UNCONDITIONAL UNIQUENESS OF SOLUTION FOR THE CAUCHY PROBLEM OF THE NONLINEAR SCHRÖDINGER EQUATION

Yoshio TSUTSUMI

Department of Mathematics, Kyoto University
Kyoto 606-8502, JAPAN

We consider the uniqueness of solution for the Cauchy problem of the nonlinear Schrödinger equation:

$$i\partial_t u + \Delta u = \lambda |u|^\alpha u, \qquad t \in [0,T], \quad x \in \mathbb{R}^n, \tag{1}$$

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}^n, \tag{2}$$

where $\lambda \in \mathbb{C}$ and $T > 0$. Let $\alpha > 0$ and $s \geq 0$ be specified later, and let $u_0 \in H^s$. Suppose that $u \in C([0,T]; H^s)$ with (2) and u satisfies equation (1) in $\mathcal{D}'((0,T) \times \mathbb{R}^n)$, that is, in the distribution sense.

We briefly recall known results on the uniqueness of solution for (1)-(2). In [3], Ginibre and Velo prove that if $s = 1$ and $\alpha < 4/(n-2)$, the solution is unique. In [2], Cazenave and Weissler show that when

$$\begin{gathered} u \in L^\gamma(0,T; B^s_{\rho,2}), \\ \rho = \frac{\alpha+2}{1+\alpha s/n}, \quad \gamma = \frac{4(\alpha+2)}{\alpha(n-2s)}, \end{gathered} \tag{3}$$

the solution is unique. The solution is often constructed within the framework of $C([0,T]; H^s)$ and an auxiliary space such as (3). Space (3) is associated with the Strichartz estimate and (ρ, γ) is an admissible pair of the Strichartz estimate (see, e.g., [1]). The uniqueness of solution in an auxiliary space such as (3) as well as in $C([0,T]; H^s)$ is called the conditional uniqueness, to which the result in [2] corresponds. On the other hand, the uniqueness without an auxiliary space is called the unconditional uniqueness. From now on, we refere to the unconditional uniqueness as (UU). In [6], T. Kato extensively studies the unconditional uniqueness of solution for (1)-(2) and shows the following result: Assume that any of the following three is satisfied.

(a) $n = 1, \quad 0 \le s < 1/2, \quad 0 < \alpha \le (1+2s)/(1-2s),$

(b) $n \ge 2, \quad 0 \le s < n/2, \quad 0 < \alpha < \min\{4/(n-2s), (2+2s)/(n-2s)\},$

(c) $n \ge 1, \quad s \ge n/2.$

Then, (UU) holds. Furioli and Terraneo [4] use the Besov spaces of negative indices to improve the result by Kato [6] and show that if

$$3 \le n \le 5, \quad 1 < \alpha < \min\left\{\frac{4}{n-2s}, \frac{n+2s}{n-2s}, \frac{2+4s}{n-2s}\right\},$$

and additionally, for $n = 3$,

$$\frac{2s}{n-2s} < \alpha \le \frac{n+2-2s}{n-2s},$$

then (UU) holds. Furthermore, as Cazenave pointed out in [1], it follows from a variant of the proof by Kato [6] that when

$$n \ge 3, \quad 1 \le s < n/2, \quad \alpha = \frac{4}{n-2s},$$

(UU) holds.

Remark 1. (i) The unconditional uniqueness does not always make sense, because equation (1) may not make sense without an auxiliary sapce. The assumption $\alpha \le (n+2s)/(n-2s)$ implies that $|u|^\alpha u \in L^1_{loc}(\mathbb{R}^n)$, which ensures that equation (3) makes sense within the framework of the distribution. Furthermore, the assumption $\alpha \le 4/(n-2s)$ comes from the scaling invariance of equation (3). Therefore, when we consider the unconditional uniqueness, the following restriction seems natural.

$$0 < \alpha \le \min\left\{\frac{4}{n-2s}, \frac{n+2s}{n-2s}\right\}. \tag{4}$$

(ii) When $1 \le s < n/2$ and $1 < \alpha \le 4/(n-2s)$, (UU) is already known (see Kato [6] and Cazenave [1]).

(iii) Even in the so-called subcritical case $\alpha < 4/(n-2s)$, the unconditonal uniqueness is not obvious (see, e.g., Kato [7], where he pointed out that if it is in $L^r(0,T;B^s_{q,2}(\mathbb{R}^n))$ for a certain addmissible pair (q,r) with sufficiently large q, the solution belongs to $L^r(0,T;B^s_{q,2}(\mathbb{R}^n))$ for all addimissible pairs (q,r) associated with the Strichartz estimate and so (UU) holds).

