Nonlinear Wave Equations

Edited by
H.Kubo, T.Ozawa, H.Takamura, and K.Tutaya

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PREFACE

This volume is intended as the proceedings of the symposium "Nonlinear Wave Equations", held on August 27 through 30 in 2007 at the Department of Mathematics, Hokkaido University.

The complete list of symposia on Nonlinear Wave Equations at Sapporo is in our website:

http://coe.math.sci.hokudai.ac.jp/sympo/nwe/index_en.html

We would like to dedicate this volume to Professor Rentaro Agemi and Professor Kōji Kubota.

Hideo Kubo
Tohru Ozawa
Hiroyuki Takamura
Kimitoshi Tsutaya

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組織委員：　久保　英夫（大阪大学）、高村　博之（はこだて未来大学）、小澤　徹（北海道大学）、
津田谷　公利（北海道大学）

Organizers: Hideo Kubo (Osaka Univ.), Hiroyuki Takamura (Future Univ.-Hakodate),
Tohru Ozawa (Hokkaido Univ.), Kimitoshi Tsutaya (Hokkaido Univ.)

Period（期間）　August 27, 2007 – August 29, 2007 （2007 年 8 月 27 日-2007 年 8 月 29 日）

Venue（場所）　Room 8-309, Faculty of Science Building #8, Hokkaido University
（北海道大学理学部 8 号館 309 号室）

URL　http://coe.math.sci.hokudai.ac.jp/sympo/nwe/index07_en.html

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Long range scattering for the Maxwell-Schrödinger system with arbitrarily large asymptotic data

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This lecture is devoted to the theory of scattering and more precisely to the construction of modified wave operators for the Maxwell-Schrödinger (MS) system in space dimension 3, namely

$$\begin{cases} i\partial_t u = -(1/2)\Delta_A u + A_e u \\ \square A_e - \partial_t \left(\partial_t A_e + \nabla \cdot A\right) = |u|^2 \\ \square A + \nabla \left(\partial_t A_e + \nabla \cdot A\right) = \mathrm{Im}\ \overline{u} \nabla_A u \end{cases} \tag{1}$$

where u and (A, A_e) are respectively a complex valued function and an $I\!R^{3+1}$ valued function defined in space time $I\!R^{3+1}$, $\nabla_A = \nabla - iA$ and $\Delta_A = \nabla_A^2$ are the covariant gradient and covariant Laplacian respectively, and $\square = \partial_t^2 - \Delta$ is the d'Alembertian. An important property of that system is its gauge invariance, namely the invariance under the transformation

$$(u, A, A_e) \to (u \exp(-i\theta), A - \nabla\theta, A_e + \partial_t\theta) \ ,$$

where θ is an arbitrary real function defined in $I\!R^{3+1}$. As a consequence of that invariance, the system (1) is underdetermined as an evolution system and has to be supplemented by an additional equation, called a gauge condition. In this lecture, we shall use exclusively the Coulomb gauge condition, namely $\nabla \cdot A = 0$. Under that condition, the equation for A_e can be solved by

$$A_e = -\Delta^{-1}|u|^2 = (4\pi|x|)^{-1} \star |u|^2 \equiv g(u) \tag{2}$$

where $\star$ denotes the convolution in $I\!R^3$. Substituting (2) and the gauge condition into (1) yields the formally equivalent system

$$\begin{cases} i\partial_t u = -(1/2)\Delta_A u + g(u)u & (3) \\ \square A = P\ \mathrm{Im}\ \overline{u}\nabla_A u & (4) \end{cases}$$

*Unité Mixte de Recherche (CNRS) UMR 8627 where $P = \mathbb{1} - \nabla^{-1}\Delta\nabla$ is the projector on divergence free vector fields.

The MS system is known to be locally well posed both in the Coulomb gauge and in the Lorentz gauge $\partial_t A_e + \nabla \cdot A = 0$ in sufficiently regular spaces [7] [8], to have weak global solutions in the energy space [6] and to be globally well posed in a space smaller than the energy space [9].

A large amount of work has been devoted to the theory of scattering and more precisely to the existence of wave operators for nonlinear equations and systems centering on the Schrödinger equation, in particular for the Maxwell-Schrödinger system [1] [2] [3] [5] [10] [11]. As in the case of the linear Schrödinger equation, one must distinguish the short range case from the long range case. In the former case, ordinary wave operators are expected and in a number of cases proved to exist, describing solutions where the Schrödinger function behaves asymptotically like a solution of the free Schrödinger equation. In the latter case, ordinary wave operators do not exist and have to be replaced by modified wave operators including an additional phase in the asymptotic behaviour of the Schrödinger function. In that respect, the MS system in $I\!R^{3+1}$ belongs to the borderline (Coulomb) long range case. General background and additional references on that matter can be found in [3] [4].

The main step in the construction of the (modified) wave operators consists in solving the local Cauchy problem with infinite initial time. In the long range case where that problem is singular, that step amounts to construct solutions with prescribed (singular) asymptotic behaviour in time, and this lecture is devoted to that question. The construction is performed by changing variables from (u, A) to new variables, replacing the original system by an auxiliary system for the new variables, solving the corresponding problems for the auxiliary system and finally returning therefrom to the original one.

The results include the uniqueness and the existence of solutions with prescribed asymptotic behaviour. The uniqueness result is fairly general and uses only a small amount of information on that behaviour. It will be stated for a slightly modified version of the system (3) (4) where the equation (4) is replaced by the associated integral equation with precribed asymptotic data $(A_+, \dot{A}_+)$, namely

$$A = A_0 - \int_t^\infty dt' \sin(\omega(t-t'))P \ \mathrm{Im} \left(\overline{u}\nabla_A u\right)(t') \qquad (5)$$

where $\omega = (-\Delta)^{1/2}$, A_0 is the solution of the free wave equation $\square A_0 = 0$ given by

$$A_0 = (\cos \omega t)A_+ + \omega^{-1}(\sin \omega t)\dot{A}_+ \qquad (6)$$

and $\nabla \cdot A_+ = \nabla \cdot \dot{A}_+ = 0$ in order to ensure the gauge condition $\nabla \cdot A = 0$.

Since the Cauchy problem at infinity for the system (3) (5) is singular, especially as regards the function u, the uniqueness result for that system takes a slightly unusual form. It states that two solutions (u_i, A_i), $i = 1, 2$ coincide provided u_i and $A_i - A_0$ do not blow up too fast and provided $u_1 - u_2$ tends to zero in a suitable sense as $t \to \infty$. We denote by $\| \cdot \|_r$ the norm in $L^r(I\!R^3)$, by $\dot{H}^1$ the usual homogeneous Sobolev space, by $V_\star$ the space

$$V_\star = \left\{ v :< x >^3 v \in L^2, \ < x >^2 \nabla v \in L^2 \right\} \tag{7}$$

where $< \cdot > = (1 + | \cdot |^2)^{1/2}$, and by S the dilation operator

$$S = t \, \partial_t + x \cdot \nabla + 1\!\!1 \ . \tag{8}$$

We use the notation

$$\widetilde{u}(t) = U(-t) \ u(t) \equiv \exp(-i(t/2)\Delta) u(t) \tag{9}$$

and we denote nonnegative integers by j, k, ℓ.

The uniqueness result can be stated as follows [5].

Proposition 1. *Let $1 \leq T < \infty$, $I = [T, \infty)$ and $\alpha \geq 0$. Let A_0 be a divergence free solution of the free wave equation satisfying*

$$\| \nabla^k S^j A_0(t) \|_\infty \ + \ \| \nabla^k x \cdot A_0(t) \|_\infty \leq C \ t^{-1} \quad \textit{for } 0 \leq j + k \leq 1 \tag{10}$$

for all $t \in I$. Let (u_i, A_i), $i = 1, 2$, be two solutions of the system (3) (5) such that $\widetilde{u}_i \in L^\infty_{loc}(I, V_\star)$, $A_i - A_0 \in L^\infty_{loc}(I, \dot{H}^1)$ and such that

$$\| \ \widetilde{u}_i(t); V^\star \ \| \ \leq C(1 + \ell n \ t)^\alpha \tag{11}$$

$$\| \ \nabla(A_i - A_0)(t) \ \|_2 \leq C \ t^{-1/2}(1 + \ell n \ t)^{2\alpha} \tag{12}$$

$$\| < x/t > (u_1 - u_2)(t) \ \|_2 \leq C \ t^{-1-\varepsilon} \tag{13}$$

for some $\varepsilon > 0$ and for all $t \in I$. Then $(u_1, A_1) = (u_2, A_2)$.

Because of the gauge condition $\nabla \cdot A_0 = 0$ and of the commutation relation $\square S = (S + 2)\square$, both $x \cdot A_0$ and SA_0 satisfy the free wave equation if A_0 does. Sufficient conditions on A_+, $\dot{A}_+$ ensuring (10) are well known.

The conditions (11)-(13) will be seen to hold for the solutions obtained in the existence result to be given below. In particular (11) will hold with $\alpha = 3$.

We now turn to the existence of solutions with prescribed asymptotic behaviour. The proof proceeds in several steps and uses extensively the auxiliary system mentioned above. One first constructs solutions of that system behaving asymptotically as given asymptotic functions by a contraction method using only general boundedness and time decay properties of those functions. One then constructs asymptotic functions satisfying the assumptions needed for the previous step by solving the auxiliary system approximately by an iteration method up to second order. One then returns to the original functions and to the original system (3) (4), thereby obtaining asymptotic functions (u_a, A_a) and solutions (u, A) of that system which behave asymptotically as (u_a, A_a). The asymptotic functions are parametrized by the asymptotic data $(A_+, \dot{A}_+)$ that occur in (5) (6) and in addition by asymptotic data u_+ for u (in a short range situation, u_+ would be the limit of $\widetilde{u}(t)$ as $t \to \infty$, a limit which does not exist in the present long range case).

We now give a simplified statement of the existence result, stripped from a number of technicalities which are too complicated to be presented here. As a consequence, that statement is somewhat incomplete. In particular (i) we do not give the explicit expression of (u_a, A_a) in terms of $(u_+, A_+, \dot{A}_+)$, (ii) we do not give the detailed assumptions on $(A_+, \dot{A}_+)$ and (iii) we do not state all the regularity properties of A_a and A, the reason being that both are sums of various parts with different properties. We refer to [3] for more details. On the other hand we give the convergence estimates of (u, A) to (u_a, A_a) in some detail. We denote by F the Fourier transform and we use the definitions (7)-(9) of $V_\star$, S and $\widetilde{u}$, and a similar definition of $\widetilde{u}_a$.

Proposition 2. *Let u_+ be such that $v_+ = \overline{Fu_+} \in H^5$, $xv_+ \in H^4$ and that v_+ satisfy the support condition*

$$\text{Supp } v_+ \subset \{x : |\, |x| - 1| \geq \eta\} \tag{14}$$

for some $\eta > 0$. Let $(A_+, \dot{A}_+)$ be sufficiently regular and decaying at infinity (in space) with $\nabla \cdot A_+ = \nabla \cdot \dot{A}_+ = 0$ and let A_0 be defined by (6). Then there exists (u_a, A_a) such that $\widetilde{u}_a \in \mathcal{C}([1, \infty), V_\star)$, $< x > \partial_t \widetilde{u}_a \in \mathcal{C}([1, \infty), L^2)$, $\nabla A_a, SA_a \in \mathcal{C}([1, \infty), H^1)$, such that

$$\| \, \widetilde{u}_a(t); V_\star \, \| \leq C(1 + \ell n \; t)^3 \;, \tag{15}$$

$$\| \nabla(A_a - A_0)(t) \|_2 \leq C\ t^{-1/2} \tag{16}$$

for all $t \geq 1$, *and such that the following holds.*

There exists $T \geq 1$ *and there exists a (unique) solution* (u, A) *of the system (3) (4) such that* $\widetilde{u} \in \mathcal{C}(I, V_\star)$, $< x > \partial_t \widetilde{u} \in \mathcal{C}(I, L^2)$, $\nabla A, SA \in \mathcal{C}(I, H^1)$, *where* $I = [T, \infty)$, *and such that the following estimates hold for all* $t \in I$.

$$\| \widetilde{u}(t); V_\star \| \leq C(1 + \ell n\ t)^3 \tag{17}$$

$$\| \nabla(A - A_0)(t) \|_2 \leq C\ t^{-1/2} \tag{18}$$

$$\| x^k \nabla^\ell (\widetilde{u} - \widetilde{u}_a)(t) \|_2 \leq C\ t^{-2+k/2}(1 + \ell n\ t)^4 \tag{19}$$

for $0 \leq \ell \leq 1$, $0 \leq k + \ell \leq 3$,

$$\| x^k \partial_t (\widetilde{u} - \widetilde{u}_a)(t) \|_2 \leq C\ t^{-3+k/2}(1 + \ell n\ t)^4 \quad \textit{for } 0 \leq k \leq 1\ , \tag{20}$$

$$\| \nabla^{k+1} S^j (A - A_a)(t) \|_2 \ + t^{-1} \| \nabla^{k+1} S^j x \cdot (A - A_a)(t) \|_2$$
$$\leq C\ t^{-5/2-k/2}(1 + \ell n\ t)^4 \tag{21}$$

for $0 \leq j, k, j + k \leq 1$.

Whereas the existence part of Proposition 2 is proved by the construction previously mentioned, the uniqueness part follows readily from Proposition 1. In fact, if (u_i, A_i), $i = 1, 2$, are two solutions of the system (3) (4) obtained from Proposition 2 with the same (u_a, A_a), then the conditions (11) (12) follow from (17) (18) with $\alpha = 3$, while the condition (19) together with the commutation relation $xU(-t) = U(-t)(x + it\nabla)$ imply

$$\| < x/t > (u_1 - u_2)(t) \|_2 \leq C\ t^{-2}(1 + \ell n\ t)^4 \tag{22}$$

which implies (13).

We finally remark that the support condition (14) which appeared in the early work [11] and had been eliminated in [2] [10] is again required in Proposition 2. However in contrast to [2] [10] [11], that proposition does not require any smallness assumption on u_+ and for that purpose relies on a more complicated method. It is an open question whether the support condition can also be eliminated in the framework of that method.

References

[1] J. Ginibre, G. Velo, Long range scattering and modified wave operators for the Maxwell-Schrödinger system I. The case of vanishing asymptotic magnetic field. Commun. Math. Phys. **236**, 395-448 (2003)

[2] J. Ginibre, G. Velo, Long range scattering for the Maxwell-Schrödinger system with large magnetic field data and small Schrödinger data. Publ. RIMS, Kyoto Univ. **42**, 421-459 (2006)

[3] J. Ginibre, G. Velo, Long range scattering and modified wave operators for the Maxwell-Schrödinger system II. The general case. Ann. H. P. (2007), in press.

[4] J. Ginibre, G. Velo, Long range scattering for some Schrödinger related nonlinear systems in "Nonlinear Dispersive Equations" (T. Ozawa and Y. Tsutsumi Eds.), GAKUTO International Series, Mathematical Sciences and Applications **26**, 35-84 (2006)

[5] J. Ginibre, G. Velo, Uniqueness at infinity for the Maxwell-Schrödinger system with arbitrarily large asymptotic data, preprint (2007).

[6] Y. Guo, K. Nakamitsu, W. Strauss, Global finite energy solutions of the Maxwell-Schrödinger system. Commun. Math. Phys. **170**, 181-196 (1995)

[7] M. Nakamitsu, M. Tsutsumi, The Cauchy problem for the coupled Maxwell-Schrödinger equations. J. Math. Phys. **27**, 211-216 (1986)

[8] M. Nakamura, T. Wada, Local wellposedness for the Maxwell-Schrödinger equations. Math. Ann. **332**, 565-604 (2005)

[9] M. Nakamura, T. Wada, Global existence and uniqueness of solutions to the Maxwell-Schrödinger equations, Preprint (2006), math.AP/0607039

[10] A. Shimomura, Modified wave operators for Maxwell-Schrödinger equations in three space dimensions. Ann. H.P. **4**, 661-683 (2003)

[11] Y. Tsutsumi, Global existence and asymptotic behaviour of solutions for the Maxwell-Schrödinger system in three space dimensions. Commun. Math. Phys. **151**, 543-576 (1993)

SMALL DATA SCATTERING FOR NONLINEAR KLEIN-GORDON EQUATIONS

NAKAO HAYASHI

1. Introduction

This talk is based on the joint work with Naumkin. I am interested in the scattering operator for the nonlinear Klein-Gordon equation

$$u_{tt} - \Delta u + u = f(u), \quad (t,x) \in \mathbf{R} \times \mathbf{R}^n \tag{1.1}$$

with a power type nonlinearity $f(u) = \mu |u|^{\sigma-1} u$ or $f(u) = \mu |u|^{\sigma}$, where $\sigma > 1 + \frac{4}{n+2}$, $\mu \in \mathbf{C}$, for space dimensions $n \geq 3$. The construction of the scattering operator implies the study of the Cauchy problem and the final state problem.

In the previous paper [7], we constructed the scattering operator in $\mathbf{H}^{1+\frac{n}{2},1}$ for the nonlinear Klein-Gordon equation (1.1) with $\sigma > 1 + \frac{2}{n}$ in the case of space dimensions $n =$ 1or 2. We applied the operator

$$\mathcal{J} = \langle i\nabla \rangle U(t) xU(-t) = E \langle i\nabla \rangle x + iAt\nabla,$$

where

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which plays the same role as the operator $x + it\nabla$ in the case of the nonlinear Schrödinger equations. Our purpose is to apply the operator $\mathcal{J}$ to the nonlinear Klein-Gordon equation (1.1) in higher space dimensions $n \geq 3$ and to prove the existence of the scattering operator similarly to the case of the nonlinear Schrödinger equations [3].

