_R)}$. Instead if we assume a further condition on f and restrict the range of q, then there is a possibility to control it.

To expedite, we introduce some function spaces. We denote by $D^{-1}(\Omega_R)$ the dual space of $D_0^1(\Omega_R)$ with $< \cdot, \cdot >$ being the dual paring of $D^{-1}(\Omega_R)$ and $D_0^1(\Omega_R)$. For $n \geq 3$ the space $D_0^1(\Omega_R)$ is defined by

$$D_0^1(\Omega_R) = \{v \in L^{\frac{2n}{n-2}}(\Omega_R) : \|v\|_{D_0^1(\Omega_R)} = \|\nabla v\|_{L^2(\Omega_R)} < \infty, v = 0 \text{ on } \partial\Omega_R\}.$$

Indeed, $H_0^1(\Omega_R) = (D_0^1 \cap L^2)(\Omega_R)$ and $D_0^1(\Omega_R) = H_0^1(\Omega_R)$, provided that Ω is bounded.

Now if we assume that $f \in (D^{-1} \cap L^q)(\Omega_R)$ and the size of norms is uniform on the parameter $R > 1$, then we can remove the term $\|u\|_{L^q(\Omega_R)}$ on the right hand side of (1.9) through a scaling invariant estimate, see Lemma 2.4 below.

Theorem 1.2. *Suppose that A, m, f and Ω satisfy the same condition as in Theorem* 1.1 *with q, r and n satisfying that*

$$q_k \le q < q_{k+1} \text{ for } q_k < \infty,$$
$$\max(q, n) \le r < \infty, \ r \ne n, \ k \ge 0, \ n \ge 3, \tag{1.10}$$

where $q_k = \frac{2n}{n-(4k+2)}$ if $n > 4k + 2$, $q_k = \infty$ by convention if $n \le 4k + 2$.

If we further assume that $f \in (D^{-1} \cap L^{q_0})(\Omega_R)$, then the solution u to the system (1.1) *satisfies that*

$$\|\nabla u\|_{L^2(\Omega_R)} \le C \|f\|_{D^{-1}(\Omega_R)}$$

and that

$$\|u\|_{W^{2, q}(\Omega_R)} \le C \|f\|_{(D^{-1} \cap L^{q_0} \cap L^q)(\Omega_R)} (1 + \|\nabla A\|_{L^r(\Omega_R)})^{\frac{2r(k+1+\delta)}{r-n}}, \tag{1.11}$$

where $\delta = \frac{n}{2}(1/q_k - 1/q)$. Here the constant C also depends on m.

If we assume that $\inf_{\Omega_R} m \ge \underline{m}$ for some positive constant $\underline{m}$ and $f \in (D^{-1} \cap L^q)(\Omega_R)$ for $2 \le q < q_0$ and $n \ge 2$, then

$$\|u\|_{H^1(\Omega_R)} \le C \|f\|_{D^{-1}(\Omega_R)},$$
$$\|u\|_{W^{2, q}(\Omega_R)} \le C \|f\|_{(D^{-1} \cap L^q)(\Omega_R)} (1 + \|\nabla A\|_{L^r})^{\frac{2r}{r-n}}. \tag{1.12}$$

Remark 1. Note that the 2-dimensional case is included for the second result.

For the estimate of the range of q below $\frac{2n}{n-2}$, in general it seems very difficult to avoid the assumption $m > \underline{m}$. To do so, we need another condition on A. For simplicity, let us assume that $m = 0$ and consider the coefficient A of the type $aB + \varepsilon E$. Here a is a scalar C^1 function with $\nabla a \in L^r(\Omega_R)$ for

$$\max(q, n) \le r < \infty, \ r \ne n$$

and satisfying that $\lambda \le a \le \Lambda$ on Ω_R, B is a constant coefficient with $|B| \le 1$ satisfying (1.4), ε is a small positive constant and E is in $C^1(\overline{\Omega_R})$ with $\nabla E \in L^r(\Omega_R)$ and $|E| \le \Lambda$ on Ω_R. Let us denote the elliptic operators corresponding to the coefficients A, B and E by L, L_B and L_E, respectively. Such L is said to be a small perturbation of the constant type elliptic operator L_B.

Theorem 1.3. *Let Ω be a bounded $C^{1,1}$ domain of $\mathbb{R}^n, n \geq 2$. If L is a small perturbation of L_B and $f \in (D^{-1} \cap L^2 \cap L^q)(\Omega_R)$ for $2 \leq q < q_0 = \frac{2n}{n-2}$, then the boundary value problem $Lu = f$ in Ω_R and $u = 0$ on $\partial \Omega_R$ has a unique solution $u \in (W_0^{1,q} \cap W^{2,q})(\Omega_R)$ such that $\|\nabla u\|_{L^2(\Omega_R)} \leq C\|f\|_{D^{-1}(\Omega_R)}$ and*

$$\|\nabla u\|_{W^{1,q}(\Omega_R)} \leq C\|f\|_{(D^{-1} \cap L^2 \cap L^q)(\Omega_R)}(1 + \|\nabla a\|_{L^r(\Omega_R)} + \|\nabla E\|_{L^r(\Omega_R)})^{\frac{2r}{r-n}}.$$
$$(1.13)$$

Remark 2. A typical example of the small perturbation of constant type elliptic operator is the Lamé operator L, which has variable coefficients $A_{ij}^{\alpha\beta}$ defined by

$$A_{ij}^{\alpha\beta}(x) = \mu_1(x)\delta_{\alpha,\beta}\delta_{i,j} + (\mu_1 + \mu_2)(x)\delta_{\alpha,i}\delta_{\beta,j}$$

and hence $Lu = -\text{div}(\mu_1\nabla u) - \nabla(\mu_2\text{div}u)$, where μ_1 and μ_2 are viscosity coefficients satisfying the relation $\mu_1(x) = \nu_1 a(x) + \varepsilon e_1(x)$ and $(\mu_1 + \mu_2)(x) = \nu_2 a(x) + \varepsilon e_2(x)$. This relation is a generalization of the Stokes relation (for instance, see [13]). We assume that a is a C^1 scalar function with $\lambda \leq a \leq \Lambda$ and $\nabla a \in L^r$, ν_i are fixed positive constants less than 1 and e_i is a C^1 scalar function with $\sum_i |e_i| \leq \Lambda$ and $\nabla e_i \in L^r$. The coefficient A can be rewritten as $A = aB + \varepsilon E$, where

$$B_{ij}^{\alpha\beta} = \nu_1\delta_{\alpha,\beta}\delta_{i,j} + \nu_2\delta_{\alpha,i}\delta_{\beta,j}$$
$$E_{ij}^{\alpha\beta} = e_1\delta_{\alpha,\beta}\delta_{i,j} + e_2\delta_{\alpha,i}\delta_{\beta,j}.$$

The elliptic estimate for Lamé operator L as in Theorem 1.3 can be applied to the heat-conducting compressible Navier-Stokes equations with variable coefficients on an unbounded domain, whole or half space, or exterior domain. One can adapt the same arguments of domain expansion of [4, 5, 6] to prove the unique solvability of strong solutions.

Remark 3. If Ω is an exterior domain with compact complement, then we denote the domain $\Omega \cap B_R$ by $\widetilde{\Omega}_R$ for sufficiently large R so that the complement of Ω is contained in B_R. If Ω is the whole space, then $\widetilde{\Omega}_R$ is B_R. If Ω is the half space $\mathbb{R}_+^n$, then we also denote $\widetilde{\Omega}_R$ by R-scaled domain of Ω_+ which is a smooth domain satisfies that $\mathbb{R}_+^n \cap B_1 \subset \Omega_+ \subset \Omega \cap B_2$. Then a solution $u \in (W_0^{1,q} \cap W^{2,q})(\widetilde{\Omega}_R)$ of the system (1.1) on $\widetilde{\Omega}_R$ satisfies the estimates (1.9), (1.11), (1.12) and (1.13) with C independent of $R > 1$.

