Coxeter multiarrangements with quasi-constant multiplicities

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Abstract

We study structures of derivation modules of Coxeter multiarrangements with quasi-constant multiplicities by using the primitive derivation. As an application, we show that the characteristic polynomial of a Coxeter multiarrangement with quasi-constant multiplicity is combinatorially computable.

1 Introduction

Let V be an ℓ-dimensional Euclidean space over $\mathbb{R}$ with inner product $I : V \times V \to \mathbb{R}$. Fix a coordinate $(x_1, \cdots, x_\ell)$ and put $S = S(V^*) \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}[x_1, \ldots, x_\ell]$. Let $W \subset O(V, I)$ be a finite irreducible reflection group with the Coxeter number h. It is proved by Chevalley in [2] that the invariant ring S^W is a polynomial ring $S^W = \mathbb{C}[P_1, \ldots, P_\ell]$ with $P_1, \ldots, P_\ell$ are homogeneous generators. Suppose that $\deg P_1 \le \cdots \le \deg P_\ell$. Then it is known that $\deg P_1 = 2 < \deg P_2 \le \cdots \le \deg P_{\ell-1} < \deg P_\ell = h$. Let $\mathcal{A}$ be the corresponding Coxeter arrangement, i.e., the collection of all reflecting hyperplanes of W. Fix a defining linear form $\alpha_H \in V^*$ for each hyperplane $H \in \mathcal{A}$. Let $m : \mathcal{A} \to \mathbb{Z}_{\ge 0}$ be a map, called a *multiplicity* on $\mathcal{A}$. Then the pair $(\mathcal{A}, m)$ is called a *Coxeter multiarrangement*. Let $\mathrm{Der}(S)$ denote the module of $\mathbb{C}$-linear derivations of S. Define a graded S-module $D(\mathcal{A}, m)$ by

$$D(\mathcal{A}, m) = \{\delta \in \mathrm{Der}(S) \mid \delta\alpha_H \in (\alpha_H)^{m(H)} \text{ for all } H \in \mathcal{A}\}.$$

We say a multiarrangement $(\mathcal{A}, m)$ is *free* if $D(\mathcal{A}, m)$ is a free S-module. When $(\mathcal{A}, m)$ is free, we can choose a homogeneous basis $\{\theta_1, \ldots, \theta_\ell\}$ for

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$D(\mathcal{A},m)$ and call the multiset $(\deg(\theta_1),\ldots,\deg(\theta_\ell))$ the *exponents* of a free multiarrangement $(\mathcal{A},m)$ and denoted by $\exp(\mathcal{A},m)$, where the degree is the polynomial degree. The module $D(\mathcal{A},m)$ was first defined by Ziegler ([18]) and deeply studied for Coxeter multiarrangements with constant multiplicity by [11, 13]. In particular, Terao proved that if m is constant, then $(\mathcal{A},m)$ is free and the exponents are expressed by using exponents of the Coxeter group and the Coxeter number h ([13]). These facts played a crucial role in the proof of Edelman-Reiner conjecture ([4, 17]).

Another aspect of the above module is a relation with the Hodge filtration of $\mathrm{Der}(S^W)$ introduced by K. Saito in [8, 9]. It is proved in [14] that if m is a constant multiplicity with $m=2k+1$, then the S^W-module $D(\mathcal{A},m)^W$ of all W-invariant vector fields is precisely equal to the k-th Hodge filtration of $\mathrm{Der}(S^W)$. Based on these results, a geometrically expressed S-basis of the module $D(\mathcal{A},m)$ for special kind of (not necessarily constant) multiplicities was constructed in [16]. The purpose of this paper is to strengthen and generalize results in [13, 16] by developing the "dual" version of [16]. Indeed, we handle the following "quasi-constant" multiplicities.

