LERAY'S PROBLEM FOR THE STATIONARY NAVIER-STOKES EQUATIONS AND THE HARMONIC VECTOR FIELDS II

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This is a continuation of Prof. Hideo Kozono's talk "Leray's problem for the stationary Navier-Stokes equations and the harmonic vector fields I" [1] of this symposium. Hence, our main concern goes on the existence of solutions to the stationary Navier-Stoke equations under the nonhomogeneous boundary condition:

$$(1) \qquad (NS) \begin{cases} -\mu\Delta v + (v\cdot\nabla)v + \nabla p = 0 \ \text{ in } \Omega, \\ \operatorname{div} v = 0 \ \text{ in } \Omega, \\ v = \beta \ \text{ on } \partial\Omega. \end{cases}$$

Here $v = v(x)$: the velocity vector field, $p = p(x)$: the pressure, β: the prescribed boundary data, $\mu(> 0)$: the coefficient of viscosity. Ω is a bounded domain in $\mathbb{R}^n$, $n = 2, 3$, with the multiply connected boundary $\partial\Omega = \cup_{j=0}^{L}\Gamma_j$ such that Γ_j, $j = 0, 1, \dots, L$, are smooth connected components and $\Gamma_1, \dots, \Gamma_L$ lie inside of Γ_0. When $n = 3$, the domain Ω is assumed to be simply connected for simplicity.

We assume that the boundary data β satisfies *the General Flux Condition*

$$(G.F.) \qquad \sum_{j=0}^{L}\int_{\Gamma_j} \beta\cdot\nu \, dS = 0,$$

where ν is the outward unit normal to $\partial\Omega$. As in the talk [I], we are going to say that β satisfies *the Restricted Flux Condition*, if

$$(R.F.) \qquad \int_{\Gamma_j} \beta\cdot\nu \, dS = 0, \ \text{for any } j = 0, 1, \dots, L,$$

hold true.

Main result of this talk is the following criterion on the sufficient conditions of existence of weak solutions of (NS).

Proposition 1. *Let the boundary data $\beta \in H^{1/2}(\partial\Omega)$ satisfy (G.F.). Let h be the same harmonic part of the divergence free extension b of β as in [I].*

Then, if the estimate

$$(2) \qquad \sup_{z\in\chi(\Omega), \nabla z\neq 0} \frac{(h, (z\cdot\nabla)z)}{\|\nabla z\|_2^2} < \mu$$

holds true, there exists at least one weak solution u of (NS).

Here

$$\chi(\Omega) = \{z \in H^1_\sigma(\Omega) \mid {}^{\exists}q \in W^{1,3/2}(\Omega) \ s.t. \ (z\cdot\nabla)z + \nabla q = 0 \ in \ \Omega\}$$

(the space of weak solutions of stationary Euler equation with Dirichlet boundary condition).

[1] The content of his talk will be cited just as [I] in what follows. The notation and function spaces defined in [I] might be used without notice.

Remark 2. *It is important that the harmonic part h above depends only on the $\int_{\Gamma_j} \beta \cdot \nu \, dS$, $j = 1, \ldots, L$, and the base of $V_{har}(\Omega)$(= the relative de Rham cohomology on Ω).*

The proof of Proposition 1 is achieved by virtue of Leray-Schauder fixed point theorem and our decomposition theorem of the vector fields on Ω stated in [I] ([4]) via *reduction to absurdity*.

The crux of the proof of Proposition 1 is the following

Lemma 3. *Given the same b and h as in Theorem 1, one has*

$$((z \cdot \nabla)z, \, b) = ((z \cdot \nabla)z, \, h), \; \text{for} \quad \text{any } z \in \chi(\Omega). \tag{3}$$

This lemma is shown by utilizing our decomposition theorem and the property of the space $\chi(\Omega)$ proven in [3].

Lemma 4 ([3]). *Suppose that $z \in H^1_{0,\sigma}(\Omega)$ and $q \in W^{1,3/2}(\Omega)$ satisfy, in the weak sense,*

$$(z \cdot \nabla)z + \nabla q = 0, \quad \text{div}\, z = 0 \quad \text{in } \Omega.$$

Then the trace $\gamma(q)$ of q exists in $W^{1/3,3/2}(\partial\Omega)$ and obeys

$$\gamma(q)|_{\Gamma_j} = c_j \, (= \text{const.}), \; j = 0, 1, \ldots, L.$$

Once Proposition 1 has been obtained, it is not difficult to show the main theorem stated in [I] by just estimating the integral $(h, (z \cdot \nabla)z)$ in (2) for any $z \in \chi(\Omega)$. Furthermore, we can derive the result shown by [2] from Proposition 1 directly.