Accordingly, (UU) has been open in the following four cases:

$$\text{(Case 1)} \qquad n=2, \quad \alpha = \frac{n+2s}{n-2s}\left(=\frac{1+s}{1-s}\right),$$

$$\text{(Case 2)} \qquad n=3, \quad (4), \quad \alpha \geq \min\left\{\frac{2+4s}{n-2s}, \frac{n+2-2s}{n-2s}\right\}$$

$$\text{or} \quad \frac{2+2s}{n-2s} \leq \alpha \leq 1,$$

$$\text{(Case 3)} \qquad n=4,5, \quad \max\left\{1, \frac{2+4s}{n-2s}\right\} \leq \alpha \leq \frac{4}{n-2s}$$

$$\text{or} \quad \frac{2+2s}{n-2s} \leq \alpha \leq 1,$$

$$\text{(Case 4)} \qquad n \geq 6, \quad \frac{2+2s}{n-2s} \leq \alpha \leq \frac{4}{n-2s}.$$

We have the following theorem concerning the unconditional uniqueness of solution for (1)-(2), which has recently been obtained in collaboration with Yin Yin Su Win, Kyoto University.

Theorem 1. *Let $0 \leq s < 1$. We assume either of the following two:*

(a) Cases 1 and 2 except for $(n,\alpha,s) = (2,1,0), (3,2,1/2),$

$$(3,\theta,0), \quad 2/3 < \theta \leq 1$$

$$\text{(b)} \qquad n=4,5, \quad \max\left\{\frac{2+2s}{n-2s}, 1\right\} \leq \alpha \leq \min\left\{\frac{4}{n-2s}, \frac{2+4s}{n-2s}\right\}.$$

Then, (UU) holds for (1)-(2).

Remark 2. (i) Case (a) in Theorem 1 is divided into three subcases. When $(n,\alpha,s) = (3,\theta,0)$ and $2/3 < \theta < 1$, our proof does not work for some technical reason. In the second subcase $(n,\alpha,s) = (2,1,0)$, $(3,1,0)$, we have

$$\alpha = \frac{n+2s}{n-2s},$$

which implies that the nonlinearity only belongs to $L^1(\mathbb{R}^n)$. This seems to be a little more serious problem. In the third subcase $(n, \alpha, s) = (3, 2, 1/2)$, we have

$$\alpha = \frac{4}{n-2s} = \frac{n+2s}{n-2s}.$$

The last subcase seems to contain an essential difficulty.

(ii) If $\alpha < 4/(n-2s)$, that is, in the subcritical case, we can replace $u \in C([0,T]; H^s)$ by $u \in L^\infty(0,T; H^s)$. However, if $\alpha = 4/(n-2s)$, the unconditional uniqueness generally breaks down without the strong continuity in the time variable of solution (see Example 1 below).

Example 1. We consider the following L^2-critical nonlinear Schrödinger equation.

$$i\partial_t u + \Delta u = -|u|^{4/n} u, \qquad t \in \mathbb{R}, \quad x \in \mathbb{R}^n, \quad n \geq 4. \tag{5}$$

We put

$$u(t,x) = \frac{1}{(2t)^{n/2}} e^{i|x|^2/(4t)} e^{i/t} \phi\left(\frac{x}{2t}\right),$$

where ϕ is a solution of the semilinear elliptic equation associated with (5).

$$-\Delta\phi + \phi - \phi^{1+4/n} = 0, \qquad \phi > 0, \quad \phi \in H^1.$$

Then, $u(t) \in C(\mathbb{R}\backslash\{0\}; H^1)$ and $u(t) \to 0$ weakly in L^2 $(t \to 0)$. Therefore, $u \in C_w(\mathbb{R}; L^2)$ and u satisfies (1)-(2) with $u_0 = 0$. But, obviously, $u \equiv 0$ is also a solution with $u(0) = 0$.

After the list of references, we draw a figure to compare our Theorem 1 with the results by Kato [6] and Furioli and Terraneo [4] for $n = 3$.