Definition 1. A multiplicity $\widetilde{m}:\mathcal{A}\to\mathbb{Z}_{\geq 0}$ is said to be *quasi-constant* if

$$\max\{\widetilde{m}(H)\mid H\in\mathcal{A}\}-\min\{\widetilde{m}(H)\mid H\in\mathcal{A}\}\leq 1.$$

It is clear that for a given quasi-constant multiplicity $\widetilde{m}$, there exist an integer k and a $\{0,1\}$-valued multiplicity $m:\mathcal{A}\to\{0,1\}$ such that $\widetilde{m}$ is either $2k+m$ or $2k-m$. The above $k\in\mathbb{Z}_{\geq 0}$ and m are uniquely determined unless $\widetilde{m}$ is the constant multiplicity with odd value. Our main results are concerning structures of derivation modules for Coxeter arrangements with quasi-constant multiplicities.

Theorem 2. *Let $\mathcal{A}$ be a Coxeter arrangement with the Coxeter number h and $m:\mathcal{A}\to\{0,1\}$ be a $\{0,1\}$-valued multiplicity. Then*

(1) $D(\mathcal{A},2k+m)\cong D(\mathcal{A},m)(-kh)$,

(2) $D(\mathcal{A},2k-m)\cong\Omega^1(\mathcal{A},m)(-kh)$, and

(3) The modules $D(\mathcal{A},2k+m)(kh)$ and $D(\mathcal{A},2k-m)(kh)$ are dual S-modules to each other,

where $M(n)$ denotes the degree shift by n for a graded S-module M.

Theorem 2 generalizes [13, 16] in the following three parts. In [16], the isomorphism $D(\mathcal{A},2k+m)\cong D(\mathcal{A},m)(-kh)$ is proved for the case $(\mathcal{A},m)$ is free. In Theorem 2, the assumption on the freeness is removed. Furthermore, considerations on $\Omega^1(\mathcal{A}, m)$ instead of $D(\mathcal{A}, m)$ enable us to treat multiarrangements of the type $(\mathcal{A}, 2k - m)$ as well (2). The structure of the module $D(\mathcal{A}, m)$ is not so much known when it is not free. Combining Theorem 2 (1) and (2), we have an interesting relation Theorem 2 (3), i.e., there exists a natural pairing between the modules $D(\mathcal{A}, 2k + m)$ and $D(\mathcal{A}, 2k - m)$. It may be simply said that a relation between multiplicities gives an algebraic relation between derivation modules.

The organization of this paper is as follows. In §2 we review Terao's result about the derivation modules of Coxeter arrangements with constant multiplicity in [13] from the viewpoint of the differential modules. In §3 we prove Theorem 2 (2) and the rest in §4. In §5 we apply these results to compute characteristic polynomials for Coxeter multiarrangements with quasi-constant multiplicities.

2 An interpretation of Terao's basis

In this section, we recall the main result of [13] and give an interpretation through the dual basis for $\Omega^1(\mathcal{A}, m)$. Let us first recall the definition of $\Omega^1(\mathcal{A}, m)$.

Definition 3. Put $Q(\mathcal{A}, m) = \prod_{H \in \mathcal{A}} \alpha_H^{m(H)}$ and denote by $\Omega_V^1 = S \otimes_{\mathbb{C}} V^* = \bigoplus_{i=1}^{\ell} S \cdot dx_i$ the module of differentials. Define

$$\Omega^1(\mathcal{A}, m) = \left\{ \omega \in \frac{1}{Q(\mathcal{A}, m)} \Omega_V^1 \,\middle|\, \begin{array}{c} d\alpha_H \wedge \omega \text{ does not have poles} \\ \text{along } H, \text{ for any } H \in \mathcal{A} \end{array} \right\}.$$

It is known that $\Omega^1(\mathcal{A}, m)$ is the dual S-module of $D(\mathcal{A}, m)$ and vice versa ([7], [18]). Next we define the affine connection ∇.