As for the validity of Leray's inequality which has been discussed in [I], we can show the following rather simple criterion, again using our decomposition theorem and Lemma 3.

Lemma 5. *If Leray's inequality (L.I.) holds true for the boundary data $\beta \in H^{1/2}(\partial\Omega)$ satisfying (G.I.), then*

$$(h, \, (z \cdot \nabla)z) = 0 \tag{4}$$

holds true for any $z \in \chi(\Omega)$.

It is easy to see form this lemma that, when Ω is the 2-D annular domain, (L.I.) holds true for $\beta \in H^{1/2}(\partial\Omega)$ satisfying (G.I.) if and only if β satisfies (R.F.). In fact, Takeshita [6] gave more general statement.

Theorem 6 ([6]). *Let Ω be a bounded domain in $\mathbb{R}^n$ with the smooth boundary $\partial\Omega = \cup_{j=0}^{L} \Gamma_j$, Γ_j being the smooth connected component of $\partial\Omega$. Assume that $\Gamma_1, \ldots, \Gamma_L$ lie inside of Γ_0 and each Γ_j, $j = 1, \ldots, L$, is diffeomorphic to a sphere contained in Ω. Then (L.I.) holds true for $\beta \in H^{1/2}(\partial\Omega)$ satisfying (G.I.) if and only if β satisfies (R.F.).*

Hence, for any domain stated in Theorem 6, the method relying upon (L.I.) as in [I] can not be used to show the existence of weak solutions of (NS) with $\beta \in H^{1/2}(\partial\Omega)$ satisfying only (G.F.). However, this fact does not signify that there exists no weak solution of (NS) in such cases.

As a matter of fact, Amick showed

Theorem 7 ([1]). *Let Ω be a bounded domain in $\mathbb{R}^2$ with the multiply connected smooth boundary $\partial\Omega = \cup_{j=0}^{L} \Gamma_j$. Suppose that Ω is symmetric with respect to x_1-axis and every boundary components Γ_j, $j = 0, 1, \ldots, L$, intersect with x_1-axis. Let $\beta \in H^{1/2}(\partial\Omega)$ satisfying (G.F.) be symmetric with respect to x_1-axis. Then there exists at least one symmetric weak solution of (NS).*

Here the function $v = (v_1, v_2)$ is called symmetric with respect to x_1-axis if v_1 is even function and v_2 is odd function with respect to x_2. We will give a simple proof of this theorem, utilizing Proposition 1 and Lemma 3.

We should mention that Fujita [5] gave another constructive proof by showing an inequality, which is similar to (L.I.), under symmetric assumption on test functions.

We might discuss the case when the domain Ω retains the other type of symmetry.

References

[1] Amick, J., *Existence of solutions to the Nonhomogeneous steady Navier-Stokes equations.* Indiana Univ. Math. J. **33**, 817-830 (1984).

[2] Borchers, W., Pileckas, K., *Note on the flux problem for stationary incompressible Navier-Stokes equations in domains with a multiply connected boundary.* Acta Appl. Math. **37**, 21-30 (1994)

[3] Kapitanskii, L.V., Pileckas, K.I., *On spaces of solenoidal vector fields and boundary value problems for the Navier-Stokes equations in domains with noncompact boundaries.* Proc. Math. Inst. Steklov, **159**, 3-34, (1984).

[4] Kozono, H., Yanagisawa, T., *L^r-variational inequality for vector fields and the Helmholtz-Weyl decompositions in bounded domains.* submitted.

[5] Fujita, H., *On stationary solutions to Navier-Stokes equation in symmetric plane domains under general outflow condition. Navier-Stokes equations: theory and numerical methods, Varenna, 1997* Pitman Res. Notes Math. Ser., **388**, Longman, Harlow, 16-30, 1998.

[6] Takeshita,A., *A remark on Leray's inequality.* Pacific J. Math. **157**, 151-158 (1993).

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