# TRAVELING WAVEFRONTS OF LATTICE DYNAMICAL SYSTEMS

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In this talk, we shall give a survey of the recent results on the traveling wavefronts of 1-D lattice dynamical systems.

Consider for an unknown $\{u_n(\cdot)\}_{n\in\mathbf{Z}}$:

$$\begin{aligned}\dot{u}_n(t) = d_{n+1}[u_{n+1}(t) - u_n(t)] - d_n[u_n(t) - u_{n-1}(t)] \\ +b_n[u_{n+1}(t) - u_{n-1}(t)] + f(n, u_n(t))\end{aligned} \tag{1}$$

for $n \in \mathbf{Z}$, $t > 0$, where $f$ is a nonlinear forcing term satisfying

$$f(n,0) = f(n,1) = 0,\ \forall n.$$

Equation (1) can be found in many biological models, see, e.g., the books of Fife [8], Shorrocks-Swingland [13], Shigesada-Kawasaki [12], etc. Equation (1) can also be regarded as a spatial-discrete version of the following parabolic partial differential equation

$$u_t = (d(x)u_x)_x + b(x)u_x + f(x,u). \tag{2}$$

We shall consider two cases:

$$\begin{aligned}&\text{monostable:}\quad f_u(x,0) > 0 > f_u(x,1),\\ &\text{bistable:}\quad f_u(x,0) < 0, f_u(x,1) < 0.\end{aligned}$$

We are interested in the wave propagation phenomenon in both homogeneous and heterogeneous media. In particular, we are interested in the traveling wavefront solutions.

## I. Monostable Nonlinearity: Homogeneous Media

In this case, we consider

$$\dot{u}_n(t) = [u_{n+1}(t) - 2u_n(t) + u_{n-1}(t)] + f(u_n(t)). \tag{3}$$

A solution of (3) is called a traveling wave with speed $c$, if there exists a function $U$ defined on $\mathbf{R}$ such that $u_n(t) = U(n + ct)$. Here $U$ is referred to as the wave profile. We are interested in solutions taking values in $[0,1]$ and connecting the steady states 0 and 1, i.e., traveling wave solutions $(c,U) \in \mathbf{R} \times C^1(\mathbf{R})$ of the problem (P):

$$\begin{cases} cU'(\cdot) = U(\cdot+1) + U(\cdot-1) - 2U(\cdot) + f(U(\cdot)) \text{ on } \mathbf{R},\\ U(-\infty) = 0,\ U(\infty) = 1,\ 0 \le U \le 1 \text{ on } \mathbf{R}.\end{cases}$$

**Theorem 1** (**Existence,** [4]). *Assume that*

$$(\mathbf{A}) : f \in C^1([0,1]),\ f(0) = f(1) = 0 < f(s)\ \forall\, s \in (0,1).$$

*Then there exists* $c_{\min} > 0$ *such that* (P) *admits a solution if and only if* $c \geq c_{\min}$.

This theorem was first proved by Zinner-Harris-Hudson [16] under the KPP-type assumption: $f(u) \leq f'(0)u\ \forall u \in [0,1]$.

**Theorem 2** (**Uniqueness**). *Assume* (**A**)*. Wave profiles of a given speed are unique up to a translation.*

**Theorem 3** (**Monotonicity**). *Assume* (**A**)*. Any wave profile is monotonic; i.e.* $U' > 0$ *on* $\mathbf{R}$.

The proof of uniqueness (Theorem 2) relies on the monotonicity (Theorem 3) and the detailed asymptotic behavior (Theorem 4 below) of wave profiles. For the details, we refer the reader to [2]. When $f'(0) > 0 > f'(1)$, see [4]. Note that, under the assumption (**A**), we only have $f'(0) \geq 0 \geq f'(1)$.