Definition 4. For a given rational vector field $\delta = \sum_{i=1}^{\ell} f_i \frac{\partial}{\partial x_i}$ and a rational differential k-form $\omega = \sum_{i_1,\ldots,i_k} g_{i_1,\ldots,i_k} dx_{i_1,\ldots,i_k}$, define $\nabla_\delta \omega$ by

$$\nabla_\delta \omega = \sum_{i_1,\ldots,i_k} \delta(g_{i_1,\ldots,i_k}) dx_{i_1,\ldots,i_k}.$$

The above ∇ defines a connection. We collect some elementary properties of ∇ which will be used later.

Proposition 5. *For a rational vector field δ, rational differential form ω and $f \in S$, ∇ has the following properties.*

- $\nabla_\delta f = \delta(f)$.

- $\nabla_{f\delta}\omega = f\nabla_\delta\omega$.
- *Leibniz rule:* $\nabla_\delta(f\omega) = f\nabla_\delta\omega + (\delta f)\omega$.
- *For any linear form* $\alpha \in V^*$, $\nabla_\delta(d\alpha \wedge \omega) = d\alpha \wedge \nabla_\delta\omega$.

Now we fix a generating system $P_1, \dots, P_\ell$ of the invariant ring $S^W = \mathbb{C}[P_1, \dots, P_\ell]$ as in §1. Note that we may choose $P_1(x) = I(x, x)$. Then $\frac{\partial}{\partial P_i}$ $(i = 1, \dots, \ell)$ can be considered as a rational vector field on V with order one poles along $H \in \mathcal{A}$. Especially, we denote $D = \frac{\partial}{\partial P_\ell}$ and call it the *primitive derivation*. Since $\deg P_i < \deg P_\ell$ for $i \leq \ell - 1$, the primitive derivation D is uniquely determined up to nonzero constant multiple independent of the choice of the generators $P_1, \dots, P_\ell$ ([8, 9]).

For any constant multiplicity $m \in \mathbb{Z}_{\geq 0}$, Terao showed the freeness of $\Omega^1(\mathcal{A}, m)$ by constructing a basis.

Theorem 6. *[13, Theorem 1.1]*

(1) If $m = 2k$, then

$$\nabla_{\frac{\partial}{\partial x_1}}\nabla_D^k dP_1, \nabla_{\frac{\partial}{\partial x_2}}\nabla_D^k dP_1, \dots, \nabla_{\frac{\partial}{\partial x_\ell}}\nabla_D^k dP_1$$

forms a basis for $\Omega^1(\mathcal{A}, 2k)$.

(2) If $m = 2k+1$, then

$$\nabla_{\frac{\partial}{\partial P_1}}\nabla_D^k dP_1, \nabla_{\frac{\partial}{\partial P_2}}\nabla_D^k dP_1, \dots, \nabla_{\frac{\partial}{\partial P_\ell}}\nabla_D^k dP_1$$

forms a basis for $\Omega^1(\mathcal{A}, 2k+1)$.

Originally in [13] a basis for $D(\mathcal{A}, m)$ is constructed. The above expression is obtained just by switching to $\Omega^1(\mathcal{A}, m)$ through ∇.

3 Main results

Lemma 7. *Let $\delta_1, \dots, \delta_\ell$ be rational vector fields. Suppose that they are linearly independent over S. Then*

$$\nabla_{\delta_1}\nabla_D^k dP_1, \nabla_{\delta_2}\nabla_D^k dP_1, \dots, \nabla_{\delta_\ell}\nabla_D^k dP_1$$

are linearly independent over S.

