Stability of ground states for nonlinear Schrödinger equations with nonlocal interaction

Masahito Ohta
Department of Mathematics, Saitama University
mohta@rimath.saitama-u.ac.jp

This talk is based on a joint work with Hiroaki Kikuchi (Kyoto University). We consider the following nonlinear Schrödinger equation

\[ i \partial_t u = -\Delta u - (W_m \ast |u|^2)u \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3, \]  

where \(*\) denotes the convolution in \(\mathbb{R}^3\), \(m \geq 0\) is a constant, and

\[ W_m(x) = \frac{e^{-m|x|}}{2\pi|x|} \]

is the Yukawa potential. Note that \(W_m \ast |u|^2 = 2(-\Delta + m^2)^{-1}|u|^2\), and (1) is a simplification of the coupled Klein-Gordon-Schrödinger equations

\[
\begin{align*}
    i\partial_t u + \Delta u &= -2uv, \\
    \partial^2_t v - \Delta v + m^2 v &= |u|^2.
\end{align*}
\]

In this talk, we consider the single equation (1) instead of the system (2), just for simplicity. We study the orbital stability of standing wave solutions \(u(t, x) = e^{i\omega t}\psi_\omega(x)\) of (1), where \(\omega > 0\) is a parameter, and \(\psi_\omega\) is a ground state of the stationary problem

\[ -\Delta \psi + \omega \psi - (W_m \ast |\psi|^2)\psi = 0, \quad x \in \mathbb{R}^3. \]  

We use the following notation.

\[
\begin{align*}
    E(v) &= \frac{1}{2} \|\nabla v\|_{L^2}^2 - \frac{1}{4} \int_{\mathbb{R}^3 \times \mathbb{R}^3} W_m(x-y)|v(x)|^2|v(y)|^2 \, dx \, dy, \\
    S_\omega(v) &= E(v) + \frac{\omega}{2} \|v\|_{L^2}^2, \\
    \mathcal{A}_\omega &= \{v \in H^1(\mathbb{R}^3) : S'_\omega(v) = 0, \; v \neq 0\}.
\end{align*}
\]
Then, the set of ground states of (3) is defined by
\[ \mathcal{G}_\omega = \{ w \in \mathcal{A}_\omega : S_\omega(w) \leq S_\omega(v), \forall v \in \mathcal{A}_\omega \}. \]

Note that the Cauchy problem for (1) is globally well-posed in \( H^1(\mathbb{R}^3) \), and the energy \( E(u) \) and \( \| u \|^2_{L^2} \) are conserved quantities of (1) (see [2]).

**Definition**

(i) For \( \Omega \subset H^1(\mathbb{R}^3) \), we say that the set \( \Omega \) is stable for (1) if for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that if \( u_0 \in H^1(\mathbb{R}^3) \) and \( \text{dist}(u_0, \Omega) < \delta \) then the solution \( u(t) \) of (1) with \( u(0) = u_0 \) satisfies \( \text{dist}(u(t), \Omega) < \varepsilon \) for all \( t \in \mathbb{R} \). Here, \( \text{dist}(v, \Omega) = \inf \{ \| v - w \|_{H^1} : w \in \Omega \} \).

(ii) For \( \varphi \in \mathcal{A}_\omega \), we say that \( e^{i\omega t} \varphi \) is stable if \( \{ e^{i\theta} \varphi(\cdot + y) : \theta \in \mathbb{R}, y \in \mathbb{R}^3 \} \) is stable, and that \( e^{i\omega t} \varphi \) is unstable if \( e^{i\omega t} \varphi \) is not stable.

When \( m = 0 \), Cazenave and Lions [1] proved that for any \( \omega > 0 \) and for \( \varphi_\omega \in \mathcal{G}_\omega \), \( e^{i\omega t} \varphi_\omega \) is stable for (1). However, little is known for the case \( m > 0 \) (see [3] for a partial result). We now state our main results.

**Theorem 1** Let \( m > 0 \), \( \omega > 0 \) and \( \psi_\omega \in \mathcal{G}_\omega \). Then, there exists \( \omega_1 > 0 \) such that \( e^{i\omega t} \psi_\omega \) is stable for (1) for any \( \omega \in (\omega_1, \infty) \).

**Theorem 2** Let \( m > 0 \), \( \omega > 0 \) and \( \psi_\omega \in \mathcal{G}_\omega \). Then, there exists \( \omega_2 > 0 \) such that \( e^{i\omega t} \psi_\omega \) is unstable for (1) for any \( \omega \in (0, \omega_2) \).

**Theorem 3** Let \( m > 0 \) and \( \varphi_m \in H^1(\mathbb{R}^3) \) is a unique positive solution of
\[ -\Delta \varphi + m^2 \varphi - \varphi^2 = 0, \quad x \in \mathbb{R}^3. \]
Then, the standing wave solution \( e^{imx^2/(1/\sqrt{2})} \varphi_m \) of (1) is stable.

**Remark** We do not know whether \( (1/\sqrt{2}) \varphi_m \in \mathcal{G}_{m^2} \) or not.

**References**