References

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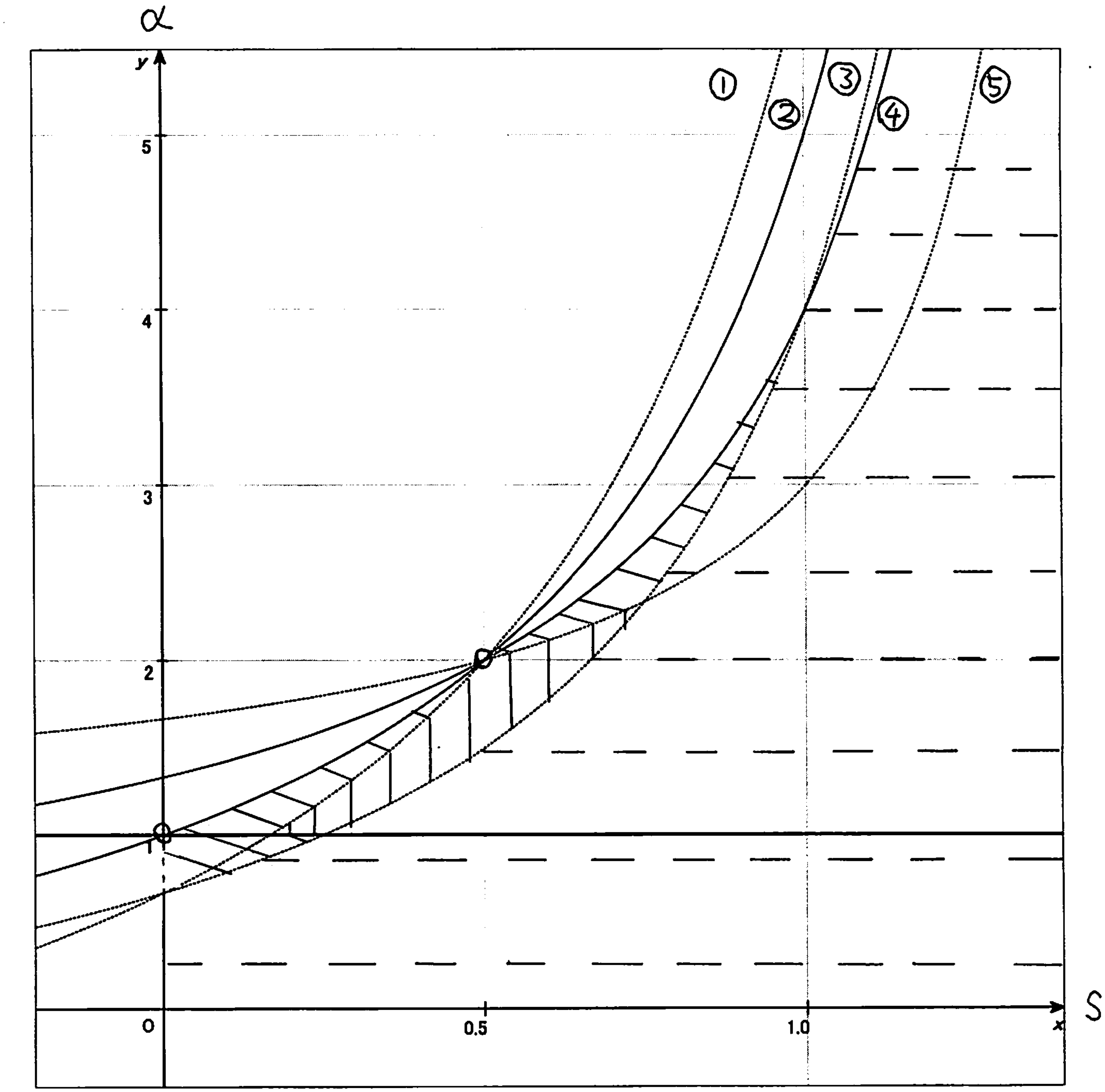

$n = 3$

① $\alpha = \dfrac{2+4s}{n-2s}$

② $\alpha = \dfrac{n+2s}{n-2s}$ $(H^s \hookrightarrow L^{\alpha+1})$

③ $\alpha = \dfrac{2+2s}{n-2s}$

④ $\alpha = \dfrac{4}{n-2s}$ (scaling critical)

⑤ $\alpha = \dfrac{n+2-2s}{n-2s}$

T. Kato [6] (horizontal dashed hatching)

Furioli and Terraneo [4] (vertical hatching)

Su Win and Tsutsumi (diagonal hatching)