Proof. Put $\delta_i = \sum_{j=1}^{\ell} a_{ij}\partial_j$, where $\partial_j = \frac{\partial}{\partial x_j}$. Then linearly independence of $\{\delta_1, \dots, \delta_\ell\}$ is equivalent to $\det(a_{ij}) \neq 0$. Now the assertion is clear from Theorem 6 (1) and

$$\nabla_{\delta_i}\nabla_D^k dP_1 = \sum_{j=1}^{\ell} a_{ij}\nabla_{\partial_j}\nabla_D^k dP_1.$$

□

Lemma 8. *The pole order of $\nabla_D^k dP_1$ is exactly equal to $2k-1$. More precisely, $\nabla_D^k dP_1 \in \frac{1}{Q(\mathcal{A},2k-1)}\Omega_V^1$ and $\alpha_H^{2k-2}\nabla_D^k dP_1$ has a pole along H for any $H \in \mathcal{A}$.*

Proof. First note that since

$$\nabla_D^k dP_1 = \nabla_{\frac{\partial}{\partial P_\ell}}\nabla_D^{k-1} dP_1,$$

Theorem 6 implies that $\nabla_D^k dP_1 \in \Omega^1(\mathcal{A}, 2k-1)$. Hence $\nabla_D^k dP_1 \in \frac{1}{Q(\mathcal{A},2k-1)}\Omega_V^1$.

Suppose that there exists $H \in \mathcal{A}$ such that $\alpha_H^{2k-2}\nabla_D^k dP_1$ does not have poles along H. Let us define the characteristic multiplicity m_H by

$$m_H(H') = \begin{cases} 1 & \text{if } H' = H, \\ 0 & \text{if } H' \neq H. \end{cases}$$

Then it is easily seen that $\nabla_D^k dP_1 \in \Omega^1(\mathcal{A}, 2k-1-m_H)$. Since $\nabla_{\frac{\partial}{\partial P_j}}$ increases the pole order at most two, we have $\nabla_{\frac{\partial}{\partial P_j}}\nabla_D^k dP_1 \in \Omega^1(\mathcal{A}, 2k+1-m_H)$. However, this contradicts to Theorem 6 (2), for $\Omega(\mathcal{A}, 2k+1) \supsetneqq \Omega^1(\mathcal{A}, 2k+1-m_H)$. □

Remark 9. Lemma 8 is a dual counterpart to [16, Lemma 4]. This property is related to the regularity of eigenvectors of the Coxeter element, which is of crucial importance in [8, 9].

Let $m : \mathcal{A} \to \{0,1\}$ be a $\{0,1\}$-valued multiplicity. The primitive derivation and ∇ enable us to compare $D(\mathcal{A}, m)$ and $\Omega^1(\mathcal{A}, 2k-m)$.

Theorem 10. *For $\delta \in D(\mathcal{A}, m)$, $\Phi_k(\delta) := \nabla_\delta\nabla_D^k dP_1$ is contained in $\Omega^1(\mathcal{A}, 2k-m)$. Furthermore, the map*

$$\begin{array}{rccc} \Phi_k : & D(\mathcal{A}, m)(kh) & \longrightarrow & \Omega^1(\mathcal{A}, 2k-m) \\ & \delta & \longmapsto & \nabla_\delta\nabla_D^k dP_1 \end{array}$$

gives an S-isomorphism.

Proof. Since ∇_δ increases pole order at most one, from Lemma 8, $\nabla_\delta \nabla_D^k dP_1 \in \frac{1}{Q(\mathcal{A},2k)}\Omega_V^1$. Let $H \in \mathcal{A}$. Then $\alpha_H^{2k-1} \cdot \nabla_D^k dP_1$ has no poles along H. Thus $\nabla_\delta \left(\alpha_H^{2k-1} \cdot \nabla_D^k dP_1\right)$ also has no poles along H. Suppose $m(H) = 1$, and put $\delta(\alpha_H) = \alpha_H g$. Then we have

$$\begin{aligned} \nabla_\delta \left(\alpha_H^{2k-1} \cdot \nabla_D^k dP_1\right) &= (2k-1)\alpha_H^{2k-2}\delta(\alpha_H)\nabla_D^k dP_1 + \alpha_H^{2k-1}\nabla_\delta\nabla_D^k dP_1 \\ &= (2k-1)\alpha_H^{2k-1} g \nabla_D^k dP_1 + \alpha_H^{2k-1}\nabla_\delta\nabla_D^k dP_1. \end{aligned}$$

