The existence of the global solutions to semilinear wave equations with cubic nonlinearities in 2-dimensional space

Akira HOSHIGA (Shizuoka University)

1 Introduction

Let us consider the following Cauchy problem:

$$\Box u = \partial_t^2 u - \triangle u = F(\partial u) \qquad (x,t) \in \mathbf{R}^2 \times (0,\infty) \tag{1.1}$$

$$u(x,0) = \varepsilon f(x), \quad \partial_t u(x,0) = \varepsilon g(x) \qquad x \in \mathbf{R}^2. \tag{1.2}$$

Here, $\partial = (\partial_0, \partial_1, \partial_2)$, $\partial_0 = \partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$ $(j = 1,2)$, ε is a positive small parameter. We assume $f, g \in C_0^\infty(\mathbf{R}^2)$, $|f| + |g| \not\equiv 0$ and
$\mathrm{supp}\{f, g\} \subset \{x \in \mathbf{R}^2 \ ; \ |x| \le M\}$. We also assume that

$$F \in C^\infty(\mathbf{R}^3)$$
$$F(\partial u) = O(|\partial u|^3) \qquad \text{near} \quad \partial u = 0.$$

More presicely, we assume

$$F(\partial u) = \sum_{\alpha,\beta,\gamma=0}^{3} A^{\alpha\beta\gamma} \partial_\alpha u \partial_\beta u \partial_\gamma u + O(|\partial u|^4) \qquad \text{near} \quad \partial u = 0,$$

where $A^{\alpha\beta\gamma}$ are constants.

The aim of this talk is to estimate the lifespan T_ε of the smooth solution to the Cauchy problem (1.1), (1.2), which is defined as follows;

$$T_\varepsilon = \sup\{\ T > 0 \ ; \ \text{there exists a smooth solution to (1.1), (1.2) in } C^\infty(\mathbf{R}^2 \times [0,T))\ \}$$

for each ε. In order to state the results which we have already known about the lifespan, we introduce some notations.

For vectors $X = (X_0, X_1, X_2) \in \mathbf{R}^3$, we define

$$C(X) = \sum_{\alpha,\beta,\gamma=0}^{2} A^{\alpha\beta\gamma} X_\alpha X_\beta X_\gamma,$$

which expresses the coefficient of the essential term of $F(\partial u)$.

On the other hand, let $u^0 = u^0(x,t)$ be the solution to the Cauchy problem;

$$\Box u^0 = 0$$
$$u^0(x,0) = f(x), \quad \partial_t u^0(x,0) = g(x)$$

and set $r = |x| \geq 0$, $\omega = \dfrac{x}{r} \in S^1$ and $\rho = r - t \in \mathbf{R}.$ Then we define

$$\mathcal{F}(\omega,\rho) = \lim_{r\to\infty} r^{\frac{1}{2}} u^0(r\omega, \rho + r),$$

which is called the Friedlander radiation field. Hörmander showed in [2] the following properties of $\mathcal{F}$.

$$|\partial_\rho^k \mathcal{F}(\omega,\rho)| \leq C(1+|\rho|)^{-\frac{1}{2}-k} \qquad \rho \in \mathbf{R} \tag{1.3}$$
$$\mathcal{F}(\omega,\rho) = 0 \qquad \text{for} \qquad \rho \geq M. \tag{1.4}$$

By (1.3) and (1.4), we find that the constant

$$H = \max_{\rho\in\mathbf{R},\ \omega\in S^1} \left\{ -\frac{1}{2} C(-1,\omega)(\partial_\rho \mathcal{F}(\omega,\rho))^2 \right\} \tag{1.5}$$

is well-defined and nonnegative.

Then Godin proved in [1] the following.

[Known results] (a) If $H > 0$, then

$$\liminf_{\varepsilon\to+0} \varepsilon^2 \log T_\varepsilon \geq \frac{1}{H}$$

holds.

(b) If $C(-1,\omega) \equiv 0$ holds for $\omega \in S^1$, then $T_\varepsilon = \infty$ holds for sufficiently small $\varepsilon > 0$.

The condition $C(-1,\omega) \equiv 0$ is called "null-condition." The only function satisfying the null-condition is

$$F(\partial u) = C((\partial_0 u)^2 - |\nabla u|^2).$$

It follows from the definition of H that the null-condition is a sufficient condition of $H = 0$ and not a necessary condition. In the case where $|f| + |g| \not\equiv 0$, we find that $H = 0$ is equivalent to the condition

$$C(-1,\omega) \geq 0 \qquad \text{for any} \quad \omega \in S^1. \tag{1.6}$$

This means that there is a nonlinearity F which does not satisfy the both assumption of (a) and (b). Thus it is usefull to consider how long the solutions to (1.1) and (1.2) exists, when the condition (1.6) holds.

2 Statement of the main theorem

We introduce generalaize differential operators;

$$\Omega = x_1\partial_2 - x_2\partial_1, \quad L_i = t\partial_i + x_i\partial_0, \quad S = t\partial_0 + x_1\partial_1 + x_2\partial_2$$

and denote

$$\Gamma_0 = \partial_0, \quad \Gamma_1 = \partial_1, \quad \Gamma_2 = \partial_2, \quad \Gamma_3 = \Omega, \quad \Gamma_4 = L_1, \quad \Gamma_5 = L_2, \quad \Gamma_6 = S.$$

We also write $\Gamma^a u = \Gamma_0^{a_0}\Gamma_1^{a_1}\cdots\Gamma_6^{a_6}u$ for a multi-index $a = (a_0, a_1, \dots, a_6)$.
Next we define some generalized Sobolev norms as

$$\begin{aligned} |v(t)|_k &= \sum_{|a|\le k} ||\Gamma^a v(\cdot,t)||_{L^\infty_x(\mathbf{R}^2)} \\ [v(t)]_k &= \sum_{|a|\le k} ||(1+|\cdot|+t)^{\frac{1}{2}}(1+||\cdot|-t|)^{\frac{1}{2}}\Gamma^a v(\cdot,t)||_{L^\infty_x(\mathbf{R}^2)} \\ ||v(t)||_k &= \sum_{|a|\le k} ||\Gamma^a v(\cdot,t)||_{L^2_x(\mathbf{R}^2)} \end{aligned}$$

Now we state the main theorem.

Theorem 2.1 *Assume that (1.6) holds. Then there exists a constant $\varepsilon_0 > 0$ such that $T_\varepsilon = \infty$ holds for $\varepsilon \in (0, \varepsilon_0)$. Moreover, the solution satisfies*

$$[\partial u(t)]_k \le C_k \varepsilon (1 + \varepsilon^2 \log t)^{M_k} \tag{2.1}$$

for some constants $M_k > 0$ which increases monotonely with respect to k.

(Remark 1) The decay estimate (2.1) means that the solution to (1.1) and (1.2) might decay slower than solutions of linear wave eqations.
(Remark 2) Kubo proved the same result in [3] in which he also showed an asymptotic behavior of solutions.

References

[1] P. Godin, *Lifespan of solutions of semilinear wave equations in two space dimensions*, Comm. in P. D. E., **18** (5 and 6), pp. 895-916, (1993).

[2] L. Hörmander, *The lifespan of classical solutions of nonlinear hyperbolic equations*, Lecture Note in Math., **1256**, pp. 214-280, (1987).

[3] H. Kubo, *Large time behavior of solutions to semilinear wave equations with dissipative structure*, preprint