When $\mu < 0$, $f(u) = \mu |u|^{\sigma-1} u$ and $1 + \frac{4}{n} < \sigma < \sigma^*(n)$, with $\sigma^*(n) = \frac{n+2}{n-2}$ for $n \geq 3$, the completeness of the scattering operator for the nonlinear Klein-Gordon equation (1.1) in the energy space was established in papers [1], [2], [6], [10], [11] by using the Morawetz type estimates and the energy conservation law. This result was extended to lower space dimensions $n = 1, 2$ in paper [9]. The condition $\mu < 0$ can be removed if we restrict our attention to small solutions (see [13] for the case of $f(u) = \mu |u|^{\sigma-1} u$, the case $f(u) = \mu |u|^{\sigma}$ can also be treated). The existence of global in time solutions to the Cauchy problem for the nonlinear Klein-Gordon equation (1.1) (i.e. the existence of the inverse wave operator $\mathcal{W}_-^{-1}$) was shown in [13] by using the $\mathbf{L}^{1+\sigma} - \mathbf{L}^{1+\frac{1}{\sigma}}$ time decay estimates for the linear problem if $\sigma_0(n) < \sigma \leq 1+\frac{4}{n}$, where $\sigma_0(n)$ is a positive root of $\frac{n}{2}\frac{\sigma-1}{\sigma+1}\sigma > 1$. The wave operator $\mathcal{W}_+$ was also constructed in [13] for $\sigma_0(n) < \sigma \leq 1 + \frac{4}{n}$. However the scattering operator $\mathcal{S}_+ = \mathcal{W}_-^{-1}\mathcal{W}_+$ was not defined since the range of the wave operator $\mathcal{W}_+$ differs from the domain of the inverse wave operator $\mathcal{W}_-^{-1}$. As far as we know the

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scattering operator $\mathcal{S}_+$ was not constructed for the nonlinearities of order less than $1+\frac{4}{n}$ except our previous work [7] for $n=1,2$. Note that $1+\frac{4}{n+2}<\sigma_0(n)$.

We now turn to the results concerning the Cauchy problem for the nonlinear Klein-Gordon equation

$$(1.2)\qquad \begin{cases} u_{tt}-\Delta u+u=f(u), & (t,x)\in\mathbf{R}\times\mathbf{R}^n, \\ u(0,x)=u_0(x),\ u_t(0,x)=u_1(x), & x\in\mathbf{R}^n, \end{cases}$$

which are related to the construction of the inverse wave operator $\mathcal{W}_-^{-1}$. When $n=3,\sigma=2$, then $\frac{n}{2}\frac{\sigma-1}{\sigma+1}\sigma=1$, so the $\mathbf{L}^{1+\sigma}-\mathbf{L}^{1+\frac{1}{\sigma}}$ time decay estimates of [13] can not be applied to the Cauchy problem (1.2) even if the nonlinearity is smooth. In the case of the Cauchy problem, the lower order σ were treated in papers [8] and [12], where the global existence of small solutions to quadratic nonlinear Klein-Gordon equations in three space dimensions was proved by the vector fields and the normal forms methods, respectively. However these methods do not work for the nonlinearity of the form $|u|\,u$. The vector field method was improved in paper [4],[5], where the global existence theorem was proved for the fractional order $\sigma>1+\frac{2}{n}$, in space dimensions $n=1,2,3$, if the initial data have a compact support. It seems that the method does not work for the data which do not have a compact support.

We put

$$w\equiv\frac{1}{2}\left(\mathbf{a}u+i\mathbf{b}\left\langle i\nabla\right\rangle^{-1}u_t\right),\ w^0\equiv\frac{1}{2}\left(\mathbf{a}u_0+i\mathbf{b}\left\langle i\nabla\right\rangle^{-1}u_1\right),$$

$$\mathcal{L}=E\partial_t+iA\left\langle i\nabla\right\rangle$$

and

$$\mathcal{N}(w)=\frac{i\mu}{2}\mathbf{b}\left\langle i\nabla\right\rangle^{-1}|(\mathbf{a}\cdot w)|^{\sigma-1}(\mathbf{a}\cdot w)\ \text{or}\ \mathcal{N}(w)=\frac{i\mu}{2}\mathbf{b}\left\langle i\nabla\right\rangle^{-1}|(\mathbf{a}\cdot w)|^{\sigma},$$

where

$$\mathbf{a}=\begin{pmatrix}1\\1\end{pmatrix},\ \mathbf{b}=\begin{pmatrix}1\\-1\end{pmatrix},\ E=\begin{pmatrix}1&0\\0&1\end{pmatrix},\ A=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

Then the nonlinear Klein-Gordon equation (1.2) can be rewritten as a system of equations

$$(1.3)\qquad \begin{cases} \mathcal{L}w=\mathcal{N}(w), & (t,x)\in\mathbf{R}\times\mathbf{R}^n, \\ w(0,x)=w^0(x), & x\in\mathbf{R}^n. \end{cases}$$

The direct Fourier transform $\hat{\phi}(\xi)$ of the function $\phi(x)$ is defined by

$$\mathcal{F}\phi=\hat{\phi}=(2\pi)^{-\frac{n}{2}}\int_{\mathbf{R}^n}e^{-i(x\cdot\xi)}\phi(x)\,dx,$$

then the inverse Fourier transformation is given by

$$\mathcal{F}^{-1}\phi=(2\pi)^{-\frac{n}{2}}\int_{\mathbf{R}^n}e^{i(x\cdot\xi)}\phi(\xi)\,d\xi.$$

Denote the usual Lebesgue space $\mathbf{L}^p=\{\phi\in\mathbf{S}';\|\phi\|_{\mathbf{L}^p}<\infty\}$, where the norm $\|\phi\|_{\mathbf{L}^p}=\left(\int_{\mathbf{R}^n}|\phi(x)|^p\,dx\right)^{\frac{1}{p}}$ if $1\le p<\infty$ and $\|\phi\|_{\mathbf{L}^\infty}=\text{vrai}\sup_{x\in\mathbf{R}^n}|\phi(x)|$ if $p=\infty$. Weighted Sobolev space is

$$\mathbf{H}_p^{m,k}=\left\{\phi\in\mathbf{S}':\|\phi\|_{\mathbf{H}_p^{m,k}}\equiv\left\|\langle x\rangle^k\langle i\partial\rangle^m\phi\right\|_{\mathbf{L}^p}<\infty\right\},$$

where $m,k \in \mathbf{R}$, $1 \le p \le \infty$, $\langle x \rangle = \sqrt{1+|x|^2}$. We also write $\mathbf{H}^{m,k} = \mathbf{H}_2^{m,k}$. The usual Sobolev space is $\mathbf{H}^m = \mathbf{H}_2^{m,0}$, so the index 0 we usually omit if it does not cause a confusion. Different positive constants we denote by the same letter C.

We introduce the free evolution group

$$U(t) = \begin{pmatrix} e^{-i\langle i\nabla\rangle t} & 0 \\ 0 & e^{i\langle i\nabla\rangle t} \end{pmatrix}.$$

The operator

$$\mathcal{J} = \langle i\nabla\rangle U(t)\, xU(-t) = E\langle i\nabla\rangle x + iAt\nabla$$

is useful for obtaining the large time decay estimates of solutions. We have $[\mathcal{L},\mathcal{J}] = 0$, since $[x, \langle i\nabla\rangle] = \langle i\nabla\rangle^{-1}\nabla$. However it is difficult to calculate the action of $\mathcal{J}$ on the nonlinearity $\mathcal{N}$. Therefore we use the first order differential operator

$$\mathcal{P} = Et\nabla + Ex\partial_t$$

which is closely related to $\mathcal{J}$ by $\mathcal{P} = \mathcal{L}x - i\mathcal{J}$, and it almost commutes with $\mathcal{L}$ since $[\mathcal{L},\mathcal{P}] = -i\langle i\nabla\rangle^{-1}\nabla\mathcal{L}$ (see [8]).

First we prove the existence of the inverse wave operator

$$\mathcal{W}_+^{-1} : \left(\mathbf{H}^{\beta,1}\right)^2 \to \left(\mathbf{H}^{\beta,1}\right)^2,$$

where $\beta = \max\left(\frac{3}{2}, 1+\frac{2}{n}\right)$.

Theorem 1.1. *Let $1+\frac{4}{n+2} < \sigma < 1+\frac{4}{n}$ and $n \ge 3$. Suppose that the initial data $w^0 \in \left(\mathbf{H}^{\beta,1}\right)^2$, $\beta = \max\left(\frac{3}{2}, 1+\frac{2}{n}\right)$ have a small norm $\|w^0\|_{\mathbf{H}^{\beta,1}}$. Then there exists a unique solution $U(-t)\,w \in \mathbf{C}\left([0,\infty); \left(\mathbf{H}^{\beta,1}\right)^2\right)$ to the Cauchy problem (1.3) such that*

$$\|w(t)\|_{\mathbf{L}^q} \le C(1+t)^{-\frac{n}{2}\left(1-\frac{2}{q}\right)}$$

for all $t \ge 0$, where $2 \le q < \frac{2n}{n-2}$. Furthermore there exists a unique final state $w^+ \in \left(\mathbf{H}^{\beta,1}\right)^2$ such that

$$\left\|U(-t)\,w(t) - w^+\right\|_{\mathbf{H}^{\beta,1}} \le C(1+t)^{-\gamma} \tag{1.4}$$

for all $t \ge 0$, where $\gamma = \frac{n}{2}(\sigma-1)\left(1-\frac{1}{q}\right) - 1 > 0$.

We now consider the final state problem for the nonlinear Klein-Gordon equation

$$\begin{cases} \mathcal{L}w = \mathcal{N}(w), \\ \|w(t) - U(t)\,w^+\|_{\mathbf{L}^2} \to 0 \text{ as } t \to \infty \end{cases} \tag{1.5}$$

with a given final state $w^+ \in \left(\mathbf{H}^{\beta,1}\right)^2$.

Theorem 1.2. *Let $1+\frac{4}{n+2} < \sigma < 1+\frac{4}{n}$ and $n \ge 3$. Suppose that the final state $w^+ \in \left(\mathbf{H}^{\beta,1}\right)^2$, $\beta = \max\left(\frac{3}{2}, 1+\frac{2}{n}\right)$. Then there exists a time $T \ge 0$ and a unique solution $U(-t)\,w \in \mathbf{C}\left([T,\infty); \left(\mathbf{H}^{\beta,1}\right)^2\right)$ of the final state problem (1.5) such that*

$$\|w(t)\|_{\mathbf{L}^q} \le C(1+t)^{-\frac{n}{2}\left(1-\frac{2}{q}\right)}$$

for all $t \ge T$, where $2 \le q < \frac{2n}{n-2}$. Furthermore the asymptotics

$$\left\|U(-t)\,w(t) - w^+\right\|_{\mathbf{H}^{\beta,1}} \le Ct^{-\gamma}$$

is valid for all $t \geq T$, *where* $\gamma = \frac{n}{2}(\sigma - 1)\left(1 - \frac{1}{q}\right) - 1 > 0$.

By Theorem 1.2, we can define the wave operator $\mathcal{W}_+$ which maps any final state $w^+ \in \left(\mathbf{H}^{\beta,1}\right)^2$ to the solution $U(-t)\, w \in \left(\mathbf{H}^{\beta,1}\right)^2$ if $t \geq T$. If we choose a sufficiently small norm $\|w^+\|_{\mathbf{H}^{\beta,1}}$, we can take $T = 0$. Namely, the wave operator

$$\mathcal{W}_+ : w^+ \in \left(\mathbf{H}^{\beta,1}\right)^2 \to w^0 \in \left(\mathbf{H}^{\beta,1}\right)^2$$

is well-defined in the neighborhood of the origin in the $\left(\mathbf{H}^{\beta,1}\right)^2$ space. Furthermore since the initial data w^0 are also sufficiently small in the norm of $\left(\mathbf{H}^{\beta,1}\right)^2$, by applying Theorem 1.1 for the negative time we can define the inverse wave operator

$$\mathcal{W}_-^{-1} : w^0 \in \left(\mathbf{H}^{\beta,1}\right)^2 \to w^- \in \left(\mathbf{H}^{\beta,1}\right)^2.$$

This means that the scattering operator

$$\mathcal{S}_+ = \mathcal{W}_-^{-1}\mathcal{W}_+ : w^+ \in \left(\mathbf{H}^{\beta,1}\right)^2 \to w^- \in \left(\mathbf{H}^{\beta,1}\right)^2$$

is well-defined in the neighborhood of the origin in the $\left(\mathbf{H}^{\beta,1}\right)^2$ space.

References

[1] P. Brenner, *On space time means and everywhere defined scattering operators for nonlinear Klein-Gordon equations*, Math. Z., **186** (1984), pp. 383-391.

[2] P. Brenner, *On scattering and everywhere defined scattering operators* for *nonlinear Klein-Gordon equations*, J. Differential Equations, **56** (1985), pp. 310-344.

[3] T.Cazenave and F.B.Weissler, *Rapidly decaying soutions of the nonlinear Schrödinger equation*, Commun. Math. Phys., **147**(1992), pp. 75-100.

[4] V. Georgiev and S. Lecente, *Weighted Sobolev spaces applied to nonlinear Klein-Gordon equation*, C. R. Acad. Sci. Paris, **329** (1999), pp. 21-26.

[5] V. Georgiev and S. Lecente, *Deacy for nonlinear Klein-Gordon equations*, Nonlinear differ. equ. appl., **11** (2004), pp. 529-555.

[6] J. Ginibre and G. Velo, *Time deacy of finite energy solutions of the nonlinear Klein-Gordon equation*, Ann. Inst. H. Poncaré Phys. Théor., **41** (1985), pp. 399-442.

[7] N.Hayashi and P.Naumkin, *Scattering operator for the nonlinear Klein-Gordon equations,* preprint 2006.

[8] S. Klainerman, *Global existence of small amplitude solutions to nonlinear Klein-Gordon equations in four space-time dimensions,* Commun. Pure Appl. Math., **38** (1985), pp. 631-641.

[9] K. Nakanishi, *Energy scattering for nonlinear Klein-Gordon and Schrödinger equations in spatial dimensions* 1 *and* 2, J. Funct. Anal. **169** (1999), pp. 201–225.

[10] H. Pecher, *Nonlinear small data scattering for the wave and Klein-Gordon equation*, Math. Z., **185** (1984), pp. 261-270.

[11] H. Pecher, *Low energy scattering for Klein-Gordon equations*, J. Funct. Anal, **63** (1985), pp. 101-122.

[12] J. Shatah, *Normal forms and quadratic nonlinear Klein-Gordon equations,* Commun. Pure Appl. Math., **38** (1985), pp. 685-696.

[13] W.A. Strauss, *Nonlinear scattering theory at low energy,* J. Funct. Anal., **41** (1981), pp. 110–133.

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Scattering for the Schrödinger-improved Boussinesq system

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This talk is based on [6], and it is concerned with the scattering theory for the Schrödinger-improved Boussinesq system (hereafter referred to as the Schrödinger-IBq system) in two space dimensions:

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u = vu, \\ \partial_t^2 v - \Delta v - \Delta\partial_t^2 v = \Delta|u|^2, \end{cases} \tag{1}$$

where $(t,x) \in \mathbb{R} \times \mathbb{R}^2$, $\partial_t = \partial/\partial t$, Δ is the Laplace operator for the space variable x, and u and v are complex-valued and real-valued unknown functions of (t,x), respectively. We prove the existence of wave operators for the system (1) when the Schrödinger data is suitably small.

There are several results on the local and global existence of solutions and the asymptotic behavior in time of solutions to the Schrödinger-IBq system (1). Ozawa and Tsutaya [5] proved the local well-posedness for the system (1) in the space $L^2 \times L^2 \times L^2 \ni (u, v, \partial_t v)$, when the space dimension $n \leq 3$ by the Strichartz estimate for the Schrödinger equation. They also showed the global well-posedness in the energy class $H^1 \times L^2 \times (L^2 \cap \dot{H}^{-1})$ when $n \leq 2$. Furthermore, in [5], when the space dimension $n = 4$, the local well-posedness for this system in the L^2-level for small initial data was shown. Cho and Ozawa [2] proved the existence of a unique global solution to the system (1) in the energy class for small initial data, when the space dimension $n = 3$ or 4, Furthermore, in [2], when $n = 4$, it was also shown that the small global solution has a free profile in $L^2 \times L^2 \times \dot{H}^{-1}$ by the Strichartz estimate for the Schrödinger equation on the time interval $[0, \infty)$. Recently, Akahori [1] proved the local well-posedness in $H^{s_1} \times H^{s_2} \times H^{s_2}$ for $-1/4 < s_1 < 0$ and $-1/2 < s_2 < 0$ when $n \leq 3$, and the global well-posedness in $L^2 \times L^2 \times L^2$ when $n \leq 2$. (For investigating the large-time behavior of solutions to the system (1) in relatively low dimensional case, (that is, $n \leq 3$), it seems that the method via the Strichartz estimate for the Schrödinger equation on the time interval $[0, \infty)$ in four-dimensional case in [2] does not work, because the Strichartz estimate for the solution u on the time interval $[0, \infty)$ does not derive the optimal time decay of the solution (u, v).)