**Theorem 4.** *Assume* (**A**)*. Any solution* $(c, U)$ *of* (P) *satisfies*

$$\lim_{x\to-\infty} \frac{U'(x)}{U(x)} = \lambda, \quad \lim_{x\to\infty} \frac{U'(x)}{U(x)-1} = \mu, \tag{4}$$

*where* $\mu \leq 0 \leq \lambda$ *are roots of the characteristic equations:*

$$\begin{cases} c\,\lambda = e^{\lambda} + e^{-\lambda} - 2 + f'(0), \\ c\,\mu = e^{\mu} + e^{-\mu} - 2 + f'(1). \end{cases}$$

*In addition,* $\lambda$ *is the smaller root when* $c > c_{\min}$ *and the larger root when* $c = c_{\min}$.

For stability, we consider the system, which is embedded from (3) to a larger one, for unknown $\{u(x,\cdot)\}_{x\in\mathbf{R}}$:

$$u_t(x,t) = u(x+1,t) - 2u(x,t) + u(x-1,t) + f(u(x,t)),\ x \in \mathbf{R},\ t > 0. \tag{5}$$

For example, we may take $u(n+h,t) := u_n(t+h)$, $h \in [0,1)$, $n \in \mathbf{Z}$. Note that the traveling wave of (3) is the same as that of (5). For the stability, we assume the following:
**(A1)** $f \in C^{1+\alpha}([0,1])$ *for some* $\alpha \in (0,1]$.
**(A2)** *There exist constants* $M_f^- > 0$ *and* $M_f^+ \in \mathbf{R}$ *such that*

$$-M_f^- u^{1+\alpha} \leq f(u) - f'(0)u \leq M_f^+ u^{1+\alpha} \quad \forall u \in [0,1].$$

**(A3)** $f'(0) > 0$ *and* $f'(1) < 0$.
We denote by $\Lambda_1(c)$ the smaller root of the characteristic equation

$$c\lambda = e^{\lambda} + e^{-\lambda} - 2 + f'(0) \tag{6}$$

for $c > c_{\min}$. Then we have the following stability theorem.

**Theorem 5** ([3]). *Assume that $f$ satisfy* **(A), (A1)–(A3)**. *Let $u$ be the solution of (5) with initial value $u_o(\cdot)$ satisfying $u_o \in C(\mathbf{R} \to [0,1])$, $\liminf_{x\to\infty} u_o(x) > 0$ and $\lim_{x\to-\infty} u_o(x)e^{-\lambda x} = 1$ for some $\lambda \in (0, \Lambda_1(c_{\min}))$. Then*

$$\lim_{t\to\infty} \sup_{x\in\mathbf{R}} \left| \frac{u(x,t)}{U(x-ct)} - 1 \right| = 0 \tag{7}$$

*where $(c, U)$ is the traveling wave satisfies*

$$\lim_{\xi\to-\infty} U(\xi)e^{-\lambda\xi} = 1 \tag{8}$$

*with $\lambda = \Lambda_1(c)$.*

The stability theorem needs to have wave profiles with the asymptotic expansion as (8) near $x = -\infty$. Indeed, by constructing suitable sub/super solutions on $(-\infty, 1]$ having special tails near $x = -\infty$, under the additional condition that $f(u) = f'(0)u + O(u^{1+\alpha})$ for some $\alpha > 0$ and all small $u$, we show in [2] that:

*If $c > c_{\min}$ and $f'(0) > 0$, then for some $x_0 \in \mathbf{R}$*

$$\lim_{x\to-\infty} e^{-\lambda x}U(x + x_0) = 1 \tag{9}$$

*with $\lambda$ the smaller root of (6). By a translation, (8) holds.*

**II. Monostable Nonlinearity: Periodic Media**

We consider

$$u_j'(t) = d_{j+1}u_{j+1}(t) + d_j u_{j-1}(t) - (d_{j+1} + d_j)u_j(t) + f(j, u_j(t)), t \in \mathbf{R}, j \in \mathbf{Z}, \tag{10}$$

$$u_j(t + N/c) = u_{j-N}(t),\ t \in \mathbf{R},\ j \in \mathbf{Z}, \tag{11}$$

$$u_j(t) \to 1 \text{ as } j \to -\infty,\quad u_j(t) \to 0 \text{ as } j \to +\infty,\ \text{locally in } t \in \mathbf{R}, \tag{12}$$