A direct application of Theorem 1.2 and Theorem 1.3 is to show the elliptic estimates on an unbounded domain Ω. To do this we introduce the homogeneous Sobolev space (for instance see [9])

$$D^{k,r}(\Omega) \equiv \{v \in L_{loc}^1(\Omega) : \|v\|_{D^{k,r}(\Omega)} < \infty\},$$
$$D^k(\Omega) = D^{k,2}(\Omega), \quad \|v\|_{D^{k,r}(\Omega)} = \|\nabla^k v\|_{L^r(\Omega)}.$$

Theorem 1.4. *Assume that Ω is an unbounded domain, a whole or half space, or an exterior domain with $C^{1,1}$ boundary and compact complement. Let $f \in (D^{-1} \cap L^{q_0} \cap L^q)(\Omega)$ for the same indices q_0, q as defined in (1.10) and A be an elliptic coefficient satisfying (1.3), (1.4) and (1.8) .*

(1) *Then there exists a unique solution $u \in D_0^1 \cap D^{2,q}$ of the elliptic system $Lu = f$ with the boundary conditions that $u = 0$ on $\partial\Omega$ and $u(x) \to 0$ as $|x| \to \infty$. Furthermore, u satisfies that*

$$\|\nabla u\|_{L^2(\Omega)} \le C\|f\|_{D^{-1}(\Omega)},$$

$$\|u\|_{W^{2,q}(\Omega)} \le C\|f\|_{(D^{-1} \cap L^{q_0} \cap L^q)(\Omega)}(1 + \|\nabla A\|_{L^r(\Omega)})^{\frac{2r(k+1+\delta)}{r-n}}. \tag{1.14}$$

(2) *If we further assume that $m \ge \underline{m}$ for some positive constant $\underline{m}$, $2 \le q < q_0$ and $n \ge 2$, then*

$$\|u\|_{H_0^1(\Omega)} \le C\|f\|_{D^{-1}(\Omega)},$$

$$\|u\|_{W^{2,q}(\Omega)} \le C\|f\|_{(D^{-1} \cap L^q)(\Omega)}(1 + \|\nabla A\|_{L^r})^{\frac{2r}{r-n}}. \tag{1.15}$$

(3) *If L is a small perturbation of L_B as in Theorem 1.3 and $f \in (D^{-1} \cap L^2 \cap L^q)(\Omega)$ for $2 \le q < q_0$, then*

$$\|\nabla u\|_{W^{1,q}(\Omega)} \le C\|f\|_{(D^{-1} \cap L^2 \cap L^q)(\Omega)}(1 + \|\nabla a\|_{L^r(\Omega)} + \|\nabla E\|_{L^r(\Omega)})^{\frac{2r}{r-n}}. \tag{1.16}$$

Secondly, we apply the above theorems to a linear parabolic system:

$$\varphi u_t + Lu = F \text{ in } [0,T] \times \Omega, \quad u(0,x) = u_0(x) \text{ in } \Omega,$$

$$(Lu)^\alpha = -\sum_{i,j,\beta} \partial_i \left(A_{ij}^{\alpha\beta} \partial_j u^\beta \right). \tag{1.17}$$

Here φ is assumed to be a nonnegative function which has a compact support or decays at space infinity. For the simplicity of presentation we only consider that the domain Ω is the whole $\mathbb{R}^3$ or half $\mathbb{R}_+^3$ space, or an exterior domain of $\mathbb{R}^3$ with $C^{1,1}$ boundary and compact complement. $A_{ij}^{\alpha\beta}$ is $C_b^1([0,T] \times \Omega)$ satisfying the conditions (1.3), (1.4), and the symmetry condition

$$A_{ji}^{\beta\alpha} = A_{ij}^{\alpha\beta} \text{ for all } i,j,\alpha,\beta. \tag{1.18}$$

Since φ can vanish on some open set or decay at space infinity, it is difficult to expect the uniqueness and regularity of solutions. For this purpose, we assume the following compatibility condition that for any $v \in C_0^1(\Omega)$

$$\sum_{i,j,\alpha,\beta} \int_\Omega A_{ij}^{\alpha\beta}(0,x) \partial_j u_0^\beta \partial_i v^\alpha \, dx = \sum_\alpha \int_\Omega (F(0)^\alpha - \sqrt{\varphi} g^\alpha) v^\alpha \, dx \tag{1.19}$$

for some $g = (g^\alpha) \in L^2(\Omega)$.

Theorem 1.5. *Assume that*

$$0 \leq \varphi \in (L^{\frac{3}{2}} \cap L^\infty)(\Omega), u_0 \in D_0^1(\Omega),$$

$$F, F_t \in L^2(0, T; D^{-1}(\Omega)), F \in L^2(0, T; L^6(\Omega))$$

and also assume that $A = A(t, x) \in C_b^1([0, T] \times \Omega)$ satisfies (1.3), (1.4) and (1.18) for each t. If we further assume the compatibility condition (1.19), then there exists a unique solution $u \in L^\infty(0, T_*; D_0^1(\Omega))$ such that $\nabla u \in L^2(0, T; W^{1,6}(\Omega))$ and $u_t \in L^2(0, T; D_0^1(\Omega))$.

For the proof we localize the problem and consider the case where φ has a positive lower bound. Then we show that the localized solutions satisfy a priori estimates uniform for the domain expansion.

2. PRELIMINARY LEMMAS

Before proving the theorems, let us introduce some lemmas which are crucial for the proof of the theorems. The first one is on the elliptic regularity in the case of constant coefficients.

Lemma 2.1. *Let A be a constant coefficient satisfying (1.3) and (1.4), and let v be a vector field in $D^2(\mathbb{R}^n)$. Then for any $1 < q < \infty$ there exists constant C depend only on $\lambda, \Lambda, N, n, q$ such that*

$$\|\nabla^2 v\|_{L^q} \leq C \|L_0 v\|_{L^q}, \tag{2.1}$$

where

$$(L_0 v)^\alpha = -\sum_{i,j,\beta} \partial_i \left(A_{ij}^{\alpha\beta} \partial_j u^\beta \right).$$

Proof of Lemma 2.1. One may find a proof using fundamental solution to the equation $L_0 v = 0$ in $\mathbb{R}^n$ in [2, 3]. Here we introduce a simplified version for a second order elliptic operator.

By density, we may assume that $u \in C_0^2$. Using Fourier transform such that $\widehat{g}(\xi) = \int e^{-ix\cdot\xi} g(x) \, dx$, we have

$$\widehat{(L_0 v)^\alpha}(\xi) = -\sum_{i,j,\beta} A_{ij}^{\alpha\beta} \xi_i \xi_j \widehat{v^\beta}(\xi) = -\sum_{i,j,\beta} A_{ij}^{\alpha\beta} \xi_i' \xi_j' |\xi|^2 \widehat{v^\beta}(\xi) \equiv \sum_\beta M^{\alpha\beta}(\xi') \widehat{\Delta v^\beta}, \tag{2.2}$$

where

$$M^{\alpha\beta}(\xi') = \sum_{i,j} A_{ij}^{\alpha\beta} \xi_i' \xi_j' \quad \text{and} \quad \xi_j' = \frac{\xi_j}{|\xi|}.$$

From the strong ellipticity (1.4), we deduce that

$$\sum_{\alpha,\beta} M^{\alpha\beta}(\xi')\eta_\alpha\eta_\beta \geq \lambda|\eta|^2 \quad \text{for all} \quad \eta \in \mathbb{R}^N,$$

$$|\det M| \geq C(N)\lambda^N,$$

$$|\nabla^k M^{\alpha\beta}| \leq C(N,n,k)\Lambda|\xi|^{-k} \quad \text{for all} \quad k \geq 0.$$

Thus using the recursion

$$\nabla M^{-1} = -M^{-1}(\nabla M)M^{-1} \quad \text{and} \quad \nabla^k M^{-1} = -\nabla^{k-1}(M^{-1}(\nabla M)M^{-1}),$$

we obtain

$$|\nabla^k M^{-1}| \leq C\Lambda^{-1}\left(\frac{\Lambda}{\lambda}\right)^{(k+1)N}|\xi|^{-k} \quad \text{for all} \quad \xi \neq 0 \quad \text{and} \quad k \geq 0. \tag{2.3}$$

From (2.2), we have

$$\widehat{\Delta v^\alpha}(\xi) = \sum_\beta (M^{-1})^{\alpha\beta}\widehat{L_0 v^\beta}(\xi)$$

and hence from (2.3) we deduce that by the Fourier multiplier theorem (for instance, see Proposition 2, p. 245 of [18]), (2.1) holds. □

The following is a half-space version of Lemma 2.1 (for the proof of scalar case see [11]).