Hence $\alpha_H^{2k-1}\nabla_\delta\nabla_D^k dP_1$ has no pole along H. This shows that $\nabla_\delta\nabla_D^k dP_1 \in \frac{1}{Q(\mathcal{A},2k-m)}\Omega_V^1$. Since $d\alpha_H \wedge \nabla_D^k dP_1$ has no poles along H, using Proposition 5, $\nabla_\delta(d\alpha_H \wedge \nabla_D^k dP_1) = d\alpha_H \wedge \nabla_\delta\nabla_D^k dP_1$ also does not have poles along H. This means $\Phi_k(\delta) = \nabla_\delta\nabla_D^k dP_1 \in \Omega^1(\mathcal{A}, 2k-m)$.

Next we prove the injectivity. Let K be the field of all rational functions. Since Φ_k is S-homomorphic, it can be extended to a K-linear map

$$\widetilde{\Phi_k} : D(\mathcal{A}, m) \otimes_S K \longrightarrow \Omega^1(\mathcal{A}, 2k-m) \otimes_S K.$$

Then $\widetilde{\Phi_k}$ is isomorphic due to Lemma 7. Hence the induced map Φ_k is obviously injective.

Finally we prove the surjectivity. Let $\omega \in \Omega^1(\mathcal{A}, 2k-m)$. Then clearly $\omega \in \Omega^1(\mathcal{A}, 2k)$. Hence from Theorem 6, there exists $\delta \in D(\mathcal{A}, 0) = \sum_i S\partial_i$ such that $\omega = \nabla_\delta\nabla_D^k dP_1$. If $m \equiv 0$, there is nothing to prove. Otherwise, choose a hyperplane $H \in \mathcal{A}$ such that $m(H) = 1$. Then $\nabla_\delta \left(\alpha_H^{2k-1} \cdot \nabla_D^k dP_1\right) = (2k-1)\alpha_H^{2k-2}\delta(\alpha_H)\nabla_D^k dP_1 + \alpha_H^{2k-1}\omega$ does not have poles along H. Hence $\alpha_H^{2k-2}\delta(\alpha_H)\nabla_D^k dP_1$ does not have poles along H. From Lemma 8, $\delta(\alpha_H)$ has to be divisible by α_H. This shows that $\delta \in D(\mathcal{A}, m)$. $\square$

4 Conclusions

By using parallel arguments to §3 in the context of [16], we can prove the following result. The notation is the same as above.

Theorem 11. *Let $m : \mathcal{A} \to \{0,1\}$ be a $\{0,1\}$-valued multiplicity and $E = \sum x_i\partial_i$ be the Euler vector field. Then for $\delta \in D(\mathcal{A}, m)$, $\Psi_k(\delta) := \nabla_\delta\nabla_D^{-k}E$ is contained in $D(\mathcal{A}, 2k+m)$. Furthermore, the map*

$$\begin{array}{rccc} \Psi_k : & D(\mathcal{A}, m)(-kh) & \longrightarrow & D(\mathcal{A}, 2k+m) \\ & \delta & \longmapsto & \nabla_\delta\nabla_D^{-k}E \end{array}$$

gives an S-isomorphism.

The action of ∇_D shifts degree by $-h$. This proves the following results.

Corollary 12. *For a $\{0,1\}$-valued multiplicity $m : \mathcal{A} \to \{0,1\}$ and an integer $k > 0$, the following conditions are equivalent.*

- $(\mathcal{A}, m)$ *is free with exponents* $(e_1, \dots, e_\ell)$.
- $(\mathcal{A}, 2k+m)$ *is free with exponents* $(kh+e_1, \dots, kh+e_\ell)$.
- $(\mathcal{A}, 2k-m)$ *is free with exponents* $(kh-e_1, \dots, kh-e_\ell)$.

Remark 13. The first condition in Corollary 12 is equivalent to say the subarrangement $m^{-1}(1) \subset \mathcal{A}$ is free. For the Coxeter arrangement of type A, free subarrangements $(\mathcal{A}, m)$ are completely classified in [12]. See also [3].