We summarize properties of the free solutions. We introduce the following operators

$$U(t) = e^{\frac{it}{2}\Delta}, \quad \Omega = (1-\Delta)^{1/2}, \quad A = (-\Delta)^{1/2}(1-\Delta)^{-1/2},$$
$$K(t) = A^{-1}\sin tA, \quad \dot{K}(t) = \cos tA.$$

We note that the solution to the Cauchy problem of the free Schrödinger equation

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(0,x) = \phi(x), & x \in \mathbb{R}^n \end{cases}$$

is given by $u(t,\cdot) = U(t)\phi$, and that the solution to the Cauchy problem of the free IBq equation

$$\begin{cases} \partial_t^2 v - \Delta v - \Delta\partial_t^2 v = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\ v(0,x) = \psi_0(x), \quad \partial_t v(0,x) = \psi_1(x), & x \in \mathbb{R}^n \end{cases}$$

is given by $v(t,\cdot) = \dot{K}(t)\psi_0 + K(t)\psi_1$. It is well-known that the solutions to the free Schrödinger equation decays as

$$\|U(t)\phi\|_{L_x^\infty(\mathbb{R}^n)} \leq C|t|^{-n/2}\|\phi\|_{L^1(\mathbb{R}^n)}. \tag{2}$$

Cho and Ozawa [3] proved the time decay estimate for the solution to the free IBq system

$$\begin{aligned} &\|\dot{K}(t)\phi\|_{L_x^\infty(\mathbb{R}^n)} + \|K(t)\psi\|_{L_x^\infty(\mathbb{R}^n)} \\ \leq &C|t|^{-n/2}(\|\Omega^{2n}A^{n/2-1}\phi\|_{\dot{B}^0_{1,1}} + \|\Omega^{2n}A^{n/2-2}\psi\|_{\dot{B}^0_{1,1}}). \end{aligned} \tag{3}$$

(For the definitions of function spaces, see below.) Hence it seems that the nonlinear interaction in the Schrödinger exponent vu in the system (1) decays like $\|u(t)v(t)\|_{L^2(\mathbb{R}^n)} \sim t^{-n/2}$, as $t \to \infty$, and that the scattering problem for the system (1) in relatively low dimension (namely, the space dimension $n \leq 3$) is rather difficult. In particular, the two-dimensional case is the borderline case between the short-range and the long range scattering. In this talk, in the case that the space dimension $n = 2$, we prove the existence of an asymptotically free solution to the system (1) when the Schrödinger data is suitably small. The uniqueness holds in a suitable class (see (4)–(7)). This implies the existence of the wave operators. (The three-dimensional case is easier than the two-dimensional case, according to the decay rate of the free solutions (see the estimates (2) and (3)).)

We remark that there is a survey [4] by Ginibre and Velo on other nonlinear coupled systems related to the Schrödinger equation in relatively low space dimensions, that is, the Maxwell-Schrödinger, the wave-Schrödinger, the Klein-Gordon-Schrödinger and the Zakharov systems.

Before stating the main result, we introduce several notations. For $\psi \in \mathcal{S}'$, we denote the Fourier transform of ψ by $\hat{\psi}$ or $\mathcal{F}\psi$. For $m, s \in \mathbb{R}$, we introduce the weighted Sobolev space:

$$H^{m,s} = \{\psi \in \mathcal{S}'; \|\psi\|_{H^{m,s}} = \|(1+|x|^2)^{s/2}(1-\Delta)^{m/2}\psi\|_{L^2} < \infty\}.$$

H^m denotes $H^{m,0}$. $\dot{H}^m$ is the homogeneous Sobolev space:

$$\dot{H}^m = \{\psi \in \mathcal{S}'; \|\psi\|_{\dot{H}^m} = \|(-\Delta)^{m/2}\psi\|_{L^2} < \infty\}.$$

For $m \in \mathbb{R}$ and $1 \leq q, r \leq \infty$, the homogeneous Besov space $\dot{B}^m_{q,r}$ is defined as follows:

$$\dot{B}^m_{q,r} = \{\psi \in \mathcal{S}'; \|\psi\|_{\dot{B}^m_{q,r}} = \|\{2^{jm}\|\varphi_j * \psi\|_{L^q}\}_j\|_{l^r_j(\mathbb{Z})} < \infty\},$$

where $\hat{\varphi}_j(\xi) = \hat{\phi}(2^{-j}\xi) - \hat{\phi}(2^{-(j+1)}\xi)$ and $\phi \in \mathcal{S}$ is a function such that $\hat{\phi} \in C_0^\infty$ with $0 \leq \hat{\phi} \leq 1$, $\hat{\phi}(\xi) = 1$ when $|\xi| \leq 1$ and $\hat{\phi}(\xi) = 0$ when $|\xi| \geq 2$.

The main result is

Theorem. *Let the space dimension $n = 2$, and let (u_+, v_{+0}, v_{+1}) be final data such that u_+ is complex valued, v_{+0} and v_{+1} are real valued, $u_+ \in L^2$, $(1+|x|)u_+ \in L^1$, $v_{+0} \in H^{0,1} \cap \dot{H}^{-2} \cap \Omega^{-4}\dot{B}^0_{1,1}$, $xv_{+0} \in \dot{H}^{-1}$, $A^{-1}v_{+1} \in L^2 \cap \dot{H}^{-2} \cap \Omega^{-4}\dot{B}^0_{1,1}$ and $xv_{+1} \in \dot{H}^{-1} \cap \dot{H}^{-3}$. Assume that $\|u_+\|_{L^1_x}$ is sufficiently small. Then the system (1) has a unique solution (u, v) satisfying*

$$u \in C(\mathbb{R}; L^2_x), \quad v \in C^1(\mathbb{R}; L^2_x), \tag{4}$$

$$\sup_{t\geq 1} t^k \|u(t) - U(t)u_+\|_{L^2_x} < \infty, \tag{5}$$

$$\sup_{t\geq 1} t^k \|u - U(\cdot)u_+\|_{L^4((t,\infty);L^4_x)} < \infty, \tag{6}$$

$$\begin{aligned}\sup_{t\geq 1} t^k (&\|v(t) - (\dot{K}(t)v_{+0} + K(t)v_{+1})\|_{L^2_x} \\ &+ \|\partial_t v(t) - \partial_t(\dot{K}(t)v_{+0} + K(t)v_{+1})\|_{L^2_x}) < \infty,\end{aligned} \tag{7}$$

where $1/2 < k \leq 3/4$. *Furthermore the wave operator*

$$W_+ : (u_+, v_{0+}, v_{1+}) \mapsto (u(0), v(0), (\partial_t v)(0))$$

is well-defined.

A similar result holds for the negative time.

Remark 1. In [6], the existence and uniqueness of an asymptotically free solution to the system (1) are proved by solving the Cauchy problem at $t = +\infty$ is solved locally on the interval $[T, \infty)$ for sufficiently large $T \geq 1$. According to the result on the global well-posedness for that system in $L^2 \times L^2 \times L^2$ by Akahori [1], we can extend the (local) asymptotically free solution obtained in [6] to the whole interval $\mathbb{R}$ as in (4)–(7).

We prove Theorem by solving the local Cauchy problem at $t = +\infty$ around some approximate solution which approaches the free solution as $t \to \infty$, and by extending it globally as in Remark 1.

References

[1] T. Akahori, *Well-posedness for Schrödinger-improved Boussinesq system and related bilinear estimates*, to appear in Funkcial. Ekvac.

[2] Y. Cho and T. Ozawa, *Global existence on nonlinear Schrödinger-IMBq equations*, J. Math. Kyoto Univ. **46** (2006), 535–552.

[3] Y. Cho and T. Ozawa, *On small amplitude solutions to the generalized Boussinesq equations*, Discrete Contin. Dyn. Syst. **17** (2007), 691–711.

[4] J. Ginibre and G. Velo, *Long range scattering for some Schrödinger related to nonlinear systems*, in "Nonlinear Dispersive Equations" (T. Ozawa and Y. Tsutsumi Eds.), GAKUTO International Series, Mathematical Sciences and Applications **26** (2006), 35–84.

[5] T. Ozawa and K. Tsutaya, *On the Cauchy problem for Schrödinger-improved Boussinesq equations*, to appear in Adv. Stud. Pure Math.

[6] A. Shimomura, *Scattering theory for the Schrödinger-improved Boussinesq system in two space dimensions*, Asymptot. Anal. **51** (2007), 167–187.

On the Schrödinger equation with dissipative nonlinearities of derivative type

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This talk is based on a joint work with Nakao Hayashi and Pavel Naumkin [8]. We consider the initial value problem for the nonlinear Shrödinger equation of the derivative type:

$$\begin{cases} i\partial_t u + \frac{1}{2}\partial_x^2 u = N(u, \partial_x u), & t > 0,\ x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{1}$$

where $i = \sqrt{-1}$, $\partial_t = \partial/\partial t$, $\partial_x = \partial/\partial x$ and u is a complex-valued unknown function. We will occasionally write u_x for $\partial_x u$, and $\overline{u}$ denotes the complex conjugate of u. The nonlinear term $N(u, u_x)$ is a cubic homogeneous polynomial in $(u, \overline{u}, u_x, \overline{u_x})$ with complex coefficients, and it satisfies so-called gauge invariance, that is,

$$N(e^{i\theta}v, e^{i\theta}q) = e^{i\theta}N(v, q) \quad \text{for} \quad v, q \in \mathbb{C} \ \text{ and } \ \theta \in \mathbb{R}. \tag{2}$$

The aim of this talk is to present a structural condition on the nonlinear term N under which the corresponding forward Cauchy problem (1) has a disspative nature. To explain the motivation, let us begin with the simplest case where N is independent of u_x, i.e., $N = \lambda|u|^2 u$ with $\lambda \in \mathbb{C}$. Then it is easy to see that

$$\|u(t)\|_{L^2}^2 - 2\,\mathrm{Im}\,\lambda \int_0^t \|u(\tau)\|_{L^4}^4\, d\tau = \|u_0\|_{L^2}^2, \tag{3}$$

which suggests dissipativity if $\mathrm{Im}\,\lambda < 0$. In fact, it is proved in [17] that the solution decays like $O((t\log t)^{-1/2})$ in L_x^∞ as $t \to +\infty$ if $\mathrm{Im}\,\lambda < 0$ and u_0 is small enough. Since the non-trivial free solution (i.e., the solution to (1) for $N \equiv 0$, $u_0 \neq 0$) only decays like $O(t^{-1/2})$, this gain of additional logarithmic time decay reflects a disspative character. Now we turn our attentions to the general gauge-invariant cubic nonlinear terms involving both u and u_x. Note that we can not expect the conservation law like (3) any more. However, as we shall show, similar time decay is still valid if $\sup_{\xi\in\mathbb{R}} \mathrm{Im}\, N(1, i\xi) < 0$.

Before stating our result, we introduce function spaces. For $s, \nu \geq 0$, let $H^{s,\nu}$ be the weighted Sobolev space given by $\{\phi \in L^2 : \|\phi\|_{H^{s,\sigma}} = \|\langle x\rangle^\nu \langle i\partial_x\rangle^s \phi\|_{L^2} < \infty\}$, where $\langle x\rangle = (1 + x^2)^{1/2}$. In particular we set $H^s = H^{s,0}$, which is usual L^2 based Sobolev space of order s. The main result is as follows.

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Theorem. *Suppose that N satisfies*

$$\operatorname{Im} N(1, i\xi) \leq 0 \quad \textit{for} \quad \xi \in \mathbb{R}. \tag{4}$$

Let $u_0 \in H^{2,1} \cap H^3$ and $\varepsilon = \|u_0\|_{H^{2,1}} + \|u_0\|_{H^3}$ is sufficiently small. Then (1) admits a unique global solution $u \in C([0,\infty); H^{2,1} \cap H^3)$. Moreover, the following asymptotic expression is valid as $t \to +\infty$ uniformly in $x \in \mathbb{R}$:

$$u(t,x) = \frac{a(\frac{x}{t}) \exp\left\{ i\frac{x^2}{2t} - i|a(\frac{x}{t})|^2 \operatorname{Re} N(1, i\frac{x}{t}) \int_0^{\log t} \frac{d\sigma}{1 - 2\operatorname{Im} N(1, i\frac{x}{t})(|a(\frac{x}{t})|^2 \sigma + b(\frac{x}{t}))} \right\}}{\sqrt{t}\sqrt{1 - 2\operatorname{Im} N(1, i\frac{x}{t})\big(|a(\frac{x}{t})|^2 \log t + b(\frac{x}{t})\big)}} + O\big(t^{-3/4+\mu}\big), \tag{5}$$

where $\mu > 0$ is an arbitrarily small constant, and $a(\xi)$, $b(\xi)$ are complex valued continuous functions of $\xi \in \mathbb{R}$ which satisfy $|a(\xi)| \leq C\varepsilon\langle\xi\rangle^{-2}$, $|b(\xi)| \leq C\varepsilon^4\langle\xi\rangle^{-4}$ with some positive constant C.

Remark 1. As an immediate consequence of (5), we can see that

$$\|u(t)\|_{L^\infty} = O((t \log t)^{-1/2}) \quad \text{as} \quad t \to +\infty$$

if $\sup_{\xi\in\mathbb{R}} \operatorname{Im} N(1, i\xi) < 0$. Moreover, we can also check that the L^2 norm of $u(t)$ also decays like $(\log t)^{-1/2}$. Therefore, by interpolation, we obtain

$$\|u(t)\|_{L^p} = O\big(t^{-(1/2-1/p)}(\log t)^{-1/2}\big) \quad \text{as} \quad t \to +\infty$$

for all $p \in [2, \infty]$.

Remark 2. Our result can be also viewed as an extension of [9] (see also [1], [2], [3], [10], [12], [14], [15], [16], [17], [20], etc.) because (5) is reduced to

$$u(t,x) = \frac{1}{\sqrt{t}} a(x/t) e^{i\left(x^2/(2t) - |a(x/t)|^2 \operatorname{Re} N(1, ix/t) \log t\right)} + O\big(t^{-3/4+\mu}\big) \quad \text{as} \quad t \to +\infty$$

when $\operatorname{Im} N(1, i\xi) \equiv 0$.

Remark 3. When $\operatorname{Im} N(1, i\xi_0) > 0$ for some $\xi_0 \in \mathbb{R}$, the authors do not know any global existence or non-existence results for (1). However, in view of the denominator of the leading term of (5), it is quite reasonable to expect that small amplitude solutions can blow up in finite time if the condition (4) is violated. Concerning the lifespan T_ε of the solution for (1) with $u_0(x) = \varepsilon\varphi(x)$, the following explicit lower bound is obtained in [19]:

$$\liminf_{\varepsilon \downarrow 0} \varepsilon^2 \log T_\varepsilon \geq \frac{1}{\sup_{\xi\in\mathbb{R}} \big(2|\hat{\varphi}(\xi)|^2 \operatorname{Im} N(1, i\xi)\big)},$$

where $\hat{\varphi}$ denotes the Fourier transform of φ. It may be an interesting open problem to consider whether the corresponding upper estimate holds or not.

Remark 4. When we put

$$A(s,\xi)=\frac{a(\xi)\exp\Big\{-i|a(\xi)|^2\operatorname{Re}N(1,i\xi)\int_0^s\frac{d\sigma}{1-2\operatorname{Im}N(1,i\xi)(|a(\xi)|^2\sigma+b(\xi))}\Big\}}{\sqrt{1-2\operatorname{Im}N(1,i\xi)(|a(\xi)|^2s+b(\xi))}},$$

the asymptotic expression (5) can be interpreted as

$$u(t,x)\overset{t\to\infty}{\sim}\frac{e^{ix^2/(2t)}}{\sqrt{t}}A\left(\log t,\frac{x}{t}\right);\quad i\partial_s A=N(1,i\xi)|A|^2A.$$

This tells us that asymptotic behavior of the solution for (1) is characterized by that of the solution for the simpler ordinary differential equation. At the level of the reduced equation, the dissiaptive nature is transparent via the identity

$$\partial_s\big(|A(s,\xi)|^2\big)=2\operatorname{Im}N(1,i\xi)|A(s,\xi)|^4.$$

Note that analogous results have been obtained in [18] for the nonlinear Klein-Gordon equations and in [11], [13] for the nonlinear wave equations.

Remark 5. It is possible to weaken (2) to a certain extent, but impossible to remove it completely. In fact, when (2) is replaced by

$$N(e^{i\theta},0)=e^{i\theta}N(1,0)\quad\text{for}\quad\theta\in\mathbb{R},\tag{6}$$

we can modify the above argument combining the idea of [4], [5] (see also Appendix of [18]) and show that the above theorem is still valid if $N(1,i\xi)$ in the statement is replaced by

$$\frac{1}{2\pi}\int_0^{2\pi}N(e^{i\theta},i\xi e^{i\theta})e^{-i\theta}\,d\theta.$$

Note that (6) is just what excludes u^3, $\overline{u}^3$, $u\overline{u}^2$ from all possible cubic nonlinear terms, but it is not a technical assumption because for these three nonlinearities we can find a class of initial data for which the solution has another kind of asymptotic profile than (5) (see [6] and [7] for the details).

References

[1] N.Hayashi and P.I.Naumkin, *Asymptotic behavior in time of solutions to the derivative nonlinear Schrödinger equation,* Ann. Inst. Henri Poincaré Phys. Théor. **68** (1998), 159–177.