Lemma 2.2. *Let A be a constant coefficient satisfying* (1.3) *and* (1.4), *and let v be a vector field in $W_0^{1,1}(\Omega^+)$ and be a weak solution of $L_0 v = f$ in Ω^+ with $u = 0$ near $(\partial\Omega)^+$ for some $f \in L^q$ with $1 < q < \infty$, where $\Omega^+ = \Omega \cap \mathbb{R}_+^n$ and $(\partial\Omega)^+ = \partial\Omega \cap \mathbb{R}_+^n$ for some bounded domain Ω containing the origin and having $C^{1,1}$ boundary. Then $v \in (W_0^q \cap W^{2,q})(\Omega^+)$ and there exists a constant C depending only on $\lambda, \Lambda, N, n, q$ such that*

$$\|\nabla^2 v\|_{L^q(\Omega^+)} \leq C\|L_0 v\|_{L^q(\Omega^+)}. \tag{2.4}$$

Proof of Lemma 2.2. We extend v and f to $\mathbb{R}_+^n$ by defining zero outside of Ω^+. Let the extended functions be still v and f. Then by the support condition v is clearly a weak solution of $L_0 v = f$ in $\mathbb{R}_+^n$. Hence by the representation of solution in half space by the fundamental solution [2, 3], we conclude that $v \in W^{2,q}(\mathbb{R}_+^n)$ and (2.4) holds.

Furthermore, since v^α is $W^{2,q}(\Omega^+)$ function for each α, there are N functions $g^\alpha \in L^q$ such that $\Delta v^\alpha = g^\alpha$ in Ω^+. Hence by setting $v^\alpha = g^\alpha = 0$ outside of Ω^+ and extending v^α and g^α via odd reflection to $\mathbb{R}^n$ and using a compactly supported even smooth function η, one can show that $v^\alpha * \eta_\delta \to v^\alpha$ in $W^{2,q}$ as $\delta \to 0$ and $(v^\alpha * \eta_\delta)(x', 0) = 0$, where $*$ denotes the convolution and $\eta_\delta(x) =$

$\delta^{-n}\eta(x/\delta)$. Therefore we conclude that $v \in W_0^{1,q}(\Omega_+)$. This completes the proof of the lemma. $\qquad \square$

The third one is on the regularity of solutions of (1.1).

Lemma 2.3. *Let $u \in (W_0^{1,q} \cap W^{2,q})(\Omega)$ be a solution to the system (1.1) on a bounded $C^{1,1}$-domain Ω with A satisfying (1.3), (1.4) and (1.8) for some $1 < q < \infty$. Then if $f \in L^s(\Omega)$ for some s with $q < s < \infty$, then $u \in (W_0^{1,s} \cap W^{2,s})(\Omega)$.*

Proof of Lemma 2.3. By applying Caledrón-Zygmund theory equipped with Lemmas 2.1 and 2.2 to the local problems over the interior and near the boundary of Ω, the regularity result follows from the contraction argument in L^p theory of [14]. See also the proof of Lemma 9.16 in [11]. We leave the details of proof to the readers. $\qquad \square$

Finally, we introduce some Sobolev inequalities invariant under domain expansion.

Lemma 2.4. *Assume that Γ is a $C^{1,1}$ domain in $\mathbb{R}^n, n \geq 2$ and $v \in W^{k,s}(\Gamma)$. Let ℓ be a number such that $s \leq \ell \leq \frac{sn}{n-ks}$ if $ks < n$, $s \leq \ell \leq \infty$ if $ks > n$ or $s \leq \ell < \infty$ if $ks = n$. Then there exists a constant C depending on Γ, ℓ, n, k, s such that for any $R > 1$*

$$\|v\|_{L^\ell(\Gamma_R)} \leq C\|v\|_{L^s(\Gamma_R)}^{1-\delta}\|v\|_{W^{k,s}(\Gamma_R)}^{\delta}, \tag{2.5}$$

where $\delta = \frac{n}{k}(1/s - 1/\ell)$ and Γ_R is the R-scaled domain of Γ defined by (1.2).

Proof of Lemma 2.4. For the proof of (2.5) with $R = 1$, see the proof of Theorem 5.8 of [1]. For the invariance, let $v(x) = w(Rx)$. Then by change of variables, we have for any $R > 1$

$$R^{-\frac{n}{\ell}}\|w\|_{L^\ell(\Gamma_R)} \leq CR^{-\frac{n}{s}(1-\delta)}\|w\|_{L^s(\Gamma_R)}^{1-\delta}R^{(k-\frac{n}{s})\delta}\|w\|_{W^{k,s}(\Gamma_R)}^{\delta}.$$

Hence the scaling invariance follows from the fact $k\delta = n(1/s - 1/\ell)$. $\qquad \square$

3. Independence of domain expansion; Proof of Theorem 1.1

In this section, we prove Theorem 1.1. If $q = 2$, then since the operator L is of divergence form, the existence and uniqueness of solution $u \in (H_0^1 \cap H^2)(\Omega_R)$ readily follows from the Lax-Milgram theorem and the standard method of difference quotient. We refer the readers to the book [10]. If $q > 2$, then since the domain is bounded, the existence and the uniqueness follow automatically from the L^2 theory and Lemma 2.3. Thus we have only to consider the estimates (1.9) and (1.11).

For the proof, we scale functions by a parameter R and define the scaled functions by $\widetilde{u}(x) = u(Rx)$, $\widetilde{A}(x) = A(Rx)$, $\widetilde{m}(x) = R^2 m(Rx)$ and $\widetilde{f}(x) = R^2 f(Rx)$. Also we define a scaled operator $\widetilde{L}$ by

$$(\widetilde{L}\widetilde{u})^\alpha \equiv -\sum_{i,j,\beta} \partial_i(\widetilde{A}_{ij}^{\alpha\beta}\widetilde{u}^\beta) + \widetilde{m}\widetilde{u}^\alpha = \widetilde{f}^\alpha \quad \text{on} \quad \Omega.$$

Now fixing $R > 1$, we freeze the coefficient $\widetilde{A}$ near the point $x_0 \in \Omega'$, where Ω' is a precompact subset of Ω (for example we can take $\Omega' = \Omega \setminus (B(0, \varepsilon_0) + \partial\Omega)$ for small ε_0, where ε_0 depends only on ρ, the modulus of continuity of A). Let L_0 be the constant coefficient operator given by $L_0 v = \widetilde{A}(x_0)\nabla^2 v$. Suppose v has support in a ball $B(x_0, \frac{\rho}{R})$ with $\rho < \frac{1}{2}\text{dist}(\partial\Omega, \Omega')$ (hereafter we denote by $B_{\frac{\rho}{R}}$ the ball $B(x_0, \frac{\rho}{R})$). Then we have

$$L_0 v = -(\widetilde{A}(x_0) - \widetilde{A})\nabla^2 v - \widetilde{A}\nabla^2 v,$$

and by (2.1)

$$\|\nabla^2 v\|_{L^q(B_{\frac{\rho}{R}})} \leq C\|L_0 v\|_{L^q(B_{\frac{\rho}{R}})}$$
$$\leq C(\sup_{B_{\frac{\rho}{R}}} |\widetilde{A}(x_0) - \widetilde{A}|\|\nabla^2 v\|_{L^q(B_{\frac{\rho}{R}})} + \|\widetilde{A}\nabla^2 v\|_{L^q(B_{\frac{\rho}{R}})}).$$

Since $\widetilde{A}$ is uniformly continuous on $\overline{\Omega'}$, there exists a positive number δ independent of R such that

$$|\widetilde{A}(x_0) - \widetilde{A}(x)| \leq \frac{1}{2C} \tag{3.1}$$

if $|x - x_0| < \frac{\delta}{R}$. Actually we can choose uniform δ for any $x_0 \in \Omega'$ from the conditions (1.3) and (1.4). Hence we have

$$\|\nabla^2 v\|_{L^q(B_{\frac{\rho}{R}})} \leq C(n, q, \lambda, \Lambda)\|\widetilde{A}\nabla^2 v\|_{L^q(B_{\frac{\rho}{R}})}, \tag{3.2}$$

provided $\rho \leq \delta$.