Another conclusion is the following.

Theorem 14. *Let $(\mathcal{A}, m)$ be a Coxeter arrangement with a $\{0,1\}$-valued multiplicity m and $k > 0$. Then $D(\mathcal{A}, 2k+m)(kh)$ and $D(\mathcal{A}, 2k-m)(kh)$ are dual S-module to each other.*

Proof. Combining Theorem 10 and 11, we have the following isomorphisms of graded S-modules.

$$D(\mathcal{A}, 2k+m)(kh) \cong D(\mathcal{A}, m) \cong \Omega^1(\mathcal{A}, 2k-m)(-kh).$$

Since $\Omega^1(\mathcal{A}, 2k-m) \cong D(\mathcal{A}, 2k-m)^*$, we have $D(\mathcal{A}, 2k+m)(kh) \cong \Omega^1(\mathcal{A}, 2k-m)(-kh) \cong D(\mathcal{A}, 2k-m)(kh)^*$. □

5 Characteristic polynomials

In a recent paper [1], the characteristic polynomial $\chi((\mathcal{A}, m), t) \in \mathbb{Z}[t]$ for a multiarrangement $(\mathcal{A}, m)$ is defined. In this section, we apply results in the previous sections to compute the characteristic polynomials. Let us first recall the definition of the characteristic polynomial briefly.

Let $(\mathcal{A}, m)$ be a multiarrangement of rank ℓ. Then the module $D^p(\mathcal{A}, m)$ and $\Omega^p(\mathcal{A}, m)$ are defined for $0 \le p \le \ell$ (see Introduction of [1] and [18]), and define functions

$$\psi(\mathcal{A}, m; t, q) = \sum_{p=0}^{\ell} H(D^p(\mathcal{A}, m), q)(t(q-1)-1)^p,$$

$$\phi(\mathcal{A}, m; t, q) = \sum_{p=0}^{\ell} H(\Omega^p(\mathcal{A}, m), q)(t(1-q)-1)^p,$$

in t and q, where $H(M, q)$ is the Hilbert series of a graded S-module M. In [1], ψ and ϕ are proved to be polynomials in t and q and $(-1)^\ell \psi(\mathcal{A}, m; t, 1) = \phi(\mathcal{A}, m; t, 1)$. The characteristic polynomial of $(\mathcal{A}, m)$ is by definition

$$\chi((\mathcal{A}, m), t) = (-1)^\ell \psi(\mathcal{A}, m; t, 1) = \phi(\mathcal{A}, m; t, 1).$$

Note that the above definition is a generalization of so called Solomon-Terao's formula ([10]), that is, $\chi((\mathcal{A}, 1), t)$ is equal to the combinatorially defined characteristic polynomial $\chi(\mathcal{A}, t)$ of $\mathcal{A}$ ([6]).

In general the computation of the characteristic polynomial $\chi((\mathcal{A}, m), t)$, especially the constant term, is difficult. One of the reasons is that $\chi((\mathcal{A}, m), t)$ is not a combinatorial invariant. However, we can compute it combinatorially for Coxeter multiarrangements with quasi-constant multiplicities.

Theorem 15. *Let $\mathcal{A}$ be a Coxeter arrangement with the Coxeter number h, and $m : \mathcal{A} \to \{0, 1\}$ be a $\{0, 1\}$-valued multiplicity as in the previous sections. Let $k \in \mathbb{Z}_{>0}$. Then*

(1) $\chi((\mathcal{A}, 2k + m), t) = \chi((\mathcal{A}, m), t - kh)$, *and*

(2) $\chi((\mathcal{A}, 2k - m), t) = (-1)^\ell \chi((\mathcal{A}, m), kh - t)$.

For the proof, we need the following lemmas.