[2] N.Hayashi and P.I.Naumkin, *Asymptotic behavior in time of solutions to the derivative nonlinear Schrödinger equation revisited,* Discrete Contin. Dynam. Systems **3** (1997), 383–400.

[3] N.Hayashi and P.I.Naumkin, *Asymptotics for large time of solutions to the nonlinear Schrödinger and Hartree equations,* Amer. J. Math. **120** (1998), 369–389.

[4] N. Hayashi and P. I. Naumkin, *Large time behavior of solutions for derivative cubic nonlinear Schrödinger equations without a self-conjugate property,* Funkcial. Ekvac. **42** (1999), 311–324.

[5] N. Hayashi and P. I. Naumkin, *Asymptotics of small solutions to nonlinear Schrodinger equations with cubic nonlinearities,* Int. J. Pure Appl. Math. **3** (2002), 255–273.

[6] N. Hayashi and P. I. Naumkin, *Large time behavior for the cubic nonlinear Schödinger equation,* Canad. J. Math. **54** (2002), 1065–1085.

[7] N. Hayashi and P. I. Naumkin, *On the asymptotics for cubic nonlinear Schrödinger equations,* Complex Var. Theory Appl. **49** (2004), 339–373.

[8] N.Hayashi, P.I.Naumkin and H.Sunagawa, *On the Schrödinger equation with dissipative nonlinearities of derivative type,* preprint, 2007.

[9] N.Hayashi, P.I.Naumkin and H.Uchida, *Large time behavior of solutions for derivative cubic nonlinear Schrödinger equations,* Publ. RIMS **35** (1999), 501–513.

[10] N. Hayashi and T. Ozawa, *Modified wave operators for the derivative nonlinear Schrödinger equation,* Math. Ann. **298** (1994), 557–576.

[11] A.Hoshiga, *The existence of the global solutions to semilinear wave equations with cubic nonlinearities in 2-dimensional space,* the abstract of the conference "Nonlinear Wave Equations," held at Sapporo, August 2007.

[12] S.Katayama and Y.Tsutsumi, *Global existence of solutions for nonlinear Schrödinger equations in one space dimension,* Comm. Partial Differential Equations, **19** (1994), 1971–1997.

[13] H.Kubo, *Large time behavior of solutions to semilinear wave equations with dissipative structure,* preprint, 2006.

[14] H.Lindblad and A.Soffer, *Scattering and small data completeness for the critical nonlinear Schrödinger equation,* Nonlinearity **19** (2006), 345–353.

[15] T. Ozawa, *Long range scattering for nonlinear Schrödinger equations in one space dimension,* Comm. Math. Phys. **139** (1991), 479–493.

[16] T.Ozawa, *On the nonlinear Schrödinger equations of derivative type,* Indiana Univ. Math. J. **45** (1996), 137–163.

[17] A.Shimomura, *Asymptotic behavior of solutions for Schrödinger equation with dissipative nonlinearities,* Comm. Partial Differential Equations **31** (2006), 1407–1423.

[18] H.Sunagawa, *Large time behavior of solutions to the Klein-Gordon equation with nonlinear dissipative terms,* J. Math. Soc. Japan **58** (2006), 379–400.

[19] H.Sunagawa, *Lower bounds of the lifespan of small data solutions to the nonlinear Schrödinger equations,* Osaka J. Math. **43** (2006), 771–789.

[20] Y.Tsutsumi, *The null gauge condition and the one dimensional nonlinear Schrödinger equation with cubic nonlinearity,* Indiana Univ. Math. J. **43** (1994), 241–254.

GLOBAL EXISTENCE AND ASYMPTOTIC BEHAVIOR FOR SYSTEMS OF SEMILINEAR WAVE EQUATIONS

SOICHIRO KATAYAMA

Let $n = 2$ or 3. We consider the Cauchy problem for a system of semilinear wave equations of the following type:

$$\Box u_i = F_i(u, \partial u) \quad \text{for } (t,x) \in (0,\infty) \times \mathbb{R}^n \tag{1}$$

with initial data

$$u_i(0,x) = \varepsilon f_i(x), \ (\partial_t u_i)(0,x) = \varepsilon g_i(x) \quad \text{for } x \in \mathbb{R}^n \tag{2}$$

$(i = 1, \ldots, N)$, where $\Box = \partial_t^2 - \Delta_x$ is the d'Alembertian, $u = (u_j)_{1 \le j \le N}$, and $\partial u = (\partial_a u_j)_{1 \le j \le N, 0 \le a \le n}$ with $\partial_0 = \partial_t$ and $\partial_k = \partial_{x_k}$ for $1 \le k \le n$, while ε is a small positive parameter.

For simplicity, we suppose that each $F_i = F_i(u, \partial u)$ $(1 \le i \le N)$ is a homogeneous polynomial of degree p in its arguments.

We say that **(SDGE)** holds if for any $f = (f_i)_{1 \le i \le N}, g = (g_i)_{1 \le i \le N} \in C_0^\infty(\mathbb{R}^n; \mathbb{R}^N)$, there exists a positive constant ε_0 such that the Cauchy problem (1)–(2) admits a unique global solution $u \in C^\infty([0,\infty) \times \mathbb{R}^n; \mathbb{R}^N)$ for any $\varepsilon \in (0, \varepsilon_0]$. Here, the word (SDGE) stands for "*small data global existence*".

On the other hand, we say **(AF)** holds if any global solution u to (1)–(2) with sufficiently small initial data is *asymptotically free*. More precisely, we say that a global solution u to (1)–(2) is *asymptotically free*, if there exists a function $\widetilde{u} = \widetilde{u}(t,x) (\not\equiv 0)$ solving $\Box \widetilde{u} = 0$ such that

$$\lim_{t \to \infty} \|u(t,\cdot) - \widetilde{u}(t,\cdot)\|_E = 0, \tag{3}$$

where $\|\cdot\|_E$ is the energy norm, that is

$$\|\phi(t,\cdot)\|_E^2 = \int_{\mathbb{R}^n} \left(|\partial_t \phi(t,x)|^2 + |\nabla_x \phi(t,x)|^2 \right) dx.$$

Let us recall the known results for the three space dimensional case $n = 3$ briefly. If $p \ge 3$, then (SDGE) and (AF) hold. On the other hand, if $p = 2$, (SDGE) does not hold in general. Hence $p = 2$

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is the critical power for $n = 3$. Klainerman [11] introduced a sufficient condition for (SDGE), which is called the null condition (see also Christodoulou [4]). If the null condition is satisfied, then F_i can be written as a linear combination of $Q_0(u_j, u_k)$ and $Q_{ab}(u_j, u_k)$, where *the null forms* Q_0 and Q_{ab} are defined by

$$Q_0(\phi, \psi) = (\partial_t \phi)(\partial_t \psi) - (\nabla_x \phi) \cdot (\nabla_x \psi), \tag{4}$$

$$Q_{ab}(\phi, \psi) = (\partial_a \phi)(\partial_b \psi) - (\partial_b \phi)(\partial_a \psi) \quad \text{for } 0 \le a, b \le 3, \tag{5}$$

respectively. It is easy to see that (AF) also holds under the null condition.

Alinhac [3] introduced a sufficient condition for (SDGE), which is weaker than the null condition. But Katayama – Kubo [10] showed that the (AF) does not hold in general under the Alinhac condition.

The "simplest example" satisfying the Alinhac condition is

$$\begin{cases} \Box u_1 = (\partial_1 u_1)(\partial_2 u_1 - \partial_1 u_2), \\ \Box u_2 = (\partial_2 u_1)(\partial_2 u_1 - \partial_1 u_2). \end{cases} \tag{6}$$

For this system, (AF) does not hold, though (SDGE) holds.

Now we turn our attention to the two space dimensional case $n = 2$. The critical power is $p = 3$ for $n = 2$. The null condition for $(n, p) = (2, 3)$ was also introduced, and (SDGE) under this null condition was obtained (see Godin [5], Hoshiga [6], and the author [8, 9]; see also Alinhac [1, 2] for the case quadratic terms are involved, and Hoshiga – Kubo [7] for the multiple propagation speeds case). It is also easy to obtain (AF) under the null condition with $(n, p) = (2, 3)$.

Our aim in this talk is to obtain the two dimensional analogue to the results by Alinhac [3] and Katayama–Kubo [10].

In the following, for a family of functions $\{\phi_\lambda\}_{\lambda \in \Lambda}$ and a function ψ, we write $\psi = \sum'_{\lambda \in \Lambda} \phi_\lambda$ if there exist some constants c_λ $(\lambda \in \Lambda)$ such that $\psi = \sum_{\lambda \in \Lambda} c_\lambda \phi_\lambda$.

We introduce the assumption

(H) There exists some $M \in \{1, \cdots, N-1\}$ such that each F_i $(1 \le i \le N)$ has the form

$$F_i(u, \partial u) = A_i(w, \partial w, \partial v) + N_i(u, \partial u) \qquad \text{for } 1 \le i \le M, \tag{7}$$

$$F_i(u, \partial u) = N_i(u, \partial u) \qquad \text{for } M+1 \le i \le N, \tag{8}$$

where

$$u = (u_i)_{1 \le i \le N} = \big((v_i)_{1 \le i \le M}, (w_i)_{1 \le i \le L}\big) = (v, w)$$

with $L = N - M$,

$$A_i(w, \partial w, \partial v) = \sum_{\substack{1\le j,k\le L,\ 1\le m\le M \\ |\alpha|,|\beta|\le 1,\ 0\le a\le 2}}{}' (\partial^\alpha w_j)(\partial^\beta w_k)(\partial_a v_m), \tag{9}$$

$$N_i(u, \partial u) = \sum_{\substack{1\le j,k,\ell\le N \\ |\alpha|\le 1}}{}' (\partial^\alpha u_j) Q_0(u_k, u_\ell) + \sum_{\substack{1\le j,k,\ell\le N \\ |\alpha|\le 1,\ 0\le a<b\le 2}}{}' (\partial^\alpha u_j) Q_{ab}(u_k, u_\ell). \tag{10}$$

Remark. (1) The assumption (H) with $A_i \equiv 0$ for all $i \in \{1, \dots, M\}$ coincides with the null condition for $(n, p) = (2, 3)$.

(2) There is a certain class of nonlinearity for which (H) is not explicitly fulfilled, but can be reduced to other nonlinearity satisfying (H). For example, consider the following system:

$$\begin{cases} \Box u_1 = (\partial_1 u_1)(\partial_2 u_1 - \partial_1 u_2)^2, \\ \Box u_2 = (\partial_2 u_1)(\partial_2 u_1 - \partial_1 u_2)^2. \end{cases} \tag{11}$$

Setting $v_1 = u_1$, $v_2 = u_2$ and $w = \partial_2 u_1 - \partial_1 u_2$, we find that solving (11) is equivalent to solve

$$\begin{cases} \Box v_1 = w^2(\partial_1 v_1), \\ \Box v_2 = w^2(\partial_2 v_1), \\ \Box w = 2w\, Q_{12}(v_1, w), \end{cases} \tag{12}$$

whose nonlinearity satisfies the assumption (H). Observe that this one corresponds to (6) for $n = 3$. Instead of (H), we can also get a sufficient condition for (SDGE) in two space dimensions, which is analogous to the Alinhac condition, but we omit the details here.

Theorem 1. *Let $n = 2$, $p = 3$ and assume that* (H) *is fulfilled. Then* (SDGE) *holds for the Cauchy problem* (1)–(2). *Moreover, there exists $(\widetilde{v}, \widetilde{w})$ solving*

$$\Box \widetilde{v}_i = A_i(\widetilde{w}, \partial\widetilde{w}, \partial\widetilde{v}) \qquad \text{for } 1 \le i \le M, \tag{13}$$

$$\Box \widetilde{w}_i = 0 \qquad \text{for } 1 \le i \le L \tag{14}$$

such that

$$\lim_{t\to\infty} \left(\|v(t,\cdot) - \widetilde{v}(t,\cdot)\|_E + \|w(t,\cdot) - \widetilde{w}(t,\cdot)\|_E \right) = 0,$$

where $u = (v, w)$ is the global solution to (1)–(2).

Theorem 2. *Let $n = 2$, and u be the global solution to* (11) *or* (12) *with initial data $u = \varepsilon f$ and $\partial_t u = \varepsilon g$ at $t = 0$, where $u = (u_1, u_2)$ for* (11), *and $u = (v_1, v_2, w)$ for* (12).

Then, there exist $f, g \in C_0^\infty(\mathbb{R}^2)$ and two positive constants C and ε_1 such that

$$\|u(t, \cdot)\|_E \geq C\varepsilon(1 + t)^{C\varepsilon^2}$$

for any $t \geq 0$, provided that $\varepsilon \in (0, \varepsilon_1]$.

If (AF) holds, then $\sup_{0 \leq t < \infty} \|u(t, \cdot)\|_E$ must be finite. Hence Theorem 2 shows that (AF) does not hold in general under the assumption (H), though Theorem 1 ensures (SDGE) under (H).

References

[1] S. Alinhac, *The null condition for quasi linear wave equations in two space dimensions I*, Invent. Math. **145** (2001), 597–618.

[2] ———, *The null condition for quasi linear wave equations in two space dimensions II*, Amer. J. Math. **123** (2001), 1071–1101.

[3] ———, *Semilinear hyperbolic systems with blowup at infinity*, Indiana Univ. Math. J. **55** (2006), 1209–1232.

[4] D. Christodoulou, *Global solutions of nonlinear hyperbolic equations for small initial data*, Comm. Pure Appl. Math. **39** (1986), 267–282.

[5] P. Godin, *Lifespan of semilinear wave equations in two space dimensions*, Commun. in Partial Differential Equations **18** (1993), 895–916.

[6] A. Hoshiga, *The initial value problems for quasi-linear wave equations in two space dimensions with small data*, Adv. Math. Sci. Appl. **5** (1995), 67–89.

[7] A. Hoshiga and H. Kubo, *Global small amplitude solutions of nonlinear hyperbolic systems with a critical exponent under the null condition*, SIAM J. Math. Anal. **31** (2000), 486–513.

[8] S. Katayama, *Global existence for systems of nonlinear wave equations in two space dimensions*, Publ. RIMS, Kyoto Univ. **29** (1993), 1021–1041.

[9] ———, *Global existence for systems of nonlinear wave equations in two space dimensions, II*, Publ. RIMS, Kyoto Univ. **31** (1995), 645–665.

[10] S. Katayama and H. Kubo, *Asymptotic behavior of solutions to semilinear systems of wave equations*, Indiana Univ. Math. J., to appear.

[11] S. Klainerman, *The null condition and global existence to nonlinear wave equations*, Lectures in Applied Math. **23** (1986), 293–326.

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UNCONDITIONAL UNIQUENESS OF SOLUTION FOR THE CAUCHY PROBLEM OF THE NONLINEAR SCHRÖDINGER EQUATION

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We consider the uniqueness of solution for the Cauchy problem of the nonlinear Schrödinger equation:

$$i\partial_t u + \Delta u = \lambda |u|^\alpha u, \qquad t \in [0,T], \quad x \in \mathbb{R}^n, \tag{1}$$

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}^n, \tag{2}$$

where $\lambda \in \mathbb{C}$ and $T > 0$. Let $\alpha > 0$ and $s \geq 0$ be specified later, and let $u_0 \in H^s$. Suppose that $u \in C([0,T]; H^s)$ with (2) and u satisfies equation (1) in $\mathcal{D}'((0,T) \times \mathbb{R}^n)$, that is, in the distribution sense.

We briefly recall known results on the uniqueness of solution for (1)-(2). In [3], Ginibre and Velo prove that if $s = 1$ and $\alpha < 4/(n-2)$, the solution is unique. In [2], Cazenave and Weissler show that when

$$\begin{aligned} &u \in L^\gamma(0,T; B^s_{\rho,2}), \\ &\rho = \frac{\alpha+2}{1+\alpha s/n}, \quad \gamma = \frac{4(\alpha+2)}{\alpha(n-2s)}, \end{aligned} \tag{3}$$

the solution is unique. The solution is often constructed within the framework of $C([0,T]; H^s)$ and an auxiliary space such as (3). Space (3) is associated with the Strichartz estimate and (ρ, γ) is an admissible pair of the Strichartz estimate (see, e.g., [1]). The uniqueness of solution in an auxiliary space such as (3) as well as in $C([0,T]; H^s)$ is called the conditional uniqueness, to which the result in [2] corresponds. On the other hand, the uniqueness without an auxiliary space is called the unconditional uniqueness. From now on, we refere to the unconditional uniqueness as (UU). In [6], T. Kato extensively studies the unconditional uniqueness of solution for (1)-(2) and shows the following result: Assume that any of the following three is satisfied.

(a) $n=1,\quad 0\leq s<1/2,\quad 0<\alpha\leq(1+2s)/(1-2s),$

(b) $n\geq 2,\quad 0\leq s<n/2,\quad 0<\alpha<\min\{4/(n-2s),(2+2s)/(n-2s)\},$

(c) $n\geq 1,\quad s\geq n/2.$

Then, (UU) holds. Furioli and Terraneo [4] use the Besov spaces of negative indices to improve the result by Kato [6] and show that if

$$3\leq n\leq 5,\quad 1<\alpha<\min\left\{\frac{4}{n-2s},\frac{n+2s}{n-2s},\frac{2+4s}{n-2s}\right\},$$

and additionally, for $n=3$,

$$\frac{2s}{n-2s}<\alpha\leq\frac{n+2-2s}{n-2s},$$

then (UU) holds. Furthermore, as Cazenave pointed out in [1], it follows from a variant of the proof by Kato [6] that when

$$n\geq 3,\quad 1\leq s<n/2,\quad \alpha=\frac{4}{n-2s},$$

(UU) holds.

