SMALL DATA SCATTERING FOR NONLINEAR KLEIN-GORDON EQUATIONS

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1. Introduction

This talk is based on the joint work with Naumkin. I am interested in the scattering operator for the nonlinear Klein-Gordon equation

$$u_{tt} - \Delta u + u = f(u), \quad (t, x) \in \mathbf{R} \times \mathbf{R}^n \tag{1.1}$$

with a power type nonlinearity $f(u) = \mu |u|^{\sigma-1} u$ or $f(u) = \mu |u|^{\sigma}$, where $\sigma > 1 + \frac{4}{n+2}$, $\mu \in \mathbf{C}$, for space dimensions $n \geq 3$. The construction of the scattering operator implies the study of the Cauchy problem and the final state problem.

In the previous paper [7], we constructed the scattering operator in $\mathbf{H}^{1+\frac{n}{2},1}$ for the nonlinear Klein-Gordon equation (1.1) with $\sigma > 1 + \frac{2}{n}$ in the case of space dimensions $n = 1$or 2. We applied the operator

$$\mathcal{J} = \langle i\nabla \rangle U(t) xU(-t) = E \langle i\nabla \rangle x + iAt\nabla,$$

where

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which plays the same role as the operator $x + it\nabla$ in the case of the nonlinear Schrödinger equations. Our purpose is to apply the operator $\mathcal{J}$ to the nonlinear Klein-Gordon equation (1.1) in higher space dimensions $n \geq 3$ and to prove the existence of the scattering operator similarly to the case of the nonlinear Schrödinger equations [3].

When $\mu < 0$, $f(u) = \mu |u|^{\sigma-1} u$ and $1 + \frac{4}{n} < \sigma < \sigma^*(n)$, with $\sigma^*(n) = \frac{n+2}{n-2}$ for $n \geq 3$, the completeness of the scattering operator for the nonlinear Klein-Gordon equation (1.1) in the energy space was established in papers [1], [2], [6], [10], [11] by using the Morawetz type estimates and the energy conservation law. This result was extended to lower space dimensions $n = 1, 2$ in paper [9]. The condition $\mu < 0$ can be removed if we restrict our attention to small solutions (see [13] for the case of $f(u) = \mu |u|^{\sigma-1} u$, the case $f(u) = \mu |u|^{\sigma}$ can also be treated). The existence of global in time solutions to the Cauchy problem for the nonlinear Klein-Gordon equation (1.1) (i.e. the existence of the inverse wave operator $\mathcal{W}_-^{-1}$) was shown in [13] by using the $\mathbf{L}^{1+\sigma} - \mathbf{L}^{1+\frac{1}{\sigma}}$ time decay estimates for the linear problem if $\sigma_0(n) < \sigma \leq 1+\frac{4}{n}$, where $\sigma_0(n)$ is a positive root of $\frac{n}{2}\frac{\sigma-1}{\sigma+1}\sigma > 1$. The wave operator $\mathcal{W}_+$ was also constructed in [13] for $\sigma_0(n) < \sigma \leq 1 + \frac{4}{n}$. However the scattering operator $\mathcal{S}_+ = \mathcal{W}_-^{-1}\mathcal{W}_+$ was not defined since the range of the wave operator $\mathcal{W}_+$ differs from the domain of the inverse wave operator $\mathcal{W}_-^{-1}$. As far as we know the

Nonlinear Wave Equations in Hokkaido University, 27,28,29 August.

scattering operator $\mathcal{S}_+$ was not constructed for the nonlinearities of order less than $1+\frac{4}{n}$ except our previous work [7] for $n=1,2$. Note that $1+\frac{4}{n+2}<\sigma_0(n)$.

We now turn to the results concerning the Cauchy problem for the nonlinear Klein-Gordon equation

$$\left\{\begin{array}{c} u_{tt}-\Delta u+u=f(u)\,,\ \ (t,x)\in\mathbf{R}\times\mathbf{R}^n, \\ u(0,x)=u_0(x)\,,\ u_t(0,x)=u_1(x)\,,\ x\in\mathbf{R}^n, \end{array}\right. \tag{1.2}$$

which are related to the construction of the inverse wave operator $\mathcal{W}_-^{-1}$. When $n=3,\sigma=2$, then $\frac{n}{2}\frac{\sigma-1}{\sigma+1}\sigma=1$, so the $\mathbf{L}^{1+\sigma}-\mathbf{L}^{1+\frac{1}{\sigma}}$ time decay estimates of [13] can not be applied to the Cauchy problem (1.2) even if the nonlinearity is smooth. In the case of the Cauchy problem, the lower order σ were treated in papers [8] and [12], where the global existence of small solutions to quadratic nonlinear Klein-Gordon equations in three space dimensions was proved by the vector fields and the normal forms methods, respectively. However these methods do not work for the nonlinearity of the form $|u|\,u$. The vector field method was improved in paper [4],[5], where the global existence theorem was proved for the fractional order $\sigma>1+\frac{2}{n}$, in space dimensions $n=1,2,3$, if the initial data have a compact support. It seems that the method does not work for the data which do not have a compact support.