Choose a smooth cutoff function $\eta \in C_0^2(B_{\frac{\rho}{R}})$ such that $\eta = 1$ on $B_{\frac{\rho}{2R}}$, $\eta = 0$ on $B_{\frac{\rho}{R}} \setminus B_{\frac{3}{4}\frac{\rho}{R}}$ and $|\nabla\eta| \leq \frac{8R}{\rho}$, $|\nabla^2\eta| \leq \frac{64R^2}{\rho^2}$. Then $v = \eta\widetilde{u} \in W_0^{2,q}(\Omega)$ and we have

$$\|\nabla^2\widetilde{u}\|_{L^q(B_{\frac{\rho}{2R}})} \leq \|\nabla^2 v\|_{L^q(B_{\frac{\rho}{R}})}$$
$$\leq C\left(\|\eta\widetilde{A}\nabla^2\widetilde{u}\|_{L^q(B_{\frac{\rho}{R}})} + \|\widetilde{A}\nabla\eta\nabla\widetilde{u}\|_{L^q(B_{\frac{\rho}{R}})} + \|A(\nabla^2\eta)\widetilde{u}\|_{L^q(B_{\frac{\rho}{R}})}\right)$$
$$\leq C\left(\|\widetilde{f}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})} + \|\widetilde{m}\widetilde{u}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})}\right.$$
$$\left. + \|\nabla\widetilde{A}\nabla\widetilde{u}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})} + \|\widetilde{A}\nabla\eta\nabla\widetilde{u}\|_{L^q(B_{\frac{\rho}{R}})} + \|\widetilde{A}(\nabla^2\eta)\widetilde{u}\|_{L^q(B_{\frac{\rho}{R}})}\right)$$
$$\leq C\left(\|\widetilde{f}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})} + \|\nabla\widetilde{A}\nabla\widetilde{u}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})} + \frac{R}{\rho}\|\nabla\widetilde{u}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})} + \frac{R^2}{\rho^2}\|\widetilde{u}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})}\right),$$

provided $\rho \leq \delta \leq 1$.

By the Hölder inequality $\|\nabla\widetilde{A}\nabla\widetilde{u}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})} \leq \|\nabla\widetilde{A}\|_{L^r(\Omega)}\|\nabla\widetilde{u}\|_{L^{\frac{rq}{r-q}}(B_{\frac{3}{4}\frac{\rho}{R}})}$ (with convention $\frac{rq}{r-q} = \infty$ if $r = q$), we have

$$\|\nabla^2\widetilde{u}\|_{L^q(B_{\frac{\rho}{2R}})} \leq C\left(\|\widetilde{f}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})} + \|\nabla\widetilde{A}\|_{L^r(\Omega)}\|\nabla\widetilde{u}\|_{L^{\frac{rq}{r-q}}(B_{\frac{3}{4}\frac{\rho}{R}})}\right.$$
$$\left. + \frac{R}{\rho}\|\nabla\widetilde{u}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})} + \frac{R^2}{\rho^2}\|\widetilde{u}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})}\right). \tag{3.3}$$

For the estimate $\|\nabla\widetilde{u}\|_{L^{\frac{rq}{r-q}}(B_{\frac{3}{4}\frac{\rho}{R}})}$, we apply the Sobolev embedding (2.5) with $s = q$, $\ell = \frac{rq}{r-q}$, $k = 1$ and $\Gamma = B_1$. Since $q < \frac{rq}{r-q} < \frac{nq}{n-q}$ for $q \leq n$ and $q < \frac{rq}{r-q} \leq \infty$ for $q > n$ (here we used the convention that $\frac{nq}{n-q} = \infty$ if $q \geq n$), we deduce that

$$\|\nabla\widetilde{u}\|_{L^{\frac{rq}{r-q}}(B_{\frac{3}{4}\frac{\rho}{R}})}$$
$$\leq C\left(\frac{\rho}{R}\right)^{\frac{n(r-q)}{rq}}\|\nabla\widetilde{u}(\frac{3}{4}\frac{\rho}{R}\cdot)\|_{L^{\frac{rq}{r-q}}(B_1)}$$
$$\leq C\left(\frac{\rho}{R}\right)^{\frac{n(r-q)}{rq}}\|\nabla\widetilde{u}(\frac{3}{4}\frac{\rho}{R}\cdot)\|_{L^q(B_1)}^{1-\frac{n}{r}}\|\nabla\widetilde{u}(\frac{3}{4}\frac{\rho}{R}\cdot)\|_{W^{1,q}(B_1)}^{\frac{n}{r}}$$
$$\leq C\left(\frac{\rho}{R}\right)^{\frac{n(r-q)}{rq}}\left(\|\nabla\widetilde{u}(\frac{3}{4}\frac{\rho}{R}\cdot)\|_{L^q(B_1)}\right.$$
$$\left. + \left(\frac{\rho}{R}\right)^{\frac{n}{r}}\|\nabla\widetilde{u}(\frac{3}{4}\frac{\rho}{R}\cdot)\|_{L^q(B_1)}^{1-\frac{n}{r}}\|\nabla^2\widetilde{u}(\frac{3}{4}\frac{\rho}{R}\cdot)\|_{L^q(B_1)}^{\frac{n}{r}}\right) \tag{3.4}$$
$$\leq C\left(\frac{\rho}{R}\right)^{\frac{n(r-q)}{rq}}\left(\left(\frac{\rho}{R}\right)^{-\frac{n}{q}}\|\nabla\widetilde{u}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})}\right.$$
$$\left. + \left(\frac{\rho}{R}\right)^{\frac{n}{r}-\frac{n}{q}}\|\nabla\widetilde{u}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})}^{1-\frac{n}{r}}\|\nabla^2\widetilde{u}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})}^{\frac{n}{r}}\right)$$
$$= C\left(\left(\frac{R}{\rho}\right)^{\frac{n}{r}}\|\nabla\widetilde{u}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})} + \|\nabla\widetilde{u}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})}^{1-\frac{n}{r}}\|\nabla^2\widetilde{u}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})}^{\frac{n}{r}}\right),$$

where the constant C does not depend on ρ and R.

Substituting (3.4) into (3.3), by Young's inequality we have that for some $\varepsilon > 0$ and small ρ

$$\|\nabla^2\widetilde{u}\|_{L^q(B_{\frac{\rho}{2R}})}$$
$$\leq C_\varepsilon\left(\|\widetilde{f}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})} + \left(\left(\frac{R}{\rho}\right)^{\frac{n}{r}}\|\nabla\widetilde{A}\|_{L^r(\Omega)} + \|\nabla\widetilde{A}\|_{L^r(\Omega)}^{\frac{r}{r-n}}\right)\|\nabla\widetilde{u}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})}\right.$$
$$\left. + \frac{R}{\rho}\|\nabla\widetilde{u}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})} + \frac{R^2}{\rho^2}\|\widetilde{u}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})}\right) + \varepsilon\|\nabla^2\widetilde{u}\|_{L^q(B_{\frac{3}{4}\frac{\rho}{R}})}.$$

Fixing ρ, choose a finite covering $\{B'_k\}_{k=1}^K$ and $\{B_k\}_{k=1}^K$ of Ω' such that $B'_k = B(x_k, \frac{\rho}{2R})$ and $B_k = B(x_k, \frac{3}{4}\frac{\rho}{R})$ with $x_k \in \Omega'$, $\Omega' \subset \cup_k B'_k \subset \Omega$ and $\sum_k \chi_{B_k}(x) \leq c_n$

for some fixed positive number c_n. The number c_n can be taken by 2^n by the finite overlapping property of balls. Hence we have

$$
\begin{aligned}
\|\nabla^2 \widetilde{u}\|^q_{L^q(\Omega')} &\leq \sum_k \|\nabla^2 \widetilde{u}\|^q_{L^q(B'_k)} \\
&\leq C_\varepsilon \sum_k \left(\|\widetilde{f}\|^q_{L^q(B_k)} + \mathbf{A}(n,r,\rho,R)^q \|\nabla \widetilde{u}\|^q_{L^q(B_k)} \right. \\
&\quad \left. + \frac{R^q}{\rho^q} \|\nabla \widetilde{u}\|^q_{L^q(B_k)} + \frac{R^{2q}}{\rho^{2q}} \|\widetilde{u}\|^q_{L^q(B_k)} \right) + \varepsilon^q \sum_k \|\nabla^2 \widetilde{u}\|^q_{L^q(B_k)} \\
&\leq C_\varepsilon c_n \left(\|\widetilde{f}\|^q_{L^q(\Omega)} + \mathbf{A}(n,r,\rho,R)^q \|\nabla \widetilde{u}\|^q_{L^q(\Omega)} \right. \\
&\quad \left. + \frac{R^q}{\rho^q} \|\nabla \widetilde{u}\|^q_{L^q} + \frac{R^{2q}}{\rho^{2q}} \|\widetilde{u}\|^q_{L^q(\Omega)} \right) + \varepsilon^q \|\nabla^2 \widetilde{u}\|^q_{L^q(\Omega)},
\end{aligned}
\tag{3.5}
$$

where

$$
\mathbf{A}(n,r,\rho,R) = \left(\frac{R}{\rho} \right)^{\frac{n}{r}} \|\nabla \widetilde{A}\|_{L^r(\Omega)} + \|\nabla \widetilde{A}\|^{\frac{r}{r-n}}_{L^r(\Omega)}.
$$