Remark 1. (i) The unconditional uniqueness does not always make sense, because equation (1) may not make sense without an auxiliary sapce. The assumption $\alpha\leq(n+2s)/(n-2s)$ implies that $|u|^{\alpha}u\in L^1_{loc}(\mathbb{R}^n)$, which ensures that equation (3) makes sense within the framework of the distribution. Furthermore, the assumption $\alpha\leq 4/(n-2s)$ comes from the scaling invariance of equation (3). Therefore, when we consider the unconditional uniqueness, the following restriction seems natural.

$$0<\alpha\leq\min\left\{\frac{4}{n-2s},\frac{n+2s}{n-2s}\right\}. \tag{4}$$

(ii) When $1\leq s<n/2$ and $1<\alpha\leq 4/(n-2s)$, (UU) is already known (see Kato [6] and Cazenave [1]).

(iii) Even in the so-called subcritical case $\alpha < 4/(n-2s)$, the unconditonal uniqueness is not obvious (see, e.g., Kato [7], where he pointed out that if it is in $L^r(0,T;B^s_{q,2}(\mathbb{R}^n))$ for a certain addmissible pair (q,r) with sufficiently large q, the solution belongs to $L^r(0,T;B^s_{q,2}(\mathbb{R}^n))$ for all addimissible pairs (q,r) associated with the Strichartz estimate and so (UU) holds).

Accordingly, (UU) has been open in the following four cases:

$$
\begin{aligned}
&\text{(Case 1)} \qquad && n=2, \quad \alpha = \frac{n+2s}{n-2s}\left(=\frac{1+s}{1-s}\right),\\
&\text{(Case 2)} && n=3, \quad (4), \quad \alpha \geq \min\left\{\frac{2+4s}{n-2s}, \frac{n+2-2s}{n-2s}\right\}\\
& && \qquad\qquad \text{or } \frac{2+2s}{n-2s} \leq \alpha \leq 1,\\
&\text{(Case 3)} && n=4,5, \quad \max\left\{1, \frac{2+4s}{n-2s}\right\} \leq \alpha \leq \frac{4}{n-2s}\\
& && \qquad\qquad \text{or } \frac{2+2s}{n-2s} \leq \alpha \leq 1,\\
&\text{(Case 4)} && n \geq 6, \quad \frac{2+2s}{n-2s} \leq \alpha \leq \frac{4}{n-2s}.
\end{aligned}
$$

We have the following theorem concerning the unconditional uniqueness of solution for (1)-(2), which has recently been obtained in collaboration with Yin Yin Su Win, Kyoto University.

Theorem 1. *Let $0 \leq s < 1$. We assume either of the following two:*

$$
\begin{aligned}
&\text{(a)} \qquad && \text{Cases 1 and 2 except for } (n,\alpha,s) = (2,1,0), (3,2,1/2),\\
& && \qquad\qquad\qquad\qquad (3,\theta,0), \quad 2/3 < \theta \leq 1\\
&\text{(b)} && n=4,5, \quad \max\left\{\frac{2+2s}{n-2s}, 1\right\} \leq \alpha \leq \min\left\{\frac{4}{n-2s}, \frac{2+4s}{n-2s}\right\}.
\end{aligned}
$$

Then, (UU) holds for (1)-(2).

Remark 2. (i) Case (a) in Theorem 1 is divided into three subcases. When $(n,\alpha,s) = (3,\theta,0)$ and $2/3 < \theta < 1$, our proof does not work for some technical reason. In the second subcase $(n,\alpha,s) = (2,1,0)$, $(3,1,0)$, we have

$$
\alpha = \frac{n+2s}{n-2s},
$$

which implies that the nonlinearity only belongs to $L^1(\mathbb{R}^n)$. This seems to be a little more serious problem. In the third subcase $(n, \alpha, s) = (3, 2, 1/2)$, we have

$$\alpha = \frac{4}{n-2s} = \frac{n+2s}{n-2s}.$$

The last subcase seems to contain an essential difficulty.

(ii) If $\alpha < 4/(n-2s)$, that is, in the subcritical case, we can replace $u \in C([0,T]; H^s)$ by $u \in L^\infty(0,T; H^s)$. However, if $\alpha = 4/(n-2s)$, the unconditional uniqueness generally breaks down without the strong continuity in the time variable of solution (see Example 1 below).

Example 1. We consider the following L^2-critical nonlinear Schrödinger equation.

$$i\partial_t u + \Delta u = -|u|^{4/n} u, \qquad t \in \mathbb{R}, \quad x \in \mathbb{R}^n, \quad n \geq 4. \tag{5}$$

We put

$$u(t,x) = \frac{1}{(2t)^{n/2}} e^{i|x|^2/(4t)} e^{i/t} \phi\left(\frac{x}{2t}\right),$$

where ϕ is a solution of the semilinear elliptic equation associated with (5).

$$-\Delta\phi + \phi - \phi^{1+4/n} = 0, \qquad \phi > 0, \quad \phi \in H^1.$$

Then, $u(t) \in C(\mathbb{R}\backslash\{0\}; H^1)$ and $u(t) \to 0$ weakly in L^2 $(t \to 0)$. Therefore, $u \in C_w(\mathbb{R}; L^2)$ and u satisfies (1)-(2) with $u_0 = 0$. But, obviously, $u \equiv 0$ is also a solution with $u(0) = 0$.

After the list of references, we draw a figure to compare our Theorem 1 with the results by Kato [6] and Furioli and Terraneo [4] for $n = 3$.

REFERENCES

[1] T. Cazenave, *Semilinear Schrödinger Equations*, Amer. Math. Soc., Providence, 2003.

[2] T. Cazenave and F.B. Weissler, *The Cauchy problem for the critical nonlinear Schrödinger equation in H^s*, Nonlinear Analysis, TMA **14** (1990), 807–836.

[3] J. Ginibre and G. Velo, *On a class of nonlinear Schrödinger equations. I. The Cauchy problem, general case*, J. Funct. Anal. **32** (1979), 1–32.

[4] G. Furioli and E. Terraneo, *Besov spaces and unconditional well-posedness for the nonlinear Schrödinger equation in $\dot{H}^s(\mathbb{R}^n)$*, Commun. Contemp. Math. **5** (2003), 349–367.

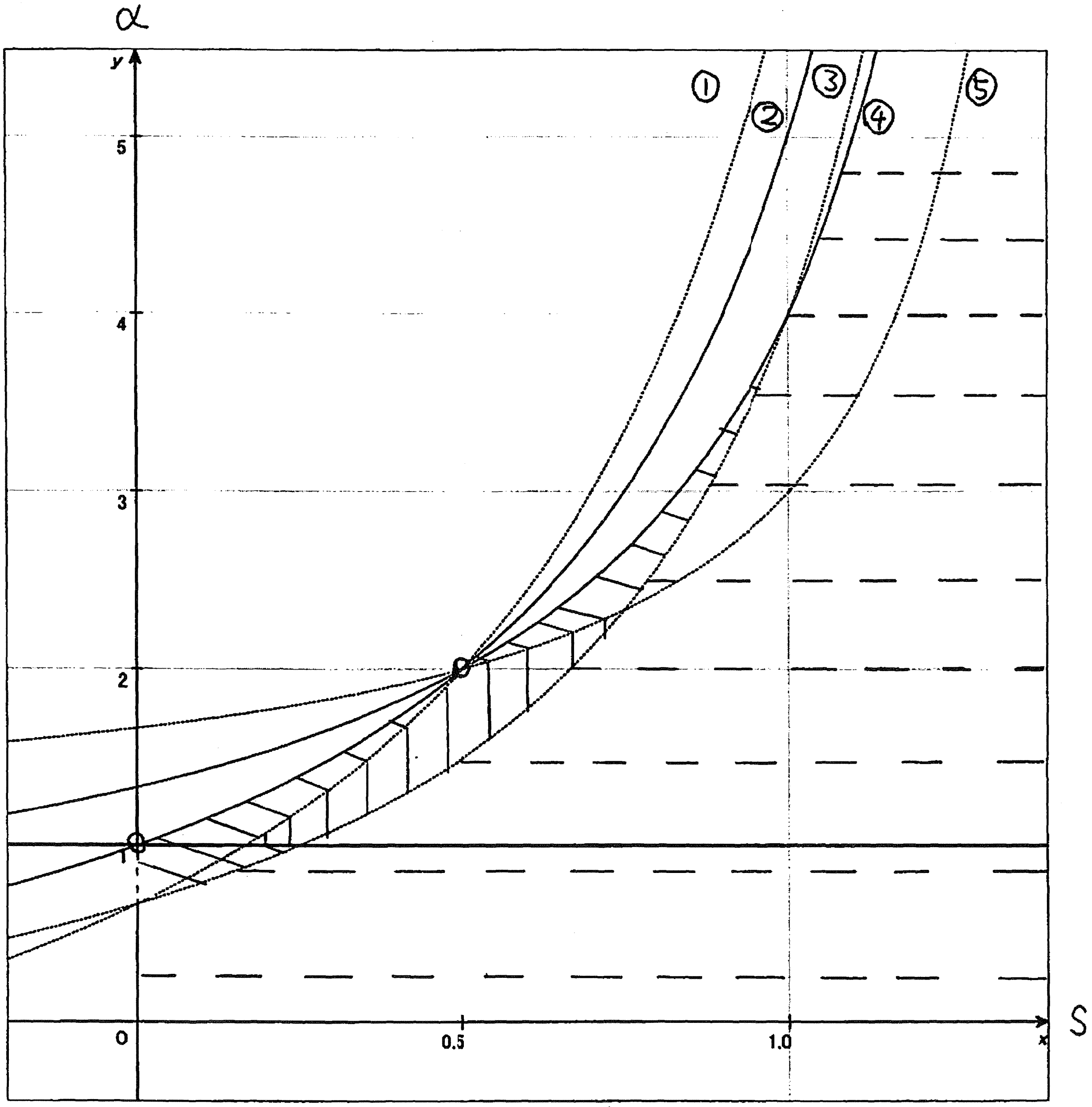
[5] G. Furioli, F. Planchon and E. Terraneo, *Unconditional well-posedness for semilinear Schrödinger and wave equations in H^s*, Contemporary Mathematics **320** (2003), 147–156.

[6] T. Kato, *On nonlinear Schrödinger equations. II. H^s-solutions and unconditional well-posedness*, J. d'Anal. Math. **67** (1995), 281–306.

[7] T. Kato, *Correction to: "On nonlinear Schrödinger equations. II. H^s-solutions and unconditional well-posedness"*, J. d'Anal. Math. **68** (1996), 305.

[8] M. Nakamura and T. Ozawa, *Nonlinear Schrödinger equations in the Sobolev space of critical order*, J. Funct. Anal. **155** (1998), 364–380.

[9] F. Planchon, *On uniqueness for semilinear wave equations*, Math. Z. **244** (2003), 587–599.



$n = 3$

① $\alpha = \dfrac{2+4s}{n-2s}$

② $\alpha = \dfrac{n+2s}{n-2s}$ $(H^s \hookrightarrow L^{\alpha+1})$

③ $\alpha = \dfrac{2+2s}{n-2s}$

④ $\alpha = \dfrac{4}{n-2s}$ (scaling critical)

⑤ $\alpha = \dfrac{n+2-2s}{n-2s}$

T. Kato [6] (dashed horizontal lines)

Furioli and Terraneo [4] (vertical lines)

Su Win and Tsutsumi (diagonal lines)

On asymptotic stability in energy space of ground states for Nonlinear Schrödinger equations

We consider even solutions of a NLS

$$iu_t + \Delta u + \beta(|u|^2)u = 0\,,\ (t,x) \in \mathbb{R}\times\mathbb{R}^D\,,\ D \geq 3 \tag{1}$$

We assume $\beta(t)$ smooth, $\beta(0)=0$ and $\limsup_{t\to+\infty} t^{-\frac{D+2}{D-2}}|\beta(t^2)t| = 0$ and assume the existence of a C^1 family $\omega \to \phi_\omega(x)$ of ground states such that $0<\phi_\omega(x) \leq ce^{-a|x|}$. We assume furthermore that these ground states are orbitally stable in $H^1(\mathbb{R})$. This means that

$$\inf\{\|u(t,x) - e^{i\gamma}\phi_\omega(x-x_0)\|_{H^1(x\in\mathbb{R}^n)} : \gamma\in\mathbb{R}\,\&\, x_0\,\in\mathbb{R}^n\}$$

is small at will for all $t\geq 0$ if the initial datum $u(0,x)$ is close enough to $\phi_\omega(x)$ in $H^1(\mathbb{R})$. An interesting open problem is whether or not the ground states are asymptotically stable, in the following sense which we state for simplicity only in the setting of even solutions of (1):

It is possible to write

$$u(t,x) = e^{i\int_0^t \omega(s)ds + i\gamma(t)}(\phi_{\omega(t)}(x) + r(t,x))$$

such that $\lim_{t\to+\infty}\omega(t)$ *exists, and*

$$r(t,x) = \zeta(t,x) + f(t,x)$$

with for fixed $a>0$ $|\zeta(t,x)| \leq C\epsilon e^{-a|x|}$ *and* $\lim_{t\to+\infty}\|\zeta(t,\cdot)\|_\infty = 0$, *and with*

$$\|f(t,x)\|_{L^p_t L^q_x} < C\epsilon$$

for some p,q . Starting from the 90's various authors have studied this problem, Buslaev, Perelman, Weinstein, Soffer, Tsai, H.T.Yau, Schlag, Krieger, Rodnianski, Sulem, Mizumachi, Nakanishi, Gustafson, Sigal, Gang Zhou, etc. All these works use the Ansatz indicated above to write an equation for the reminder,

$$\begin{aligned} ir_t &= -\Delta r + \omega(t) r - \beta(\phi^2_{\omega(t)})r - \beta'(\phi^2_{\omega(t)})\phi^2_{\omega(t)}r \\ &- \beta'(\phi^2_{\omega(t)})\phi^2_{\omega(t)}\overline{r} + \dot\gamma(t)\phi_{\omega(t)} - i\dot\omega(t)\partial_\omega\phi_{\omega(t)} + \dot\gamma(t)r + O(r^2). \end{aligned}$$

Because of $\overline{r}$ it is natural to introduce the vectors $R = \begin{bmatrix} r \\ \overline{r}\end{bmatrix}$ and $\Phi = \begin{bmatrix}\phi_\omega \\ \phi_\omega\end{bmatrix}$, and the operator

$$H_{\omega(t)} = \sigma_3\left[-\Delta + \omega(t) - \beta(\phi^2_{\omega(t)}) - \beta'(\phi^2_{\omega(t)})\phi^2_{\omega(t)}\right] + i\beta'(\phi^2_{\omega(t)})\phi^2_{\omega(t)}\sigma_2,$$

and write the above as

$$iR_t = H_{\omega(t)}R + \sigma_3\dot{\gamma}R + \sigma_3\dot{\gamma}\Phi - i\dot{\omega}\partial_\omega\Phi + O(R^2). \tag{2}$$

Buslaev & Perelman, Cuccagna proved a version of asymptotic convergence $R(t) \to 0$, picking initial data in a space much smaller than H^1, and under the assumption that the family H_ω does not have discrete spectrum except for 0. Buslaev & Perelman, Buslaev & Sulem, Cuccagna prove a similar result, allowing for a pair of eigenvalues close to the continuous spectrum. Similar results where obtained by Soffer & Weinstein, and by Tsai & Yau, in a different problem involving small ground states obtained by bifurcation from a linear operator. In the latter setting Gustafson *et al.* and later Mizumachi proved some asymptotic stability results in energy space. When H_ω has eigenvalues other than 0, system (2) splits in a coupled system involving ODE's for the discrete modes in R and PDE's the continuous modes in R. Even though the spectrum is in the imaginary axis, the continuous spectrum acts like the stable part of the system while the discrete modes give what looks like a center manifold. It is not possible to tell from linear theory whether in the center manifold one has asymptotic convergence to the equilibrium or there are invariant tori. Various papers described how, in the presence of eigenvalues close to the continuous spectrum, the interaction of the discrete modes with the continuous modes produces leaking of energy from the discrete to the continuous, and hence asymptotic stability. However, until 2006, the general case looked like a serious problem. Last year Sigal & Gang Zhou envisaged the correct normal form expansion to treat the case when there are eigenvalues not close to the continuous spectrum. In this talk we present the result by Cuccagna & Mizumachi, with some simplifications and strengthening of the result by Sigal & Zhou. In particular we show that, if the discrete spectrum consists of exactly three eigenvalues, in the absence of translation invariance, generically orbital stability of ground states leads to asymptotic stability in energy space. Furthermore, in the presence of more eigenvalues, the same is true, assuming the so called Fermi Golden Rule hypothesis. While the validity of this hypothesis is still an open problem, it is likely that a careful analysis of the normal form expansion will prove the FGR to be correct. Notice that we have excluded translation. We hope that with an intelligent choice of gauge this limitation might be resolved

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Asymptotic lower bound for solutions to a class of Schrödinger and wave equations

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We shall present some results obtained in collaboration with Luis Vega and connected with the so called local smoothing for solutions to a class of Schrödinger equations. More precisely we shall look for a–priori lower bound of solutions to

$$\mathbf{i}\partial_t u - \Delta u + V(x)u = 0$$
$$u(0) = f,$$

of the following type:

$$\lim_{R\to\infty} \frac{1}{R}\int_{-\infty}^{\infty}\int_{|x|<R} |\partial_{|x|}u|^2 dxdt = \|f\|^2_{\dot{H}_V^{\frac{1}{2}}(\mathbf{R}^n)}, \tag{1}$$

where $\dot{H}_V^{\frac{1}{2}}(\mathbf{R}^n)$ denotes the perturbed Sobolev space of order $\frac{1}{2}$.