Lemma 16. *Let $\mathfrak{m} = (x_1, \ldots, x_\ell) \subset S$ be the graded maximal ideal of S. Let $(\mathcal{A}, m)$ be any multiarrangement. Then $\Omega^p(\mathcal{A}, m)$ is saturated in the following sense, that is, if $\omega \in \frac{1}{Q(\mathcal{A},m)}\Omega_V^p$ satisfies $\mathfrak{m} \cdot \omega \subset \Omega^p(\mathcal{A}, m)$, then $\omega \in \Omega^p(\mathcal{A}, m)$. Similarly, if $\delta \in \mathrm{Der}^p(S)$ satisfies $\mathfrak{m} \cdot \delta \subset D^p(\mathcal{A}, m)$, then $\delta \in D^p(\mathcal{A}, m)$.*

Proof. We may assume the coordinate $(x_1, \ldots, x_\ell)$ is generic so that no coordinate hyperplane $\{x_i = 0\}$ is contained in $\mathcal{A}$. From the assumption, $d\alpha_H \wedge x_i\omega$ has no poles along H, obviously, so does $d\alpha_H \wedge \omega$. Hence $\omega \in \Omega^p(\mathcal{A}, m)$. For $D^p(\mathcal{A}, m)$ the proof is similar. □

Lemma 17. *Let $(\mathcal{A}, m)$ be as in Theorem 15.*

$$\begin{aligned} D^p(\mathcal{A}, 2k + 2 \pm m) &\cong D^p(\mathcal{A}, 2k \pm m)(-ph), \text{ and} \\ \Omega^p(\mathcal{A}, 2k + 2 \pm m) &\cong \Omega^p(\mathcal{A}, 2k \pm m)(ph). \end{aligned}$$

Proof. We only give a proof for Ω^p. The other case is immediate from the fact that D^p and Ω^p are dual S-modules to each other.

The case $p = 1$ is obvious from Theorem 10 and 11. Put $m' = 2k \pm m$. Consider the coherent sheaf $\mathcal{E}^p(\mathcal{A}, m') := \widetilde{\Omega^p(\mathcal{A}, m')}$ on $\mathbb{P}^{\ell-1} = \operatorname{Proj} S$ corresponding to the graded S-module $\Omega^p(\mathcal{A}, m')$ ([5]). Recall that $\mathcal{E}^p(\mathcal{A}, m')$

is known to be a reflexive $\mathcal{O}$-module, and from Lemma 16 $\Omega^p(\mathcal{A}, m')$ can be recovered from $\mathcal{E}^p(\mathcal{A}, m')$ by taking the global section $\Gamma_*(\mathcal{E}^p(\mathcal{A}, m')) := \bigoplus_{d\in\mathbb{Z}} \Gamma(\mathbb{P}^{\ell-1}, \mathcal{E}^p(\mathcal{A}, m')(d)) = \Omega^p(\mathcal{A}, m')$. Let $L(\mathcal{A})$ be the intersection lattice, and denote by $L_k(\mathcal{A})$ the set of intersections of codimension k. For $X \in L_2(\mathcal{A})$, denote by $\overline{X} \subset \mathbb{P}^{\ell-1}$ the corresponding flat. Consider the open subset

$$U = \mathbb{P}^{\ell-1} \setminus \bigcup_{X\in L_2(\mathcal{A})} \overline{X}$$

with the inclusion $i : U \hookrightarrow \mathbb{P}^{\ell-1}$. Since $\mathcal{E}^p(\mathcal{A}, m')$ is reflexive, hence normal, we have $i_*\mathcal{E}^p(\mathcal{A}, m')_U \cong \mathcal{E}^p(\mathcal{A}, m')$. Furthermore, since $\mathcal{E}^p(\mathcal{A}, m')_U$ is locally free on U, we have

$$\mathcal{E}^p(\mathcal{A}, m')_U \cong \wedge^p \mathcal{E}^1(\mathcal{A}, m')_U.$$