Now let us denote the $2\varepsilon_0$ neighborhood of the boundary $\partial\Omega$ by Ω_{ε_0} for some fixed ε_0 to be chosen later, that is, $\Omega_{\varepsilon_0} = \partial\Omega + B(0, 2\varepsilon_0)$. Then $\Omega \setminus \Omega' \subset \Omega_{\varepsilon_0}$. Since $\partial\Omega$ is uniformly smooth, there are finite number of points $x_k \in \partial\Omega, 1 \leq k \leq N$ with a neighborhood $\mathcal{N}_k = \mathcal{N}_{x_k}$ containing a ball $B(x_k, 3\varepsilon_0)$ and hence $\Omega_{\varepsilon_0} \subset \bigcup_{1 \leq k \leq K'} B(x_k, 3\varepsilon_0)$, and with a diffeomorphism $\Psi = \Psi_k$ from $\mathcal{N}_k$ onto unit ball $B = B(0,1)$ in $\mathbb{R}^n$ such that $\Psi_k(\mathcal{N}_k \cap \Omega) \subset \mathbb{R}^n_+$, $\Psi_k(\mathcal{N}_k \cap \partial\Omega) \subset \partial\mathbb{R}^n_+$. Write $y = \Psi_k(x), \mathfrak{u}(y) = \widetilde{u}(x), x \in \mathcal{N}_k, y \in B$. Fixing k, we denote $\mathcal{N}_k$ and Ψ_k by $\mathcal{N}$ and Ψ, respectively. Then we have

$$
\mathfrak{L}\mathfrak{u} = \mathfrak{f} \quad \text{in} \quad B^+ = B \cap \mathbb{R}^n_+,
$$

where

$$
\begin{aligned}
(\mathfrak{L}\mathfrak{u})^\alpha &= \sum_{\beta,i,j} \mathfrak{A}^{\alpha\beta}_{ij} \partial_i \partial_j \mathfrak{u}^\beta + \sum_{\beta,k} \mathfrak{B}^{\alpha\beta}_k \partial_k \mathfrak{u}^\beta + \mathfrak{m}\mathfrak{u}^\alpha, \\
\mathfrak{f}(y) &= \widetilde{f}(\Psi^{-1}(y)), \quad \mathfrak{m}(y) = \widetilde{m}(\Psi^{-1}(y)), \\
\mathfrak{A}^{\alpha\beta}_{ij}(y) &= -\sum_{k,l} \widetilde{A}^{\alpha\beta}_{kl}(\Psi^{-1}(y)) \partial_k \Psi_i(y) \partial_l \Psi_j(y) \\
\mathfrak{B}^{\alpha\beta}_k(y) &= -\sum_{i,j} \widetilde{A}^{\alpha\beta}_{ij}(\Psi^{-1}(y)) \partial_i \partial_j \Psi_k(y) - \sum_{i,j,l} \partial_k \widetilde{A}^{\alpha\beta}_{ij}(\Psi^{-1}(y)) \partial_i \Psi_l(y) \partial_j \Psi_k(y).
\end{aligned}
$$

Since the Jacobian matrix $\nabla\Psi$ is invertible, there exists a constant $C(\Psi)$ such that $C(\Psi)^{-1}|x| \leq |\nabla\Psi(\Psi^{-1}(y))x| \leq C(\Psi)|x|$ for all $y \in B$ and any $x \in \mathbb{R}^n$. Thus from this we deduce that $\sum_{\alpha,\beta,i,j} \mathfrak{A}^{\alpha\beta}_{ij} \xi^\alpha_i \xi^\beta_j \geq \lambda |(\nabla\Psi)\xi|^2 \geq C(\Psi)^{-2}\lambda|\xi|^2$ for any $\xi \in \mathbb{R}^{nN}$ and also that $\mathfrak{A}$ is uniformly continuous on B_+ and has the modulus continuity $\frac{\rho^*}{R}$, where ρ^* is a small number depending on ρ and $\nabla\Psi$.

Now we choose a half ball $B'_+ = B(0,s) \cap \mathbb{R}^n_+$ with $0 < s < 1$ such that $B(x_k, 2\varepsilon_0) \subset \Psi^{-1}(B'_+) \subset B(x_k, 3\varepsilon_0)$. Then we have only to apply the same proof for interior estimate on the domain Ω' to the half ball B'_+. For this purpose, we take $\frac{\rho^*}{R}$ scaled balls covers the half ball B'_+ and cut-off functions supported in the balls. For the estimates on the half balls which are half balls of $\frac{\rho^*}{R}$ scale covering B'_+ with centers $B(0,s) \cap \partial \mathbb{R}^n_+$, we use the half space estimate of Lemma 2.2. Actually we have

$$\|\nabla^2 \mathfrak{u}\|^q_{L^q(B'_+)}$$
$$\leq C \left(\|\mathfrak{f}\|^q_{L^q(B_+)} + \left(\left(\frac{R}{\rho^*}\right)^{\frac{nq}{r}} \|\nabla \mathfrak{A}\|^q_{L^r(B_+)} + \|\nabla \mathfrak{A}\|^{\frac{qr}{r-n}}_{L^r(B_+)} \right) \|\nabla \mathfrak{u}\|^q_{L^q(B_+)} \right.$$
$$\left. + \frac{R^q}{\rho^{*q}} \|\nabla \mathfrak{u}\|^q_{L^q(B_+)} + \frac{R^{2q}}{\rho^{*2q}} \|\mathfrak{u}\|^q_{L^q(B_+)} \right) + \varepsilon^q \|\nabla^2 \mathfrak{u}\|^q_{L^q(B_+)}.$$

Changing the variables via pullback Ψ, we have that

$$\|\nabla^2 \widetilde{u}\|^q_{L^q(\Omega \cap B(x_k, 2\varepsilon_0))}$$
$$\leq C \left(\|\widetilde{f}\|^q_{L^q(\mathcal{N})} + \mathbf{A}(n, r, \rho^*, R)^q \|\nabla \widetilde{u}\|^q_{L^q(\mathcal{N})} \right.$$
$$\left. + \frac{R^q}{\rho^{*q}} \|\nabla \widetilde{u}\|^q_{L^q(\mathcal{N})} + \frac{R^{2q}}{\rho^{*2q}} \|\widetilde{u}\|^q_{L^q(\mathcal{N})} \right) + \varepsilon^q \|\nabla^2 \widetilde{u}\|^q_{L^q(\mathcal{N})}.$$

Summing all estimates with respect to k, by the choice ε_0 and ε such that $\Omega_{\varepsilon_0} \subset \bigcup_k B(x_k, \varepsilon)$ we have that

$$\|\nabla^2 \widetilde{u}\|^q_{L^q(\Omega_{\varepsilon_0})}$$
$$\leq C \left(\|\widetilde{f}\|^q_{L^q(\Omega)} + \mathbf{A}(n, r, \rho^*, R)^q \|\nabla \widetilde{u}\|^q_{L^q(\Omega)} \right. \tag{3.6}$$
$$\left. + \frac{R^q}{\rho^{*q}} \|\nabla \widetilde{u}\|^q_{L^q(\Omega)} + \frac{R^{2q}}{\rho^{*2q}} \|\widetilde{u}\|^q_{L^q(\Omega)} \right) + \varepsilon^q \|\nabla^2 \widetilde{u}\|^q_{L^q(\Omega)}.$$

Finally combining the estimates (3.5) with (3.6), we get

$$\|\nabla^2 \widetilde{u}\|_{L^q(\Omega)} \leq C \left(\|\widetilde{f}\|_{L^q(\Omega)} + \mathbf{A}(n, r, \widetilde{\rho}, R) \|\nabla \widetilde{u}\|_{L^q(\Omega)} \right.$$
$$\left. + \frac{R}{\widetilde{\rho}} \|\nabla \widetilde{u}\|_{L^q(\Omega)} + \frac{R^2}{\widetilde{\rho}^2} \|\widetilde{u}\|_{L^q(\Omega)} \right), \tag{3.7}$$

where $\widetilde{\rho} = \max(\rho, \rho^*)$. Then by converting (3.7) back into the unscaled variables such that $Rx \mapsto x$, we obtain

$$\|\nabla^2 u\|_{L^q(\Omega_R)} \leq C \left(\|f\|_{L^q(\Omega_R)} + R^{-1} \mathbf{A}(n, r, \widetilde{\rho}, R) \|\nabla u\|_{L^q(\Omega_R)} \right.$$
$$\left. + \|\nabla u\|_{L^q(\Omega_R)} + \|u\|_{L^q(\Omega_R)} \right), \tag{3.8}$$
$$A(n, r, \widetilde{\rho}, R) = R \left(\widetilde{\rho}^{-\frac{n}{r}} \|\nabla A\|_{L^r(\Omega_R)} + \|\nabla A\|^{\frac{r}{r-n}}_{L(\Omega_R)} \right).$$