In fact we shall deduce such a type of asymptotic estimates from the following identities:

$$\int_{\mathbf{R}}\int_{\mathbf{R}^n} \left[\nabla_x \bar{u} D^2\psi \nabla_x u - (\Delta^2\psi + 4\partial_{|x|}V\partial_{|x|}\psi)\frac{|u|^2}{4}\right] dxdt \tag{2}$$
$$= \psi'(\infty)\|f\|^2_{\dot{H}_V^{\frac{1}{2}}(\mathbf{R}^n)},$$

where ψ is a radially symmetric function and $\lim_{|x|\to\infty}\partial_{|x|}\psi = \psi'(\infty) \in [0,\infty)$.

Notice that (2) are a precised version of the classical Morawetz estimates (simply choose $\psi \equiv |x|$ in (2)). In fact the main point in (2) is that they are identities and not inequalities.

We shall present also some extensions of those results for solutions to the conformally invariant NLS with small initial data in L^2. As a by–product we shall deduce that the $\dot{H}^{\frac{1}{2}}$–norm of solutions to NLS is essentially constant in time, provided that the initial data are regular enough.

We shall also present some related results for critical NLW:

$$\Box u + \lambda u|u|^{2^*-2} = 0,\ \lambda \geq 0 \tag{3}$$
$$u(0) = f \in \dot{H}^1, \partial_t u(0) = g \in L^2.$$

More precisely we shall show that the solutions to (3) satisfy the following Morawetz type identities:

$$\int_{\mathbf{R}}\int_{\mathbf{R}^n} (\nabla_x u D^2\psi\nabla_x u - \frac{1}{4}|u|^2\Delta^2\psi) + \frac{\lambda}{n}|u|^{2^*}\Delta\psi\ dxdt$$
$$= \psi'(\infty)\left(\int_{\mathbf{R}^n} |\nabla_x f|^2 + \frac{2\lambda}{2^*}|f|^{2^*} + |g|^2\ dx\right),$$

where

$$\lim_{|x|\to\infty}\partial_{|x|}\psi = \psi'(\infty)$$

and ψ is a general radially symmetric function. In particular, by choosing carefully the function ψ, one can deduce the following asymptotic identities for solutions to (3):

$$\lim_{R\to\infty}\frac{1}{R}\int_{\mathbf{R}}\int_{|x|<R} |\partial_{|x|}u|^2\ dxdt = \int_{\mathbf{R}^n} |\nabla_x f|^2 + \frac{2\lambda}{2^*}|f|^{2^*} + |g|^2\ dx.$$

On small amplitude global solutions for the nonlinear Klein-Gordon equation

Jun Kato (Nagoya University)

This talk is based on the joint work with Tohru Ozawa (Hokkaido University). In this talk, we consider the Cauchy problem of the quadratic nonlinear Klein-Gordon equation in two space dimensions,

$$\partial_t^2 u - \Delta u + u = Q(u, \partial u), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^2, \tag{1}$$

$$u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x), \quad x \in \mathbb{R}, \tag{2}$$

where $\partial = (\partial_t, \partial_1, \partial_2)$, Q is the quadratic nonlinearity in u and ∂u.

There are many studies concerning the global existence and asymptotic behavior of solutions for nonlinear Klein-Gordon equations. Let n be the spatial dimensions. When $n \geq 5$, Klainerman-Ponce [5] and Shatah [8] showed that the Cauchy problem (1)-(2) has the unique global solution for small initial data and the solution asymptotically approaches to the free solution of the linear Klein-Gordon equation as $t \to \infty$. The proofs in [5] and [8] are based on the $L^p - L^q$ estimate of the solution to the linear Klein-Gordon equation.

When $n \leq 4$, the $L^p - L^q$ estimate does not provide us a sufficient time decay to construct global solutions. To overcome this difficulty, Klainerman [4] introduced the invariant Sobolev space with respect to the generators of the Lorentz group and showed the existence of global solution to (1)-(2) when $n = 3, 4$. Independently, Shatah [9] introduced the method of the normal forms, which is the extension of the Poincaré's theory of normal forms for the ordinary differential equations to the nonlinear Klein-Gordon equations, and showed the existence of global solution to (1)-(2) when $n = 3$, 4.

When $n = 2$, Georgiev-Popivanov [2] and Kosecki [3] showed the existence of global solution provided that the nonlinearity in (1) is the special form. The general nonlinearities are treated by Simon-Taflin [10] and Ozawa-Tsutaya-Tsutsumi [7]. In particular, the proof in [7] is based on the method of normal forms and the decay estimate due to Georgiev [1], and requires the following conditions on the initial data other than the smallness,

$$u_0 \in H^{k+16,k+15}(\mathbb{R}^2), \quad u_1 \in H^{k+15,k+15}(\mathbb{R}^2), \quad k \geq 21, \tag{3}$$

where $H^{m,s}$ denotes the weighted Sobolev space whose norm is defined by

$$\|f\|_{H^{m,s}} = \|(1+|x|^2)^{s/2}(1-\Delta)^{m/2} f\|_{L^2}.$$

The purpose of this talk is to give a simple proof and to relax the condition of the initial data for the existence of global solutions to (1)-(2) by using the endpoint Strichartz estimates in mixed norms on the polar coordinates. Such estimates for the wave and the Klein-Gordon equation in three space dimensions are introduced by Machihara-Nakamura-Nakanishi-Ozawa in [6].

References

[1] V. Georgiev, Decay estimates for the Klein-Gordon equation, Comm. Partial Differential Equations **17** (1992), 1111–1139.

[2] V. Georgiev, P, Popivanov, Global solution to the two-dimensional Klein-Gordon equation, Comm. Partial Differential Equations **16** (1991), 941–995.

[3] R. Kosecki, The unit condition and global existence for a class of nonlinear Klein-Gordon equations, J. Differential Equations **100** (1992), 257–268.

[4] S. Klainerman, Global existence of small amplitude solutions to nonlinear Klein-Gordon equations in four space-time dimensions, Comm. Pure Appl. Math. **38** (1985), 631–641.

[5] S. Klainerman, G. Ponce, Global, small amplitude solutions to nonlinear evolution equations, Comm. Pure Appl. Math. **36** (1983),133–141.

[6] S. Machihara, M. Nakamura, K. Nakanishi, T. Ozawa, Endpoint Strichartz estimates and global solutions for the nonlinear Dirac equation, J. Funct. Anal. **219** (2005), 1–20.

[7] T. Ozawa, K. Tsutaya, Y. Tsutsumi, Global existence and asymptotic behavior of solutions for the Klein-Gordon equations with quadratic nonlinearity in two space dimensions, Math. Z. **222** (1996), 341–362.

[8] J. Shatah, Global existence of small solutions to nonlinear evolution equations, J. Differential Equations **46** (1982), 409–425.

[9] J. Shatah, Normal forms and quadratic nonlinear Klein-Gordon equations, Comm. Pure Appl. Math. **38** (1985), 685–696.

[10] J. C. H. Simon, E. Taflin, The Cauchy problem for nonlinear Klein-Gordon equations, Comm. Math. Phys, **152** (1993), 433–478.

The existence of the global solutions to semilinear wave equations with cubic nonlinearities in 2-dimensional space

Akira HOSHIGA (Shizuoka University)

1 Introduction

Let us consider the following Cauchy problem:

$$\Box u = \partial_t^2 u - \triangle u = F(\partial u) \qquad (x,t) \in \mathbf{R}^2 \times (0,\infty) \tag{1.1}$$

$$u(x,0) = \varepsilon f(x), \quad \partial_t u(x,0) = \varepsilon g(x) \qquad x \in \mathbf{R}^2. \tag{1.2}$$

Here, $\partial = (\partial_0, \partial_1, \partial_2)$, $\partial_0 = \partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$ $(j = 1, 2)$, ε is a positive small parameter. We assume $f, g \in C_0^\infty(\mathbf{R}^2)$, $|f| + |g| \not\equiv 0$ and $\mathrm{supp}\{f, g\} \subset \{x \in \mathbf{R}^2 \ ; \ |x| \le M\}$. We also assume that

$$F \in C^\infty(\mathbf{R}^3)$$

$$F(\partial u) = O(|\partial u|^3) \qquad \text{near} \quad \partial u = 0.$$

More presicely, we assume

$$F(\partial u) = \sum_{\alpha,\beta,\gamma=0}^{3} A^{\alpha\beta\gamma} \partial_\alpha u \partial_\beta u \partial_\gamma u + O(|\partial u|^4) \qquad \text{near} \quad \partial u = 0,$$

where $A^{\alpha\beta\gamma}$ are constants.

The aim of this talk is to estimate the lifespan T_ε of the smooth solution to the Cauchy problem (1.1), (1.2), which is defined as follows;

$$T_\varepsilon = \sup\{\ T > 0 \ ; \ \text{there exists a smooth solution to (1.1), (1.2) in } C^\infty(\mathbf{R}^2 \times [0,T))\ \}$$

for each ε. In order to state the results which we have already known about the lifespan, we introduce some notations.

For vectors $X = (X_0, X_1, X_2) \in \mathbf{R}^3$, we define

$$C(X) = \sum_{\alpha,\beta,\gamma=0}^{2} A^{\alpha\beta\gamma} X_\alpha X_\beta X_\gamma,$$

which expresses the coefficient of the essential term of $F(\partial u)$.

On the other hand, let $u^0 = u^0(x,t)$ be the solution to the Cauchy problem;

$$\Box u^0 = 0$$
$$u^0(x,0) = f(x), \quad \partial_t u^0(x,0) = g(x)$$

and set $r = |x| \geq 0$, $\omega = \dfrac{x}{r} \in S^1$ and $\rho = r - t \in \mathbf{R}$. Then we define

$$\mathcal{F}(\omega,\rho) = \lim_{r\to\infty} r^{\frac{1}{2}} u^0(r\omega, \rho + r),$$

which is called the Friedlander radiation field. Hörmander showed in [2] the following properties of $\mathcal{F}$.

$$|\partial_\rho^k \mathcal{F}(\omega,\rho)| \leq C(1+|\rho|)^{-\frac{1}{2}-k} \qquad \rho \in \mathbf{R} \tag{1.3}$$
$$\mathcal{F}(\omega,\rho) = 0 \quad \text{for} \quad \rho \geq M. \tag{1.4}$$

By (1.3) and (1.4), we find that the constant

$$H = \max_{\rho\in\mathbf{R},\ \omega\in S^1} \left\{ -\frac{1}{2} C(-1,\omega)(\partial_\rho \mathcal{F}(\omega,\rho))^2 \right\} \tag{1.5}$$

is well-defined and nonnegative.

Then Godin proved in [1] the following.

[Known results] (a) If $H > 0$, then

$$\liminf_{\varepsilon\to+0} \varepsilon^2 \log T_\varepsilon \geq \frac{1}{H}$$

holds.

(b) If $C(-1,\omega) \equiv 0$ holds for $\omega \in S^1$, then $T_\varepsilon = \infty$ holds for sufficiently small $\varepsilon > 0$.

The condition $C(-1,\omega) \equiv 0$ is called "null-condition." The only function satisfying the null-condition is

$$F(\partial u) = C((\partial_0 u)^2 - |\nabla u|^2).$$

It follows from the definition of H that the null-condition is a sufficient condition of $H = 0$ and not a necessary condition. In the case where $|f| + |g| \not\equiv 0$, we find that $H = 0$ is equivalent to the condition

$$C(-1,\omega) \geq 0 \qquad \text{for any} \quad \omega \in S^1. \tag{1.6}$$

This means that there is a nonlinearity F which does not satisfy the both assumption of (a) and (b). Thus it is usefull to consider how long the solutions to (1.1) and (1.2) exists, when the condition (1.6) holds.

2 Statement of the main theorem

We introduce generalaize differential operators;

$$\Omega = x_1\partial_2 - x_2\partial_1, \quad L_i = t\partial_i + x_i\partial_0, \quad S = t\partial_0 + x_1\partial_1 + x_2\partial_2$$

and denote

$$\Gamma_0 = \partial_0, \quad \Gamma_1 = \partial_1, \quad \Gamma_2 = \partial_2, \quad \Gamma_3 = \Omega, \quad \Gamma_4 = L_1, \quad \Gamma_5 = L_2, \quad \Gamma_6 = S.$$

We also write $\Gamma^a u = \Gamma_0^{a_0}\Gamma_1^{a_1}\cdots\Gamma_6^{a_6}u$ for a multi-index $a = (a_0, a_1, \ldots, a_6)$.
Next we define some generalized Sobolev norms as

$$\begin{aligned} |v(t)|_k &= \sum_{|a|\le k} ||\Gamma^a v(\cdot,t)||_{L^\infty_x(\mathbf{R}^2)} \\ [v(t)]_k &= \sum_{|a|\le k} ||(1+|\cdot|+t)^{\frac{1}{2}}(1+||\cdot|-t|)^{\frac{1}{2}}\Gamma^a v(\cdot,t)||_{L^\infty_x(\mathbf{R}^2)} \\ ||v(t)||_k &= \sum_{|a|\le k} ||\Gamma^a v(\cdot,t)||_{L^2_x(\mathbf{R}^2)} \end{aligned}$$

Now we state the main theorem.

Theorem 2.1 *Assume that (1.6) holds. Then there exists a constant* $\varepsilon_0 > 0$ *such that* $T_\varepsilon = \infty$ *holds for* $\varepsilon \in (0, \varepsilon_0)$. *Moreover, the solution satisfies*

$$[\partial u(t)]_k \le C_k\varepsilon(1+\varepsilon^2\log t)^{M_k} \tag{2.1}$$

for some constants $M_k > 0$ *which increases monotonely with respect to* k.

(Remark 1) The decay estimate (2.1) means that the solution to (1.1) and (1.2) might decay slower than solutions of linear wave eqations.
(Remark 2) Kubo proved the same result in [3] in which he also showed an asymptotic behavior of solutions.

References

[1] P. Godin, *Lifespan of solutions of semilinear wave equations in two space dimensions*, Comm. in P. D. E., **18** (5 and 6), pp. 895-916, (1993).

[2] L. Hörmander, *The lifespan of classical solutions of nonlinear hyperbolic equations*, Lecture Note in Math., **1256**, pp. 214-280, (1987).

[3] H. Kubo, *Large time behavior of solutions to semilinear wave equations with dissipative structure*, preprint

Bilinear Strichartz estimates and applications to 2D NLS*

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1 Introduction

We consider the wellposedness for the Cauchy problem of the 2D defocusing NLS with cubic nonlinearity:

$$\begin{cases} i\partial_t u + \Delta u = |u|u, & (t,x) \in \mathbb{R}^{1+2} \\ u|_{t=0} = u_0. \end{cases} \tag{1.1}$$

We can summarize the known results on H^s-local wellposedness as follows. We have local wellposedness in H^s for all $s \geq 0$, and if $s > 0$ then the time interval of existence of solution can be obtained in term of H^s-norm of the initial data and the solution-map is uniformly continuous (indeed, analytic). For $s < 0$, the solution-map is not uniformly continuous. The critical regularity with respect to the scale invariance is $s = 0$ (it is also a Galilean invariant regularity).

We immediately have global existence of solution for the initial date in H^1 by using the conservation laws: *Energy*

$$E(u) = \int_{\mathbb{R}^2} \frac{1}{2}|\nabla u|^2 + \frac{1}{4}|u|\,dx, \tag{1.2}$$

and *Mass*

$$\int_{\mathbb{R}^2} |u|^2\,dx.$$

For rough data $0 < s < 1$, the energy is infinite, and the use of conservation law is meaningless. The pioneer work exploiting the apriori estimate was by Bourgain [1] who obtained the apriori estimate for $s > 3/5$, involving polynomial growth of H^s-norm. In [5], the *I-method* was used to establish the apriori estimate for $s > 4/7$.

We prove the following theorem under weaker regularity condition on s.

Theorem 1 *The Cauchy problem (1.1) is globally wellposed in H^s for all $s > 1/2$.*

*Joint work with J. Colliander, M. Keel, G. Staffilani, T. Tao, and based on the paper [6].