We put

$$w\equiv\frac{1}{2}\left(\mathbf{a}u+i\mathbf{b}\,\langle i\nabla\rangle^{-1}u_t\right),\ w^0\equiv\frac{1}{2}\left(\mathbf{a}u_0+i\mathbf{b}\,\langle i\nabla\rangle^{-1}u_1\right),$$

$$\mathcal{L}=E\partial_t+iA\,\langle i\nabla\rangle$$

and

$$\mathcal{N}(w)=\frac{i\mu}{2}\mathbf{b}\,\langle i\nabla\rangle^{-1}|(\mathbf{a}\cdot w)|^{\sigma-1}(\mathbf{a}\cdot w)\ \text{ or } \mathcal{N}(w)=\frac{i\mu}{2}\mathbf{b}\,\langle i\nabla\rangle^{-1}|(\mathbf{a}\cdot w)|^{\sigma},$$

where

$$\mathbf{a}=\begin{pmatrix}1\\1\end{pmatrix},\ \mathbf{b}=\begin{pmatrix}1\\-1\end{pmatrix},\ E=\begin{pmatrix}1&0\\0&1\end{pmatrix},\ A=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

Then the nonlinear Klein-Gordon equation (1.2) can be rewritten as a system of equations

$$\left\{\begin{array}{c} \mathcal{L}w=\mathcal{N}(w)\,,\ \ (t,x)\in\mathbf{R}\times\mathbf{R}^n, \\ w(0,x)=w^0(x)\,,\ x\in\mathbf{R}^n. \end{array}\right. \tag{1.3}$$

The direct Fourier transform $\hat{\phi}(\xi)$ of the function $\phi(x)$ is defined by

$$\mathcal{F}\phi=\hat{\phi}=(2\pi)^{-\frac{n}{2}}\int_{\mathbf{R}^n}e^{-i(x\cdot\xi)}\phi(x)\,dx,$$

then the inverse Fourier transformation is given by

$$\mathcal{F}^{-1}\phi=(2\pi)^{-\frac{n}{2}}\int_{\mathbf{R}^n}e^{i(x\cdot\xi)}\phi(\xi)\,d\xi.$$

Denote the usual Lebesgue space $\mathbf{L}^p=\{\phi\in\mathbf{S}';\|\phi\|_{\mathbf{L}^p}<\infty\}$, where the norm $\|\phi\|_{\mathbf{L}^p}=\left(\int_{\mathbf{R}^n}|\phi(x)|^p\,dx\right)^{\frac{1}{p}}$ if $1\le p<\infty$ and $\|\phi\|_{\mathbf{L}^\infty}=\text{vrai}\sup_{x\in\mathbf{R}^n}|\phi(x)|$ if $p=\infty$. Weighted Sobolev space is

$$\mathbf{H}_p^{m,k}=\left\{\phi\in\mathbf{S}':\|\phi\|_{\mathbf{H}_p^{m,k}}\equiv\left\|\langle x\rangle^k\langle i\partial\rangle^m\phi\right\|_{\mathbf{L}^p}<\infty\right\},$$

where $m,k\in\mathbf{R}$, $1\leq p\leq\infty$, $\langle x\rangle=\sqrt{1+|x|^2}$. We also write $\mathbf{H}^{m,k}=\mathbf{H}_2^{m,k}$. The usual Sobolev space is $\mathbf{H}^m=\mathbf{H}_2^{m,0}$, so the index 0 we usually omit if it does not cause a confusion. Different positive constants we denote by the same letter C.

We introduce the free evolution group

$$U(t)=\begin{pmatrix} e^{-i\langle i\nabla\rangle t} & 0 \\ 0 & e^{i\langle i\nabla\rangle t}\end{pmatrix}.$$

The operator

$$\mathcal{J}=\langle i\nabla\rangle U(t)\,xU(-t)=E\langle i\nabla\rangle x+iAt\nabla$$

is useful for obtaining the large time decay estimates of solutions. We have $[\mathcal{L},\mathcal{J}]=0$, since $[x,\langle i\nabla\rangle]=\langle i\nabla\rangle^{-1}\nabla$. However it is difficult to calculate the action of $\mathcal{J}$ on the nonlinearity $\mathcal{N}$. Therefore we use the first order differential operator

$$\mathcal{P}=Et\nabla+Ex\partial_t$$

which is closely related to $\mathcal{J}$ by $\mathcal{P}=\mathcal{L}x-i\mathcal{J}$, and it almost commutes with $\mathcal{L}$ since $[\mathcal{L},\mathcal{P}]=-i\langle i\nabla\rangle^{-1}\nabla\mathcal{L}$ (see [8]).