Using a well-known interpolation inequality (for instance see Theorem 5.2 in [1]) that $\|\nabla v\|_{L^q(\Omega)} \leq C\|v\|_{L^q(\Omega)}^{\frac{1}{2}}\|v\|_{W^{2,q}(\Omega)}^{\frac{1}{2}}$ for any $v \in W^{2,q}(\Omega)$ and for some constant C independent of v. Now by scaling $(u(x) \mapsto u(Rx))$ and rescaling $(u(Rx) \mapsto u(x))$ with parameter $R > 1$ we have

$$\|\nabla u\|_{L^q(\Omega_R)} \leq C\|u\|_{L^q(\Omega_R)}^{\frac{1}{2}}(R^{-2}\|u\|_{L^q(\Omega_R)} + R^{-1}\|\nabla u\|_{L^q(\Omega_R)} + \|\nabla^2 u\|_{L^q(\Omega_R)})^{\frac{1}{2}}$$
$$\leq C\|u\|_{L^q(\Omega_R)}^{\frac{1}{2}}\|u\|_{W^{2,q}(\Omega_R)}^{\frac{1}{2}}.$$

Substituting this into (3.8) and applying Young's inequality, we finally conclude the estimate (1.9).

4. Proof of Theorems 1.2 and 1.3

Multiplying u to (1.1) and integrating over Ω_R, by the ellipticity of L and non-negativity of m we have

$$\|\nabla u\|_{L^2(\Omega_R)} \leq C\|f\|_{D^{-1}(\Omega_R)}. \tag{4.1}$$

Since $f \in L^{\frac{2n}{n-2}}(\Omega_R)$, Theorem 1.1 holds for $q = q_0 = \frac{2n}{n-2}$. Using the Sobolev embedding $D_0^1 \hookrightarrow L^{\frac{2n}{n-2}}$, we easily get from (4.1)

$$\|u\|_{L^{q_0}(\Omega_R)} \leq C\|f\|_{D^{-1}(\Omega_R)}. \tag{4.2}$$

Since the embedding $D_0^1 \hookrightarrow L^{\frac{2n}{n-2}}$ is invariant with respect to the scaling, the constant C in (4.2) does not depend on R. Thus we first get

$$\|u\|_{W^{2,q_0}(\Omega_R)} \leq C\|f\|_{(D^{-1}\cap L^{q_0})(\Omega_R)}(1 + \|\nabla A\|_{L^r(\Omega_R)})^{\frac{2r}{r-n}}.$$

If q_k is finite, then there holds the embedding $\|v\|_{L^{q_{j+1}}(\Omega)} \leq C\|v\|_{W^{2,q_j}(\Omega)}$ for any $0 \leq j \leq k-1$. Here the constant C does not depend on v. By scaling invariance of the above embedding for $R > 1$ we have

$$\|u\|_{L^{q_{j+1}}(\Omega_R)} \leq C\|u\|_{W^{2,q_j}(\Omega_R)},$$

where C does not depend on R. Thus by induction

$$\|u\|_{L^{q_k}(\Omega_R)} \leq C\|f\|_{(D^{-1}\cap L^{q_0}\cap L^q)(\Omega_R)}(1 + \|\nabla A\|_{L^r(\Omega_R)})^{\frac{2rk}{r-n}},$$
$$\|u\|_{W^{2,q_k}(\Omega_R)} \leq C\|f\|_{(D^{-1}\cap L^{q_0}\cap L^q)(\Omega_R)}(1 + \|\nabla A\|_{L^r(\Omega_R)})^{\frac{2r(k+1)}{r-n}}. \tag{4.3}$$

Using the embedding (2.5) with $s = q_k$, $t = q$, $k = 2$ and $\Gamma = \Omega$, from the invariance under scaling with parameter $R > 1$ and the estimates (4.3) we deduce that

$$\|u\|_{L^q(\Omega_R)} \leq C\|f\|_{(D^{-1}\cap L^{q_0}\cap L^q)(\Omega_R)}(1 + \|\nabla A\|_{L^r(\Omega_R)})^{\frac{2r(k+\delta)}{r-n}}.$$

Therefore the inequality (1.11) follows from (1.9).

Now we consider the case $m \geq \underline{m}$. Similarly to the above argument, the integration by parts shows that

$$\|u\|_{H^1(\Omega_R)} \leq C\|f\|_{D^{-1}(\Omega_R)}.$$

Using the estimate (2.5) with $s = 2$, $\ell = q$, $k = 1$ and the scale invariance with respect to $R > 1$, we have for $2 \leq q < \frac{2n}{n-2}$

$$\|u\|_{L^q(\Omega_R)} \leq C\|u\|_{L^2(\Omega_R)}^{1-n(1/2-1/q)}\|u\|_{H^1(\Omega_R)}^{n(1/2-1/q)} \leq C\|f\|_{D^{-1}(\Omega_R)}.$$

Substituting this into (1.9), we obtain the desired estimate (1.12).

From now on we prove Theorem 1.3. Since L is an elliptic operator for small $\varepsilon > 0$, it is clear that a unique solution u exists in $(W_0^{1,q} \cap W^{2,q})(\Omega_R)$ such that $\|\nabla u\|_{L^2(\Omega_R)} \leq C\|f\|_{D^{-1}(\Omega_R)}$. For the estimate (1.13), we rewrite the equation $Lu = f$ by $L_B u = a^{-1}(f + \nabla aB\nabla u - \varepsilon L_E u)$. Since L_B is a constant elliptic operator, by the estimate (1.7), we have for some small ε that

$$\|\nabla^2 u\|_{L^2(\Omega_R)}$$
$$\leq C\underline{a}^{-1}(\|f\|_{L^2(\Omega_R)} + \|\nabla a\|_{L^r(\Omega_R)}\|\nabla u\|_{L^{\frac{2r}{r-2}}(\Omega_R)}$$
$$\qquad + \varepsilon(\|\nabla E\|_{L^r(\Omega_R)}\|\nabla u\|_{L^{\frac{2r}{r-2}}(\Omega_R)} + \Lambda\|\nabla^2 u\|_{L^2(\Omega_R)})) \qquad (4.4)$$
$$\leq C(\|f\|_{L^2(\Omega_R)} + (\|\nabla a\|_{L^r(\Omega_R)} + \|\nabla E\|_{L^r(\Omega_R)})\|\nabla u\|_{L^{\frac{2r}{r-2}}(\Omega_R)}) + \frac{1}{4}\|\nabla^2 u\|_{L^2(\Omega_R)}.$$

Using the scale invariant estimate (2.5) with $s = 2, \ell = \frac{2r}{r-2}, k = 1$, by the fact $2 < \frac{2r}{r-2} < q_0$ we have

$$\|\nabla u\|_{L^{\frac{2r}{r-2}}} \leq C\|\nabla u\|_{L^2(\Omega_R)}^{1-\frac{n}{r}}\|\nabla u\|_{H^1(\Omega_R)}^{\frac{n}{r}}$$
$$\leq C\|f\|_{D^{-1}(\Omega_R)} + C\|f\|_{D^{-1}(\Omega_R)}^{1-\frac{n}{r}}\|\nabla^2 u\|_{L^2(\Omega_R)}^{\frac{n}{r}}.$$

Substituting this into (4.4) and using Young's inequality we obtain

$$\|\nabla^2 u\|_{L^2(\Omega_R)}$$
$$\leq C\|f\|_{(D^{-1}\cap L^2)(\Omega_R)}(1 + \|\nabla a\|_{L^r(\Omega_R)} + \|\nabla E\|_{L^r(\Omega_R)})^{\frac{r}{r-n}} + \frac{1}{2}\|\nabla^2 u\|_{L^2(\Omega_R)}$$

and hence

$$\|\nabla u\|_{H^1(\Omega_R)} \leq C\|f\|_{(D^{-1}\cap L^2)(\Omega_R)}(1 + \|\nabla a\|_{L^r(\Omega_R)} + \|\nabla E\|_{L^r(\Omega_R)})^{\frac{r}{r-n}}. \qquad (4.5)$$