For the proof of Theorem 1, we use, with a *resonant* correction term to (1.2), the following quantity:

$$\widetilde{E}(u) = \Lambda_2(\sigma_2; u) + \Lambda_4(\sigma_4; u)$$

where

$$\Lambda_k(\sigma; u) = \int_{\xi_1+\ldots+\xi_k=0} \sigma(\xi_1, \ldots, \xi_k)\widehat{u}(\xi_1)\ldots\widehat{\overline{u}}(\xi_k).$$

$$\sigma_2 = -\frac{1}{2}\xi_1 m_1 \xi_2 m_2$$

$$\sigma_4 = \frac{|\xi_1|^2 m_1^2 - |\xi_2|^2 m_2^2 + |\xi_3|^2 m_3^2 - |\xi_4|^2 m_4^2}{4(|\xi_1|^2 - |\xi_2|^2 + |\xi_3|^2 - |\xi_4|^2)} 1_{\Omega_{nr}}$$

$$\Omega_{nr} = \{(\xi_1, \ldots, \xi_4) : \max|\xi_k| \le N \text{ or } |\cos\angle(\xi_1 + \xi + 2, \xi + 1 + \xi_4)| \ge \theta > 0\}$$

and $m_k = m(\xi_k)$ is a smooth non-negative radial function which takes the value 1 for $|\xi| \le N$, and takes $(|\xi|/N)^{s-1}$ for $|\xi| \ge 2N$.

The set Ω_{nr} appears in the resonance condition for nonlinear interaction (more precisely when ξ_{12} and ξ_{14} are almost orthogonal). Exploiting the presence of resonance condition is our improvement of the previous work in [5]. The interest of the resonance condition lies in the following estimate.

Theorem 2 *(Angularly refined bilinear Strichartz estimate) Let* $0 < N_1 < N_2$ *and* $\theta \in (0, 1/100)$. *For* $\phi_1, \phi_2 \in L^2$ *with Fourier frequencies* N_1, N_2, *respectively, we have*

$$\|F\|_{L^2_{t,x}} \lesssim \min\left\{\theta, \frac{N_1}{N_2}\right\}^{1/2} \|\phi_1\|_{L^2}\|\phi_2\|_{L^2},$$

where

$$F(t,x) = \int_{\xi_1+\xi_2=0} e^{ix(\xi_1+\xi_2)} 1_{|\cos\angle(\xi_1,\xi_2)| \le \theta} \widehat{S(t)\phi_1}(\xi_1)\widehat{S(t)\phi_2}(\xi_2).$$

Remark 1 The above estimate without angularly constrained was already obtained in [1].

Remark 2 Fang and Grillakis obtained the global wellposedness for $s \ge 1/2$ by using the interaction Morawetz estimate [7], and Colliander, Grillakis and Tzirakis improved this for $s > 2/5$. For radial data, Kllip, Tao and Visan recently obtained global wellposedness and scattering for all $s \ge 0$. But Theorem 1 (in particular Theorem 2) seems interesting, because the above angularly constrained Strichatz estimate in conjunction with [3, 7] may improve the global wellposedness for $s > 4/13$ without radial condition on the initial data.

References

[1] J. Bourgain, *Refinement of Strichartz inequality and applications to 2D-NLS with critical nonlinearity*, Internat. Math. Res. Notices, 5 (1998), 253–283.

[2] J. Bourgain, *Global solutions of nonlinear Schrödinger equations*, AMS Colloquium Publications, 46 (1999).

[3] J. Colliander, M. Grillakis, N. Tzirakis, *Improved interaction Morawetz inequalities for the cubic nonlinear Schrödinger equation on* $\mathbb{R}^2$, preprint (2007).

[4] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, *Sharp Global well-posedness of KdV and modified KdV on the* $\mathbb{R}$ *and* $\mathbb{T}$, J. Amer. Math. Soc., 16 (2003), 705–749.

[5] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, *Almost conservation laws and global rough solutions to a Nonlinear Schrödinger Equation*, Math. Res. Letters, 9 (2002), 1 - 24.

[6] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, *Resonant decompositions and the I-method for cubic nonlinear Schrödinger on* $\mathbb{R}^2$, preprint (2007).

[7] M. Grillakis, Y. Fang, *On the global existence of rough solutions of the cubic defocusing Schrödinger equation in* $\mathbb{R}^{2+1}$, to appear in J.H.D.E., 2007.

[8] R. Kllip, T. Tao, M. Visan, *The cubic nonlinear Schrödinger equation in two dimensions with radial data*, preprint, 2007.

Energy scattering for 2D nonlinear Klein-Gordon equation in the critical case

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This is joint work with S. Ibrahim, M. Majdoub and N. Masmoudi. We study global behavior of solutions to the nonlinear Klein-Gordon equation on $\mathbb{R}^{1+2}$:

$$\ddot{u} - \Delta u + u + f(u) = 0, \quad u : \mathbb{R}^{1+2} \to \mathbb{R}, \tag{1}$$

with the nonlinearity f of the Sobolev critical growth order for $|u| \to \infty$:

$$f(u) = (e^{4\pi u^2} - 1 - 4\pi u^2)u, \tag{2}$$

where we subtract the cubic term in order to avoid another critical problem at $|u| \to 0$. The equation has the conserved energy

$$E(u) = E_0(u;t) + F(u(t)),$$

$$E_0(u;t) := \int_{\mathbb{R}^2} |\dot{u}(t)|^2 + |\nabla u(t)|^2 + |u(t)|^2 dx, \quad F(\varphi) = \int_{\mathbb{R}^2} \sum_{k\geq 3} \frac{(4\pi\varphi^2)^k}{2\pi k!} dx. \tag{3}$$

By using the Trudinger-Moser type inequality, we deduce that

$$\|\varphi\|_{H^1}^2 := \int_{\mathbb{R}^2} |\nabla\varphi|^2 + |\varphi|^2 dx \leq 1 \implies F(\varphi) < \infty, \tag{4}$$

but otherwise $F(\varphi)$ can be infinite. In this sense we can call the case $E(u) = 1$ critical, and $E(u) < 1$ subcritical. Global wellposedness in those cases has been established by [2] in the class

$$u \in C(\mathbb{R}; H^1) \cap L^4_{loc}(\mathbb{R}; C^{1/4}), \quad \dot{u} \in C(\mathbb{R}; L^2), \tag{5}$$

where $C^{1/4}$ denotes the Hölder space. The main result in this talk is the following scattering together with global integrability:

Theorem 1. *For any solution u of (1) satisfying $E(u) \leq 1$ and (5), we have $u \in L^4(\mathbb{R}; C^{1/4})$ and there exist free Klein-Gordon solutions $u_\pm$ such that*

$$E_0(u - u_\pm; t) \to 0 \quad (t \to \pm\infty). \tag{6}$$

Moreover, the maps $(u(0), \dot{u}(0)) \mapsto (u_\pm(0), \dot{u}_\pm(0))$ are homeomorphisms from $B_N = \{(\varphi, \psi) \in H^1 \times L^2 \mid \|\varphi\|_{H^1}^2 + \|\psi\|_{L^2}^2 + F(\varphi) \leq 1\}$ onto $B_L = \{(\varphi, \psi) \in H^1 \times L^2 \mid \|\varphi\|_{H^1}^2 + \|\psi\|_{L^2}^2 \leq 1\}$.

In particular, we have the scattering operator defined as a norm-preserving homeomorphism of the closed unit ball in $H^1 \times L^2$. For the power nonlinearity $f(u) = |u|^{p-1}u$, the same scattering result holds in the whole energy space, and has been proved for all $p > 3$ (subcritical) in [4]. In the higher dimensions $\mathbb{R}^n$ ($n \geq 3$), the subcritical case $1 + 4/n < p < 1 + 4/(n-2)$ had been solved in [1], and the Sobolev critical case $p = 1 + 4/(n-2)$ in [5]. (The case $n = 1$ was also solved in [4] for $p > 5$.) Those large data results are based on highly nontrivial collaborations of local estimates using the Strichartz, Sobolev and interpolation inequalities on the one hand, and global nonlinear arguments relying on the Morawetz-type estimates and finite propagation property on the other hand.

The essential difference in the subcritical case between the power and the exponential nonlinearities appears only in the local estimates. To treat the solutions with energy close to the threshold, we need some log-Sobolev inequality to interpolate the Trudinger-Moser and the Strichartz inequalities. We modify the argument in [2] to get estimates for the Duhamel formula independent of the time interval, where a key ingredient is the uniform local H^1 norm:

$$\|\varphi\|_{H^1[R]} = \sup_{c\in\mathbb{R}^2} \int_{|x-c|<R} |\nabla\varphi(x)|^2 + |\varphi(x)|^2 dx. \tag{7}$$

However in the critical case $E(u) = 1$, we encounter an essentially new difficulty: we cannot get local estimates independent of the time interval. More precisely, in the power case we have the estimates of the form

$$\left\| \int_{t_0}^{t} \frac{\sin((t-s)\sqrt{1-\Delta})}{\sqrt{1-\Delta}} f(u(s))ds \right\|_{L^\infty H^1 \cap X} \leq C(\|u\|_{L^\infty H^1}, \|u\|_X), \tag{8}$$

for some appropriate space-time norm X and on any time interval containing t_0, with a function C which depends only on n, p and X; whereas in the exponential critical case $E(u) = 1$, the constant C has to depend on how much the energy concentrate, specifically on the upper bound of the local norm $H^1[R]$ for some $R > 4$. This forces us to track the degree of concentration in terms of the concentration radius

$$r_\varepsilon(t) = \inf\{r > 0 \mid \exists c \in \mathbb{R}^2, \int_{|x-c|<r} e(u;t)dx < 1 - \varepsilon\}, \tag{9}$$

where $e(u;t)$ denotes the energy density. We classify its global behavior into the following three cases: the dispersive case where we have

$$\exists\varepsilon \in (0,1),\ \liminf_{t\to\infty} r_\varepsilon(t) > 6, \tag{10}$$

the stationary case where we have

$$\forall\varepsilon \in (0,1),\ \limsup_{t\to\infty} r_\varepsilon(t) < \infty, \tag{11}$$

and the waving concentration case where neither of them holds. Applying nonlinear global estimates to each case in different ways, we finally conclude that the solution scatters anyway.

The above result can be easily adapted to the nonlinear Schrödinger equations (NLS) in the subcritical case (cf. [3] for small data case), but we are not sure about the critical case , where the argument becomes much more delicate, especially relying on the finite propagation property which does not strictly hold for NLS.

References

[1] J. Ginibre and G. Velo, *Time decay of finite energy solutions of the non linear Klein-Gordon and Schrödinger equations*, Ann. Inst. Henri. Poincaré, **43** (1985), 399–442.

[2] S. Ibrahim, M. Majdoub and N. Masmoudi, *Global solutions for a semilinear, two-dimensional Klein-Gordon equation with exponential-type nonlinearity.* Comm. Pure Appl. Math. **59** (2006), no. 11, 1639–1658.

[3] M. Nakamura and T. Ozawa, *Nonlinear Schrödinger equations in the Sobolev space of critical order.* J. Funct. Anal. **155** (1998), no. 2, 364–380.

[4] K. Nakanishi, *Energy scattering for nonlinear Klein-Gordon and Schrödinger equations in spatial dimensions* 1 *and* 2. J. Funct. Anal. **169** (1999), no. 1, 201–225.

[5] K. Nakanishi, *Scattering theory for the nonlinear Klein-Gordon equation with Sobolev critical power.* Internat. Math. Res. Notices **1999**, no. 1, 31–60.

Stability of ground states for nonlinear Schrödinger equations with nonlocal interaction

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This talk is based on a joint work with Hiroaki Kikuchi (Kyoto University). We consider the following nonlinear Schrödinger equation

$$i\partial_t u = -\Delta u - (W_m * |u|^2)u \quad (t,x) \in \mathbb{R} \times \mathbb{R}^3, \tag{1}$$

where $*$ denotes the convolution in $\mathbb{R}^3$, $m \geq 0$ is a constant, and

$$W_m(x) = \frac{e^{-m|x|}}{2\pi|x|}$$

is the Yukawa potential. Note that $W_m * |u|^2 = 2(-\Delta + m^2)^{-1}|u|^2$, and (1) is a simplification of the coupled Klein-Gordon-Schrödinger equations

$$\begin{cases} i\partial_t u + \Delta u = -2uv, \\ \partial_t^2 v - \Delta v + m^2 v = |u|^2. \end{cases} \tag{2}$$

In this talk, we consider the single equation (1) instead of the system (2), just for simplicity. We study the orbital stability of standing wave solutions $u(t,x) = e^{i\omega t}\psi_\omega(x)$ of (1), where $\omega > 0$ is a parameter, and ψ_ω is a ground state of the stationary problem

$$-\Delta\psi + \omega\psi - (W_m * |\psi|^2)\psi = 0, \quad x \in \mathbb{R}^3. \tag{3}$$

We use the following notation.

$$E(v) = \frac{1}{2}\|\nabla v\|_{L^2}^2 - \frac{1}{4}\iint_{\mathbb{R}^3\times\mathbb{R}^3} W_m(x-y)|v(x)|^2|v(y)|^2\,dx\,dy,$$
$$S_\omega(v) = E(v) + \frac{\omega}{2}\|v\|_{L^2}^2,$$
$$\mathcal{A}_\omega = \{v \in H^1(\mathbb{R}^3) : S'_\omega(v) = 0,\ v \neq 0\}.$$

Then, the set of ground states of (3) is defined by

$$\mathcal{G}_\omega = \{w \in \mathcal{A}_\omega : S_\omega(w) \le S_\omega(v),\ \forall v \in \mathcal{A}_\omega\}.$$

Note that the Cauchy problem for (1) is globally well-posed in $H^1(\mathbb{R}^3)$, and the energy $E(u)$ and $\|u\|_{L^2}^2$ are conserved quantities of (1) (see [2]).

Definition (i) For $\Omega \subset H^1(\mathbb{R}^3)$, we say that the set Ω is stable for (1) if for any $\varepsilon > 0$ there exsits $\delta > 0$ such that if $u_0 \in H^1(\mathbb{R}^3)$ and $\mathrm{dist}(u_0, \Omega) < \delta$ then the solution $u(t)$ of (1) with $u(0) = u_0$ satisfies $\mathrm{dist}(u(t), \Omega) < \varepsilon$ for all $t \in \mathbb{R}$. Here, $\mathrm{dist}(v, \Omega) = \inf\{\|v - w\|_{H^1} : w \in \Omega\}$.
(ii) For $\varphi \in \mathcal{A}_\omega$, we say that $e^{i\omega t}\varphi$ is stable if $\{e^{i\theta}\varphi(\cdot + y) : \theta \in \mathbb{R},\ y \in \mathbb{R}^3\}$ is stable, and that $e^{i\omega t}\varphi$ is unstable if $e^{i\omega t}\varphi$ is not stable.

When $m = 0$, Cazenave and Lions [1] proved that for any $\omega > 0$ and for $\varphi_\omega \in \mathcal{G}_\omega$, $e^{i\omega t}\psi_\omega$ is stable for (1). However, little is known for the case $m > 0$ (see [3] for a partial result). We now state our main results.

Theorem 1 *Let $m > 0$, $\omega > 0$ and $\psi_\omega \in \mathcal{G}_\omega$. Then, there exists $\omega_1 > 0$ such that $e^{i\omega t}\psi_\omega$ is stable for (1) for any $\omega \in (\omega_1, \infty)$.*

Theorem 2 *Let $m > 0$, $\omega > 0$ and $\psi_\omega \in \mathcal{G}_\omega$. Then, there exists $\omega_2 > 0$ such that $e^{i\omega t}\psi_\omega$ is unstable for (1) for any $\omega \in (0, \omega_2)$.*

Theorem 3 *Let $m > 0$ and $\varphi_m \in H^1(\mathbb{R}^3)$ is a unique positive solution of*

$$-\Delta\varphi + m^2\varphi - \varphi^2 = 0, \quad x \in \mathbb{R}^3.$$

Then, the standing wave solution $e^{im^2t}(1/\sqrt{2})\varphi_m$ of (1) is stable.

Remark We do not know whether $(1/\sqrt{2})\varphi_m \in \mathcal{G}_{m^2}$ or not.

References

[1] T. Cazenave and P. L. Lions, *Orbital stability of standing waves for some nonlinear Schrödinger equations*, Comm. Math. Phys. **85** (1982), 549–561.

[2] J. Ginibre and G. Velo, *On a class of nonlinear Schrödinger equations with nonlocal interaction*, Math. Z. **170** (1980), 109–136.

[3] M. Ohta, *Stability of stationary states for the coupled Klein-Gordon-Schrödinger equations*, Nonlinear Anal. **27** (1996), 455-461.

ON THE SEMIRELATIVISTIC EQUATIONS

YONGGEUN CHO

In this note, we consider an equation describing a boson star which is a quantum system of boson particles with high speed and potential of gravitational or electric type. Let u be a one-particle wave function of boson star which is a complex-valued function of $(t,x) \in [0,T] \times \mathbb{R}^n, n \geq 1$ and $V_\gamma(x) = \lambda|x|^{-\gamma}, 0 < \gamma < n, \lambda \in \mathbb{R}$ be the interaction potential. Then it satisfies the equation:

$$
\begin{aligned}
&i\partial_t u = \sqrt{m^2 - \Delta}\, u + F(u),\\
&u(0,x) = \varphi(x), \quad F(u) = (V_\gamma * |u|^2)u.
\end{aligned} \tag{1}
$$

Here m is the mass of particle. We assumed that the Planck constant $\hbar =$ the light speed $c = 1$. For a rigorous derivation, see [7] and [6].

The solution u also satisfies the equivalent integral equation:

$$
\begin{aligned}
&u(t) = U(t)\varphi - i\int_0^t U(t-s)F(u)(s)\,ds,\\
&U(t) = e^{-it\sqrt{m^2-\Delta}}.
\end{aligned} \tag{2}
$$

If the solution u is sufficiently smooth, then it satisfies the mass and energy conservation laws:

$$
\begin{aligned}
&\|u(t)\|_{L^2} = \|\varphi\|_{L^2},\\
&E(u(t)) \equiv < \sqrt{m^2-\Delta}u, u >_{L^2} + \frac{1}{2} < F(u), u > = E(u(0)) = E(\varphi),
\end{aligned} \tag{3}
$$

where $< \cdot, \cdot >_{L^2}$ is the complex inner product of L^2 space.