First we prove the existence of the inverse wave operator

$$\mathcal{W}_+^{-1}:\left(\mathbf{H}^{\beta,1}\right)^2\to\left(\mathbf{H}^{\beta,1}\right)^2,$$

where $\beta=\max\left(\frac{3}{2},1+\frac{2}{n}\right)$.

Theorem 1.1. *Let $1+\frac{4}{n+2}<\sigma<1+\frac{4}{n}$ and $n\geq 3$. Suppose that the initial data $w^0\in\left(\mathbf{H}^{\beta,1}\right)^2$, $\beta=\max\left(\frac{3}{2},1+\frac{2}{n}\right)$ have a small norm $\left\|w^0\right\|_{\mathbf{H}^{\beta,1}}$. Then there exists a unique solution $U(-t)\,w\in\mathbf{C}\left([0,\infty);\left(\mathbf{H}^{\beta,1}\right)^2\right)$ to the Cauchy problem (1.3) such that*

$$\|w(t)\|_{\mathbf{L}^q}\leq C(1+t)^{-\frac{n}{2}\left(1-\frac{2}{q}\right)}$$

for all $t\geq 0$, where $2\leq q<\frac{2n}{n-2}$. Furthermore there exists a unique final state $w^+\in\left(\mathbf{H}^{\beta,1}\right)^2$ such that

$$\left\|U(-t)\,w(t)-w^+\right\|_{\mathbf{H}^{\beta,1}}\leq C(1+t)^{-\gamma}\tag{1.4}$$

for all $t\geq 0$, where $\gamma=\frac{n}{2}(\sigma-1)\left(1-\frac{1}{q}\right)-1>0$.

We now consider the final state problem for the nonlinear Klein-Gordon equation

$$\begin{cases}\mathcal{L}w=\mathcal{N}(w), \\ \|w(t)-U(t)\,w^+\|_{\mathbf{L}^2}\to 0 \text{ as } t\to\infty\end{cases}\tag{1.5}$$

with a given final state $w^+\in\left(\mathbf{H}^{\beta,1}\right)^2$.

Theorem 1.2. *Let $1+\frac{4}{n+2}<\sigma<1+\frac{4}{n}$ and $n\geq 3$. Suppose that the final state $w^+\in\left(\mathbf{H}^{\beta,1}\right)^2$, $\beta=\max\left(\frac{3}{2},1+\frac{2}{n}\right)$. Then there exists a time $T\geq 0$ and a unique solution $U(-t)\,w\in\mathbf{C}\left([T,\infty);\left(\mathbf{H}^{\beta,1}\right)^2\right)$ of the final state problem (1.5) such that*

$$\|w(t)\|_{\mathbf{L}^q}\leq C(1+t)^{-\frac{n}{2}\left(1-\frac{2}{q}\right)}$$

for all $t\geq T$, where $2\leq q<\frac{2n}{n-2}$. Furthermore the asymptotics

$$\left\|U(-t)\,w(t)-w^+\right\|_{\mathbf{H}^{\beta,1}}\leq Ct^{-\gamma}$$

is valid for all $t \geq T$, *where* $\gamma = \frac{n}{2}(\sigma - 1)\left(1 - \frac{1}{q}\right) - 1 > 0$.

By Theorem 1.2, we can define the wave operator $\mathcal{W}_+$ which maps any final state $w^+ \in \left(\mathbf{H}^{\beta,1}\right)^2$ to the solution $U(-t)\,w \in \left(\mathbf{H}^{\beta,1}\right)^2$ if $t \geq T$. If we choose a sufficiently small norm $\|w^+\|_{\mathbf{H}^{\beta,1}}$, we can take $T = 0$. Namely, the wave operator

$$\mathcal{W}_+ : w^+ \in \left(\mathbf{H}^{\beta,1}\right)^2 \to w^0 \in \left(\mathbf{H}^{\beta,1}\right)^2$$

is well-defined in the neighborhood of the origin in the $\left(\mathbf{H}^{\beta,1}\right)^2$ space. Furthermore since the initial data w^0 are also sufficiently small in the norm of $\left(\mathbf{H}^{\beta,1}\right)^2$, by applying Theorem 1.1 for the negative time we can define the inverse wave operator

$$\mathcal{W}_-^{-1} : w^0 \in \left(\mathbf{H}^{\beta,1}\right)^2 \to w^- \in \left(\mathbf{H}^{\beta,1}\right)^2.$$

This means that the scattering operator

$$\mathcal{S}_+ = \mathcal{W}_-^{-1}\mathcal{W}_+ : w^+ \in \left(\mathbf{H}^{\beta,1}\right)^2 \to w^- \in \left(\mathbf{H}^{\beta,1}\right)^2$$

is well-defined in the neighborhood of the origin in the $\left(\mathbf{H}^{\beta,1}\right)^2$ space.

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