On the other hand, it follows from the hypothesis $2 \leq q < q_0$ and Lemma 2.4 that

$$\|\nabla u\|_{L^q(\Omega_R)} \leq \|\nabla u\|_{L^2(\Omega_R)}^{1-n(\frac{1}{2}-\frac{1}{q})}\|\nabla u\|_{H^1(\Omega_R)}^{n(\frac{1}{2}-\frac{1}{q})}$$
$$\leq C\|f\|_{(D^{-1}\cap L^2)(\Omega_R)}(1 + \|\nabla a\|_{L^r(\Omega_R)} + \|\nabla E\|_{L^r(\Omega_R)})^{\frac{r}{r-n}}. \qquad (4.6)$$

Now using the elliptic estimate (1.7) again for any $2 \leq q < q_0$, similarly to (4.4) we have

$$
\begin{aligned}
\|\nabla^2 u\|&_{L^q(\Omega_R)} \\
&\leq C(\|f\|_{L^q(\Omega_R)} + (\|\nabla a\|_{L^r(\Omega_R)} + \|\nabla E\|_{L^r(\Omega_R)})\|\nabla u\|_{L^{\frac{qr}{r-q}}(\Omega_R)}) \\
&\qquad + \frac{1}{4}\|\nabla^2 u\|_{L^q(\Omega_R)}.
\end{aligned} \tag{4.7}
$$

Since $q < \frac{qr}{r-q} < \frac{nq}{n-q}$ (here $\frac{nq}{n-q} = \infty$ if $q \geq n$), an application of Lemma 2.4 yields that

$$
\|\nabla u\|_{L^{\frac{qr}{r-q}}(\Omega_R)} \leq C\|\nabla u\|_{L^q(\Omega_R)}^{1-\frac{n}{r}}\|\nabla u\|_{W^{1,q}(\Omega_R)}^{\frac{n}{r}}.
$$

Substituting this into (4.7) and using Young's inequality together with (4.6), we obtain (1.13).

5. PROOF OF THEOREM 1.4

Since $f \in D^{-1}(\Omega)$, by the Lax-Milgram theorem, there exists a unique weak solution $u \in D_0^1(\Omega)$ with $\|\nabla u\|_{L^2(\Omega)} \leq C\|f\|_{D^{-1}}$ to the boundary value problem:

$$
Lu = f, \quad u = 0 \text{ on } \partial\Omega, \ u(x) \to 0 \text{ as } |x| \to \infty.
$$

Now let f^R be a restriction of f to the domain $\widetilde{\Omega}_R$ as defined in Remark 3. Then clearly $f^R \in (D^{-1} \cap L^{q_0} \cap L^q)(\widetilde{\Omega}_R)$ and the size of its norm is uniform on R. From Theorems 1.1 and 1.2 it follows that there exists a unique solution $u^R \in (W_0^{1,q} \cap W^{2,q})(\widetilde{\Omega}_R)$ to the boundary value problem $Lu^R = f^R$ in $\widetilde{\Omega}_R$ and $u^R = 0$ on $\partial\widetilde{\Omega}_R$ such that

$$
\|\nabla u^R\|_{L^2(\widetilde{\Omega}_R)} \leq C\|f^R\|_{D^{-1}(\widetilde{\Omega}_R)} \leq C\|f\|_{D^{-1}(\Omega)},
$$

$$
\|u^R\|_{W^{2,q}(\widetilde{\Omega}_R)} \leq C\|f\|_{(D^{-1} \cap L^{q_0} \cap L^q)(\Omega)}(1 + \|\nabla A\|_{L^r(\Omega)})^{\frac{2r(k+1+\delta)}{r-n}},
$$

where $\delta = \frac{n}{2}(1/q_k - 1/q)$. If we extend u^R by defining 0 outside $\widetilde{\Omega}_R$, then by the weak compactness, there exists a subsequence u^{R_j} and w such that $u^{R_j} \to w$ weakly in $(D_{0,loc}^1 \cap W_{loc}^{2,q})(\Omega)$ and for any R suitably large

$$
\|\nabla w\|_{L^2(\widetilde{\Omega}_R)} \leq C\|f\|_{D^{-1}(\Omega)},
$$

$$
\|w\|_{W^{2,q}(\widetilde{\Omega}_R)} \leq C\|f\|_{(D^{-1} \cap L^{q_0} \cap L^q)(\Omega)}(1 + \|\nabla A\|_{L^r(\Omega)})^{\frac{2r(k+1+\delta)}{r-n}}.
$$

In view of the weak formulation, w becomes a strong solution satisfying the equations $Lw = f$. The uniqueness assertion shows that $u = w$. Therefore the first part of Theorem 1.4 follows. The second and third parts are proved similarly. Details are omitted.

6. APPLICATION TO A PARABOLIC SYSTEM: PROOF OF THEOREM 1.5

For sufficiently large R, consider the parabolic system

$$\varphi^R u_t^R + L u^R = F^R \ \text{ on } \ [0,T] \times \widetilde{\Omega}_R,$$
$$u^R(0) = u_0^R \ \text{ on } \ \widetilde{\Omega}_R, \tag{6.1}$$

where $\varphi^R = \varphi + R^{-2}$ on $\widetilde{\Omega}_R$, $F^R = \psi^R F$ for some nonnegative cut-off function ψ^R supported in $\widetilde{\Omega}_R$ with $|\nabla \psi^R| \leq 100 R^{-1}$, $u_0^R \in (H_0^1 \cap H^2)(\widetilde{\Omega}_R)$ is a solution to the boundary value problem

$$(L|_{t=0}) u_0^R = F^R(0) - \sqrt{\varphi^R} g \ \text{ on } \ \widetilde{\Omega}_R. \tag{6.2}$$

Then the unique existence of smooth solution u^R follows readily from the standard argument of parabolic system[1].

Now we consider some a priori estimates for the solution u^R. Multiplying u_t to (6.1) and integrating over $\widetilde{\Omega}_R$, we get

$$\int_{\widetilde{\Omega}_R} \varphi^R |u_t^R|^2 \, dx + \sum_{i,j,\alpha,\beta} \int_{\widetilde{\Omega}_R} A_{ij}^{\alpha\beta} \partial_i u_t^{R,\alpha} \partial_j u^{R,\beta} \, dx = \, < F^R, u_t^R > .$$

The symmetry of A (1.18) shows that

$$\int_{\widetilde{\Omega}_R} \varphi^R |u_t^R|^2 \, dx + \frac{1}{2} \frac{d}{dt} \sum_{i,j,\alpha,\beta} \int_{\widetilde{\Omega}_R} A_{ij}^{\alpha\beta} \partial_i u^{R,\alpha} \partial_j u^{R,\beta} \, dx$$
$$= \, < F^R, u_t^R > + \frac{1}{2} \sum_{i,j,\alpha,\beta} \int_{\widetilde{\Omega}_R} (A_{ij}^{\alpha\beta})_t \partial_i u^{R,\alpha} \partial_j u^{R,\beta} \, dx.$$

Integrating over $[0,t]$, we have from the ellipticity of A and Hölder inequality that

$$\int_0^t \|\sqrt{\varphi^R} u_t\|_{L^2(\widetilde{\Omega}_R)}^2 \, ds + \frac{\lambda}{2} \|\nabla u^R\|_{L^2(\widetilde{\Omega}_R)}^2$$
$$\leq C \|\nabla u_0^R\|_{L^2(\widetilde{\Omega}_R)}^2 + \int_0^t \left(\varepsilon^{-1} \|F^R\|_{D^{-1}(\widetilde{\Omega}_R)}^2 + \varepsilon \|\nabla u_t\|_{L^2(\widetilde{\Omega}_R)}^2 \right. \tag{6.3}$$
$$\left. + C \|A_t\|_{L^6(\Omega)} \|\nabla u^R\|_{L^2(\widetilde{\Omega}_R)} \|\nabla u^R\|_{L^3(\widetilde{\Omega}_R)} \right) ds.$$

Hereafter we will use C as a generic constant independent of R.