Our main concern is to establish the global time existence of u. At first let us introduce a local existence result. See [8] and [2].

Proposition 1. *Let $0 < \gamma < n$ and $n \geq 1$. Suppose $\varphi \in H^s(\mathbb{R}^n)$ with $s \geq \frac{\gamma}{2}$. Then there exists a positive number T independent of m such that (2) has a unique solution $u \in C([0,T]; H^s)$ with $\|u\|_{L_T^\infty H^s} \leq C\|\varphi\|_{H^s}$, where C does not depend on m.*

The key estimate for the proof is the Hardy-Sobolev inequality:

$$
\|V_\gamma * |u|^2\|_{L^\infty} \leq C\|u\|^2_{\dot{H}^{\frac{\gamma}{2}}}. \tag{4}
$$

From the uniform H^s-estimate on the mass m, we can consider a non-relativistic limit problem. Let u_m be the solution of (1) for each m. Using the phase modulation $v_m = e^{imt}u_m$, the function v_m satisfies the equation

$$
i\partial_t v_m = (\sqrt{m^2-\Delta} - m)v_m + F(v_m), \;\; v_m(0) = \varphi.
$$

Since $\sqrt{m^2-\Delta}-m \sim -\frac{1}{2m}\Delta$ at least at a formal level (for a rigorous proof, see [2]), we can expect that v_m is very close to a function w_m in $L_T^\infty(H^s)$, where w_m is a solution of the nonlinear Schrödinger equation:

$$i\partial_t w_m = -\frac{1}{2m}\Delta w_m + F(w_m), \;\; w_m(0) = \varphi. \tag{5}$$

Now let $T_{v_m}^*$ and $T_{w_m}^*$ be the maximal existence time of the solutions u_m and w_m, respectively. Then from the local existence result (Proposition 1) we deduce that $T^* \equiv \inf_{m>1}\min(T_{v_m}^*, T_{w_m}^*)$ is strictly positive and have the following.

Proposition 2. *If $s \ge \frac{\gamma}{2}$ and $T < T^*$, then $v_m - w_m \to 0$ in $L_T^\infty(H^s)$ as $m \to \infty$.*

We now look into a global theory. Using the local result (Proposition 1) and conservation laws (3), we have the following theorem. Let the maximal existence time of Proposition 1 be T^*.

Theorem 1. *If $0 < \gamma < 1$ and $\varphi \in H^s, s \ge \frac{1}{2}$, then $T^* = \infty$. Let $\gamma = 1$, $n \ge 2$ and $s \ge \frac{1}{2}$. Then if $\lambda > 0$, or $\lambda < 0$ and $\|\varphi\|_{L^2}$ is sufficiently small, then $T^* = \infty$.*

The key estimate for the case $\lambda < 0$ is

$$\|u\|_{\dot H^{\frac12}}^2 \le |E(\varphi)| + \frac{1}{2}| < F(u), u > | \le |E(\varphi)| + C\|\varphi\|_{L^2}^{4-2\gamma}\|u\|_{\dot H^{\frac12}}^{2\gamma}. \tag{6}$$

In case that $\gamma < 1$, we just use Young's inequality for the second term of RHS of (6) to get $\|u\|_{\dot H^{\frac12}}^2 \le 2|E(\varphi)| + C\|\varphi\|_{L^2}^2$. If $\gamma = 1$, then the smallness of $\|\varphi\|_{L^2}$ is absolutely necessary but its size is independent of the size of $\|\varphi\|_{\dot H^{\frac12}}$.

To deal with the case $\gamma > 1$, we need another tools, so to speak, Strichartz estimates ([9, 10]) such that

$$\begin{aligned} &\|U(t)\varphi\|_{L_T^{q_0}H_{r_0}^{s_0-\sigma_0}} \le C\|\varphi\|_{H^{s_0}}, \\ &\left\|\int_0^t U(t-t')f(t')\,dt'\right\|_{L_T^{q_1}H_{r_1}^{s_1-\sigma_1}} \le C\|f\|_{L_T^1H^{s_1}}, \end{aligned} \tag{7}$$

where $(q_i, r_i), i = 0, 1$, satisfy that for any $\theta \in [0, 1]$

$$\begin{gathered} \frac{2}{q_i} = (n-1+\theta)\left(\frac12 - \frac{1}{r_i}\right), \;\; 2\sigma_i = (n+1+\theta)\left(\frac12 - \frac{1}{r_i}\right), \\ 2 \le q_i, r_i \le \infty, \;\; (q_i, r_i) \ne (2, \infty) \end{gathered} \tag{8}$$

and a refined Hardy-Sobolev inequality such that

$$\|V_\gamma * |u|^2\|_{L^\infty} \le C\|u\|_{L^{\frac{2n}{n-(\gamma-\varepsilon)}}}\|u\|_{L^{\frac{2n}{n-(\gamma+\varepsilon)}}}. \tag{9}$$

If $\theta = 0$, the pair (q_i, r_i) is called wave admissible pair and if $\theta = 1$, then Schrödinger admissible one.

The first global result is the following (see [2]).

Theorem 2. *Let $0 < \gamma < \frac{2n}{n+1}$, $n \ge 2$. Then there exists an α with $0 < \alpha < \gamma$ such that if $\varphi \in H^{\frac12}$ and if $\lambda > 0$, or $\lambda < 0$ but $\|\varphi\|_{L^2}$ is sufficiently small, then (2) has a unique solution $u \in C([0,\infty); H^{\frac12}) \cap L_{loc}^q(H_r^{\frac12-\sigma})$, where $q = \frac{4n}{(n-1)\alpha}$, $r = \frac{2n}{n-\alpha}$ and $\sigma = \frac{(n+1)\alpha}{4n}$.*

For the proof, the wave admissible pair is useful. If $\lambda < 0$, then one can use the following estimate instead of (6)

$$\begin{aligned} \|u\|^2_{\dot H^{\frac12}} &\le |E(\varphi)| + \frac{1}{2}| < F(u), u > | \le |E(\varphi)| + C\|\varphi\|_{L^2}^{4-2\gamma}\|u\|^{2\gamma}_{\dot H^{\frac12}} \\ &\le C(1+\|\varphi\|^2_{\dot H^{\frac12}})^{\gamma} + C\|\varphi\|_{L^2}^{4-2\gamma}\|u\|^{2\gamma}_{\dot H^{\frac12}}. \end{aligned} \tag{10}$$

Hence the choice of the initial data φ such that

$$\|\varphi\|_{L^2} \le \min\left(1, (8^{\gamma}C^{\gamma}(1+\|\varphi\|^2_{\dot H^{\frac12}})^{\gamma(\gamma-1)})^{-\frac{1}{4-2\gamma}}\right),$$

enables us to have

$$\|u(t)\|^2_{\dot H^{\frac12}} \le 2C(1+\|\varphi\|^2_{\dot H^{\frac12}})^{\gamma} \tag{11}$$

for any existence time t.

If we use Schrödinger admissible pair (actually end-point Strichartz estimate) and (9), then we have a small data global existence theorem (see [2]).

Theorem 3. *Let $2 < \gamma < n$, $n \ge 3$ and $s > \frac{\gamma}{2} - \frac{n-2}{2n}$. Then there exists $\rho > 0$ such that for any $\varphi \in H^s$ with $\|\varphi\|_{H^s} \le \rho$, (2) has a unique solution $u \in (C \cap L^\infty)(H^s) \cap L^2(H^{s-\frac{n+2}{2n}}_{\frac{2n}{n-2}})$. Moreover there is $\varphi^+ \in H^s$ such that*

$$\|u(t) - U(t)\varphi^+\|_{H^s} \to 0 \text{ as } t \to \infty.$$

Up to now, we have looked into global theory of (1) according to the value of γ. There are some gaps for this such that $\frac{2n}{n+1} \le \gamma < 2$ for $n = 2$ and $\frac{2n}{n+1} \le \gamma \le 2$ for $n \ge 3$. These gaps can be filled partly by assuming the radial symmetry of data. To introduce them, we need to investigate two types of Strichartz estimates. The first one is on the Strichartz estimate of radial functions.

Lemma 1 (see [3]). *For any radial φ and F*

$$\begin{aligned} &\|U(t)\varphi\|_{L^p_T H^{s-\sigma}_p} \lesssim \|\varphi\|_{H^s}, \\ &\left\| \int_0^t U(t-t')F(t')\,dt' \right\|_{L^p_T H^{s-\sigma}_p} \lesssim \|F\|_{L^1_T H^s}, \end{aligned} \tag{12}$$

where $s \in \mathbb{R}$ and $\frac{2n}{n-1} < p < \infty, \sigma = \frac{n}{2} - \frac{n+1}{p}$.

The second one is on a weighted Strichartz estimate.

Lemma 2 (see [4]). *Let $\frac12 < a < \frac{n}{2}$ and $n \ge 2$. Then for any $\varphi \in H^s$ and $F \in L^1_T H^s$, $s \ge 0$, we have*

$$\begin{aligned} &\|U(\cdot)\varphi\|_{L^2_T(\tilde H^s_a)} \lesssim \|\varphi\|_{H^s}, \\ &\left\| \int_0^{(\cdot)} U(\cdot - t')F(t')\,dt' \right\|_{L^2_T(\tilde H^s_a)} \lesssim \|F\|_{L^1_T H^s}. \end{aligned} \tag{13}$$

The constants in the estimates can be chosen independently of T.

Here we denote the weighted Sobolev space $\widetilde{H}^s_a$ by

$$\widetilde{H}^s_a = \{v : \|v\|_{\widetilde{H}^s_a} \equiv \||\cdot|^{-a}L_a(\Delta)(1-\Delta)^{\frac{s}{2}}v\|_{L^2} < \infty\},$$

where a is a positive real number and the pseudo-differential operator $L_a(\Delta)$ is $(-\Delta)^{\frac{1}{2}(1-a)}(1-\Delta)^{-\frac{1}{4}}$.

The proof of the second parts of (12) and (13) does not follow from the first parts. One may use the low-diagonal operator estimate for this. See [1], [5] and [11].

Interpolating the estimate of Lemma 1 and (7), we get that given $\varepsilon > 0$ we can find q and σ such that $\frac{2n}{n-1} < q < \frac{2n}{n-1} + \varepsilon$, $\frac{1}{2n} < \sigma < \frac{1}{2n} + \varepsilon$ and

$$\begin{aligned} \|U(t)\varphi\|_{L^q_T H^{\frac{1}{2}-\sigma}_{\frac{2n}{n-1}}} &\lesssim \|\varphi\|_{H^{\frac{1}{2}}}, \\ \left\|\int_0^t U(t-t')F(t')\,dt'\right\|_{L^q_T H^{\frac{1}{2}-\sigma}_{\frac{2n}{n-1}}} &\lesssim \|F\|_{L^1_T H^{\frac{1}{2}}}, \end{aligned} \tag{14}$$

With these pairs we can make the value of σ close to $\frac{1}{2n}$ and the value γ to $\frac{2n-1}{n}$. Using (14) via (10) and (11), we get an improvement as follows (see [3]):

Theorem 4. *Let γ satisfy $1 < \gamma < \frac{2n-1}{n}, n \geq 2, s \geq \frac{1}{2}$. If $\lambda > 0$, then for any radially symmetric function $\varphi \in H^s$ (2) has a unique radially symmetric solution $u \in C(\mathbb{R}; H^s) \cap L^q_{loc}H^{\frac{1}{2}-\sigma}_{\frac{2n}{n-1}}$ for $q = \frac{2n}{n-1} + \varepsilon$ and $\sigma = \frac{1}{2} + \varepsilon'$ with sufficiently small $\varepsilon, \varepsilon' > 0$. For all time the energy and L^2 norm of $u(t)$ are conserved. If $\lambda < 0$, then there exists $\rho > 0$ such that the same conclusion holds for φ with $\|\varphi\|_{L^2} \leq \rho$.*

Finally, using the radial symmetry one can show the following lemmas (see [4]).

Lemma 3. *Let $n \geq 3$ and $0 < \gamma < n-1$.*

(i) *If f and g are radial functions with $f, |x|^{-\delta}f, |x|^{-(\gamma-\delta)}g \in L^2$ for some $0 < \delta \leq \gamma$, then we have*

$$\sup_{x\in\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(y)||g(y)|}{|x-y|^\gamma}\,dy \lesssim \||\cdot|^{-\delta}f\|_{L^2}\||\cdot|^{-(\gamma-\delta)}g\|_{L^2}. \tag{15}$$

(ii) *If f, g are radially symmetric and $f, |x|^{-(\gamma-\delta)}g \in L^2$ for some $0 < \delta \leq \gamma$, then for any $x \neq 0$*

$$\int_{\mathbb{R}^n} \frac{|f(y)||g(y)|}{|x-y|^\gamma}\,dy \lesssim |x|^{-\delta}\|f\|_{L^2}\||\cdot|^{-(\gamma-\delta)}g\|_{L^2}. \tag{16}$$

Lemma 4. *Let $1 < \gamma < 2$ if $n \geq 4$ and $1 < \gamma < \frac{3}{2}$ if $n = 3$. Then for any radial functions $f, g \in L^2$ with $|x|^{-1}f, |x|^{-1}g \in L^2$ we have for some positive small $\varepsilon < \min(\gamma-1, 2-\gamma)$*

$$\begin{aligned} &\left\||\cdot|^{-\gamma-\frac{1}{2}} * (fg)\right\|_{L^{2n}} \\ &\lesssim (\|f\|_{L^2}^{2-\gamma-\varepsilon}\||\cdot|^{-1}f\|_{L^2}^{\gamma-1+\varepsilon} + \|f\|_{L^2}^{2-\gamma+\varepsilon}\||\cdot|^{-1}f\|_{L^2})\||\cdot|^{-1}g\|_{L^2}. \end{aligned} \tag{17}$$

Using two lemmas above together with Lemma 2 with $a = 1$, we can fill the gap completely for $n \geq 4$ as follows (see [4]):

Theorem 5. (1) *Let* $1 < \gamma < 2$ *and* $n \geq 4$. *Let* $\varphi \in H^{\frac{1}{2}}$ *be radially symmetric and assume that* $\|\varphi\|_{L^2}$ *is sufficiently small if* $\lambda < 0$. *Then there exists a unique radial solution* $u \in C_b H^{\frac{1}{2}}$ *such that* $|x|^{-1}u \in L^2_{loc}L^2$ *of* (2) *satisfying the energy and* L^2 *conservations* (3).

(2) *Let* $\gamma = 2$ *and* $n \geq 4$. *Let* $\varphi \in H^1$ *be radially symmetric. If* $\|\varphi\|_{H^1}$ *is sufficiently small, then there exists a unique radial solution* $u \in C_b H^1$ *such that* $|x|^{-1}u \in L^2L^2$ *to* (2). *Moreover, there exist radial functions* φ^+ *and* φ^- *such that*

$$\|u(t) - U(t)\varphi^{\pm}\|_{H^1} \to 0 \quad \text{as} \quad t \to \pm\infty.$$

It should be noted that in the course of proof for the case $\lambda < 0$ and $1 < \gamma < 2$, the estimates (10) and (11) are always needed.

REFERENCES

[1] C. AHN AND Y. CHO, *Lorentz space extension of Strichartz estimat*, Proc. Amer. Math. Soc. 133 (2005) pp. 3497–3503.

[2] Y. CHO AND T. OZAWA, *On the semi-relativistic Hartree type equation*, SIAM J. Math. Anal. 38 (2006), pp. 1060–1074.

[3] Y. CHO AND T. OZAWA, *On radial solutions of semi-relativistic Hartree equations*, to appear in 2007 DCDS supplement.

[4] Y. CHO, T. OZAWA, H. SASAKI AND Y. SHIM, *Remarks on the relativistic Hartree equations*, Hokkaido University Preprint Series in Math ♯827 (2007).

[5] M. CHRIST AND A. KISELEV, *Maximal functions associated to filtrations*, J. Func. Anal., 179 (2001), pp. 409–425.

[6] A. ELGART AND B. SCHLEIN, *Mean field dynamics of boson stars*, Comm. Pure Appl. Math. 60 (2007), pp. 500–545.

[7] E. H. LIEB AND H.-T. YAU, *The Chandrasekhar theory of stellar collapse as the limit of quantum mechanics*, Commun. Math. Phys., 112 (1987), pp. 147–174.

[8] E. LENZMANN, *Well-posedness for semi-relativistic Hartree equations of critical type*, arXiv:math.AP/0505456.

[9] S. MACHIHARA, K. NAKANISHI AND T. OZAWA, *Nonrelativistic limit in the energy space for nonlinear Klein-Gordon equations*, Math. Ann., 322 (2002), pp. 603–621.

[10] S. MACHIHARA, K. NAKANISHI AND T. OZAWA, *Small global solutions and the nonrelativistic limit for the nonlinear Dirac equation*, Rev. Mat. Iberoamericana, 19 (2003), pp. 179–194.

[11] H. SMITH AND C. D. SOGGE, *Global strichartz estimates for nontrapping perturbations of the Laplacian*, Comm. Partial Differential Equations 25 (2000), pp. 2171–2183.

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