Combining these facts, we have

$$\begin{aligned}
\mathcal{E}^p(\mathcal{A}, m'+2) &= i_*\mathcal{E}^p(\mathcal{A}, m'+2)_U \\
&= i_*\left(\wedge^p \mathcal{E}^1(\mathcal{A}, m'+2)_U\right) \\
&= i_*\left(\wedge^p \left(\mathcal{E}^1(\mathcal{A}, m')_U \otimes \mathcal{O}(h)_U\right)\right) \\
&= i_*\left(\mathcal{E}^p(\mathcal{A}, m')_U \otimes \mathcal{O}(ph)_U\right) \\
&= \mathcal{E}^p(\mathcal{A}, m') \otimes \mathcal{O}(ph).
\end{aligned}$$

By taking the global section, we have $\Omega^p(\mathcal{A}, 2k+2\pm m) \cong \Omega^p(\mathcal{A}, 2k \pm m)(ph)$. $\square$

Proof of Theorem 15. Let us prove (2). From Theorem 10 and Lemma 17, we obtain the isomorphism $\Omega^p(\mathcal{A}, 2k-m) \cong D^p(\mathcal{A}, m)(pkh)$ of graded S-modules. Hence their Hilbert series are related by the relation

$$H(\Omega^p(\mathcal{A}, 2k-m), q) = H(D^p(\mathcal{A}, m), q)q^{-pkh}.$$

From the definitions of ϕ and ψ,

$$\begin{aligned}
\phi(\mathcal{A}, 2k-m; t, q) &= \sum_{p=0}^{\ell} H(\Omega^p(\mathcal{A}, 2k-m), q)(t(1-q)-1)^p \\
&= \sum_{p=0}^{\ell} H(D^p(\mathcal{A}, m), q)q^{-pkh}(t(1-q)-1)^p \\
&= \sum_{p=0}^{\ell} H(D^p(\mathcal{A}, m), q)\{q^{-kh}(t(1-q)-1)\}^p, \\
&= \psi(\mathcal{A}, m; \frac{q^{-kh}-1}{1-q} - q^{-kh}t, q).
\end{aligned}$$

Now we have $\phi(\mathcal{A}, 2k-m; t, 1) = \psi(\mathcal{A}, m; kh-t, 1)$ as $q \to 1$ and obtain (2). The proof of (1) is similar. $\square$

Example 18. *Suppose $\mathcal{A}$ is defined by $xyz(x+y)(y+z)(x+y+z)$, which is linearly isomorphic to the Coxeter arrangement of type A_3 and $h=4$. Let $m : \mathcal{A} \to \{0,1\}$ be defined by $m^{-1}(1) = xyz(x+y+z)$. Then $\chi((\mathcal{A}, m), t) = t^3 - 4t^2 + 6t - 3$. Thus we have from Theorem 15 that*

$$\begin{aligned}
\chi((\mathcal{A}, 2k+m), t) &= (t-4k)^3 - 4(t-4k)^2 + 6(t-4k) - 3 \\
\chi((\mathcal{A}, 2k-m), t) &= (t-4k)^3 + 4(t-4k)^2 + 6(t-4k) + 3.
\end{aligned}$$

Theorem 15 says that for any quasi-constant multiplicity m' on a Coxeter arrangement $\mathcal{A}$ with the Coxeter number h, the formula

$$\chi((\mathcal{A}, m'+2k+2), t) = \chi((\mathcal{A}, m'+2k), t-h)$$

holds. Some computational examples show that similar formula holds for any multiplicity $m' : \mathcal{A} \to \mathbb{Z}_{\geq 0}$, namely, supporting the following conjecture.

Conjecture 19. *Let $\mathcal{A}$ be a Coxeter arrangement with the Coxeter number h. Let $m : \mathcal{A} \to \mathbb{Z}_{\geq 0}$ be a multiplicity. Then there exists a constant $N = N(\mathcal{A}, m)$ such that*

$$\chi((\mathcal{A}, m+2k+2), t) = \chi((\mathcal{A}, m+2k), t-h)$$

is satisfied for any integer $k > N$.

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