Now after taking ∂_t to (6.1), if we multiply u_t on both sides and integrate them over $\widetilde{\Omega}_R$, then we obtain from the ellipticity of L that

$$\frac{1}{2} \frac{d}{dt} \int_{\widetilde{\Omega}_R} \varphi^R |u_t^R|^2 \, dx + \lambda \|\nabla u_t^R\|_{L^2(\widetilde{\Omega}_R)}^2$$
$$\leq \, < F_t^R, u_t^R > - \sum_{i,j,\alpha,\beta} \int_{\widetilde{\Omega}_R} (A_{ij}^{\alpha\beta})_t \partial_i u_t^{R,\alpha} \partial_j u^{R,\beta} \, dx \tag{6.4}$$

---

[1] One can use the Galerkin approximation as in [8].

Since $\varphi^R u_t^R(0) = F^R(0) - (L|_{t=0})u_0^R = \sqrt{\varphi^R}g$, one can show that

$$\frac{1}{2}\|\sqrt{\varphi^R}u_t^R(0)\|_{L^2(\widetilde{\Omega}_R)} \leq \frac{1}{2}\|g\|_{L^2(\Omega)}.$$

Hence integrating (6.4) over $[0,t]$, we have by Hölder's and Young's inequalities that

$$\|\sqrt{\varphi^R}u_t^R\|_{L^2(\widetilde{\Omega}_R)}^2 + \lambda \int_0^t \|\nabla u_t^R\|_{L^2(\widetilde{\Omega}_R)}^2 \, ds$$
$$\leq \|g\|_{L^2(\Omega)}^2 + C \int_0^t \left(\|F_t^R\|_{D^{-1}(\widetilde{\Omega}_R)}^2 + \|\nabla u^R\|_{L^3(\widetilde{\Omega}_R)}^2 \right) ds. \tag{6.5}$$

Combining (6.3) and (6.5) for $\varepsilon = \frac{\lambda}{2}$, we have for any $t \in [0,T]$

$$\|\sqrt{\varphi^R}u_t^R\|_{L^2(\widetilde{\Omega}_R)}^2 + \frac{\lambda}{2}\|\nabla u^R(t)\|_{L^2(\widetilde{\Omega}_R)}^2 + \frac{\lambda}{2}\int_0^t \|\nabla u_t^R\|_{L^2(\widetilde{\Omega}_R)}^2 \, ds$$
$$\leq C\|\nabla u_0^R\|_{L^2(\widetilde{\Omega}_R)}^2 + \|g\|_{L^2(\Omega)}^2 + C\int_0^t(\|F^R\|_{D^{-1}(\widetilde{\Omega}_R)}^2 + \|F_t^R\|_{D^{-1}(\widetilde{\Omega}_R)}^2)\,ds \tag{6.6}$$
$$+ C\int_0^t(\varepsilon^{-1} + \|A_t\|_{L^6(\Omega)}^2)\|\nabla u^R\|_{L^2(\widetilde{\Omega}_R)}^2 \, ds + \varepsilon\int_0^t \|\nabla u^R\|_{L^6(\widetilde{\Omega}_R)}^2 \, ds.$$

Here for the last integral we used the Hölder inequality that $\|\nabla u^R\|_{L^3(\widetilde{\Omega}_R)} \leq \|\nabla u^R\|_{L^2(\widetilde{\Omega}_R)}^{\frac{1}{2}}\|\nabla u^R\|_{L^6(\widetilde{\Omega}_R)}^{\frac{1}{2}}$ and Young's inequality.

By the elliptic estimate (1.11) with $k = 0, q_0 = 6$ and $r = 6$ (hence $\delta = 0$),

$$\|\nabla u^R\|_{W^{1,6}(\widetilde{\Omega}_R)}$$
$$\leq C(\|F^R\|_{(D^{-1}\cap L^6)(\widetilde{\Omega}_R)} + \|\varphi^R u_t^R\|_{(D^{-1}\cap L^6)(\widetilde{\Omega}_R)})(1 + \|\nabla A\|_{L^6(\Omega)})^4. \tag{6.7}$$

From the choice of ψ^R we observe that

$$\|F^R\|_{(D^{-1}\cap L^6)(\widetilde{\Omega}_R)} \leq C\|F\|_{(D^{-1}\cap L^6)(\Omega)}.$$

Moreover, from the assumption $\varphi \in L^{\frac{3}{2}}(\Omega)$ that for any $v \in D_0^1(\widetilde{\Omega}_R)$

$$< \varphi^R u_t^R, v > = < (\varphi + R^{-2})u_t^R, v >$$
$$\leq (\|\varphi\|_{L^{\frac{3}{2}}(\Omega)} + CR^{-2+2})\|u_t^R\|_{L^6(\widetilde{\Omega}_R)}\|v\|_{L^6(\widetilde{\Omega}_R)}$$
$$\leq C(1 + \|\varphi\|_{L^{\frac{3}{2}}(\Omega)})\|\nabla u_t^R\|_{L^2(\widetilde{\Omega}_R)}\|\nabla v\|_{L^2(\widetilde{\Omega}_R)}.$$

Hence

$$\|\varphi^R u_t^R\|_{(D^{-1}\cap L^6)(\widetilde{\Omega}_R)} \leq C(1 + \|\varphi\|_{(L^{\frac{3}{2}}\cap L^\infty)(\Omega)})\|\nabla u_t^R\|_{L^2(\widetilde{\Omega}_R)}.$$

Now let us define a constant C_0 as

$$C_0 \equiv 1 + \|\nabla u_0\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)} + \|\varphi\|_{(L^{\frac{3}{2}}\cap L^\infty)(\Omega)} + \sup_{[0,T]}(\|\nabla A\|_{L^2(\Omega)} + \|A_t\|_{L^6(\Omega)})$$
$$+ \|F\|_{L^2(0,T;(D^{-1}\cap L^6)(\Omega))} + \|F_t\|_{L^2(0,T;D^{-1}(\Omega))}.$$

Then substituting the above two estimates into (6.7), we have

$$\|\nabla u^R\|_{W^{1,6}(\widetilde{\Omega}_R)} \leq CC_0^4(\|F\|_{(D^{-1}\cap L^6)(\Omega)} + \|\nabla u_t^R\|_{L^2(\widetilde{\Omega}_R)}). \qquad (6.8)$$

Plugging this into (6.6), we have for any $t \in [0, T]$

$$\|\sqrt{\varphi^R}u_t^R\|_{L^2(\widetilde{\Omega}_R)}^2 + \frac{\lambda}{2}\|\nabla u^R(t)\|_{L^2(\widetilde{\Omega}_R)}^2 + \frac{\lambda}{2}\int_0^t \|\nabla u_t^R\|_{L^2(\widetilde{\Omega}_R)}^2\, ds$$

$$\leq CC_0^{10} + C(\varepsilon^{-1} + C_0^2)\int_0^t \|\nabla u^R\|_{L^2(\widetilde{\Omega}_R)}^2\, ds + \varepsilon CC_0^8\int_0^t \|\nabla u_t^R\|_{L^2(\widetilde{\Omega}_R)}^2\, ds.$$

If we choose ε with $\varepsilon CC_0^8 \leq \frac{\lambda}{4}$, then by Gronwall's inequality we get for any $t \in [0, T]$

$$\|\sqrt{\varphi^R}u_t^R(t)\|_{L^2(\widetilde{\Omega}_R)}^2 + \|\nabla u^R(t)\|_{L^2(\widetilde{\Omega}_R)}^2 + \int_0^t \|\nabla u_t^R\|_{L^2(\widetilde{\Omega}_R)}^2\, ds \leq C_T, \qquad (6.9)$$

where C_T is a constant depending on C, C_0 and T. From (6.8) it follows that

$$\int_0^T \|\nabla u^R\|_{W^{1,6}(\widetilde{\Omega}_R)}^2\, dt \leq C_T. \qquad (6.10)$$

Now extending φ^R, u_0^R, F^R and g by defining zero outside $\widetilde{\Omega}_R$ to Ω, it is an easy matter to show u_0^R is a weak solution of (6.2) replaced by Ω and $u_0^R \to u_0$ in $D_0^1(\Omega)$. Similarly, extending the solution u^R by defining zero outside $\widetilde{\Omega}_R$, by the above a priori estimates (6.9) and (6.10) we deduce that there exists a subsequence u^{R_j} converging in weak or weak$*$ sense to a solution $u \in L^\infty(0, T; D_0^1)$ of (1.17) satisfying that $\nabla u \in L^2(0, T; W^{1,6}(\Omega))$, $u_t \in L^2(0, T; D_0^1(\Omega))$. The uniqueness follows immediately from the weak formulation of (1.17). This completes the proof of the theorem.

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