where $d_j = d_{j-N}$ for all $j \in \mathbf{Z}$, $f(j, s) = f(j - N, s)$ for all $(j, s) \in \mathbf{Z} \times [0, 1]$, $N \in \mathbf{N}$, $c$ is a (positive) constant. Assume that $f : \mathbf{Z} \times [0, 1] \to \mathbf{R}$ is of class $C^1$ in $s$ for each $j \in \mathbf{Z}$ and

$$\begin{cases} \forall\ j \in \mathbf{Z}, \quad f(j, 0) = f(j, 1) = 0 \\ \forall\ j \in \mathbf{Z}, \quad f_s'(j, 0) := \partial f/\partial s(j, 0) > 0, \\ \forall\ (j, s) \in \mathbf{Z} \times (0, 1), \quad 0 < f(j, s) \le f_s'(j, 0)s, \\ \exists\ \alpha > 0, \exists\ \gamma \ge 0,\ \forall\ (j, s) \in \mathbf{Z} \times [0, 1], \\ \qquad f(j, s) \ge f_s'(j, 0)s - \gamma s^{1+\alpha}, \\ \exists\ \rho \in (0, 1),\ \forall\ j \in \mathbf{Z},\ \forall\ \rho \le s \le s' \le 1, \\ \qquad f(j, s) \ge f(j, s') \quad \text{(e.g., } f_s'(j, 1) < 0). \end{cases} \tag{13}$$

We always assume $0 \le u_j(t) \le 1$ for all $j \in \mathbf{Z}, t \in \mathbf{R}$.

**Theorem 6** ([9]). *A traveling wave solution of (10)-(12) with a speed $c$ exists if and only if $c \ge c^*$ for some $c^* > 0$. Furthermore, $u_j'(t) > 0$ and $u_j(-\infty) = 0 < u_j(t) < 1 = u_j(+\infty)$ for all $(j, t) \in \mathbf{Z} \times \mathbf{R}$.*

The sufficient condition for the existence of traveling wave for the periodic media with monostable nonlinearity was derived before by Hudson-Zinner [10]. We provide in [9] another totally different proof for the sufficient condition. Also, the necessary condition is proved in [9].

• At this moment, the uniqueness and stability of traveling wave for the periodic media with monostable nonlinearity are still open.

• Unlike the homogeneous case, the wave profile is not a single function for the periodic case. It consists $N$ functions. This makes the study of traveling wave in periodic media more complicated. On the other hand, periodic media is the most simple case for the heterogeneous media.

## III. Bistable Nonlinearity: Homogeneous Media

We consider

$$\dot{u}_n(t) = d[u_{n+1}(t) - 2u_n(t) + u_{n-1}(t)] + f(u_n(t)),$$

where $f$ is of bistable type. In [15], Zinner proved that there exists $d_* > 0$ such that a TW with $c \neq 0$ exists, if $d > d_*$. On the other hand, Keener [11] shows that only stationary wave exists, if $d$ is small. This is so-called the phenomenon of **propagation failure**. Moreover, Zinner [14] proved that if $d$ is large enough, then there exists a unique speed $c$ such that the wave profile is unique up to a shift in time. Moreover, this traveling wave is globally stable.

We also mention that Chow-(Mallet-Paret)-Shen studied 1-D lattice dynamical systems and fully discrete dynamical systems in [6, 7]. Finally, Bates-Chen-Chmaj [1] also consider the 1-D lattice dynamical systems which allows infinite-range couplings with both positive and negative weights.

## IV. Bistable Nonlinearity: Periodic Media

In [5], we consider

$$\dot{u}_j(t) = \sum_k a_{j,k}\, u_{j+k}(t) + f_j(u_j(t)), \quad t \in \mathbf{R}, j \in \mathbf{Z}. \tag{14}$$

We are looking for a traveling wave solution of (14) with speed $c$ such that

$$\lim_{j\to\infty} u_j(t) = 1, \ \lim_{j\to-\infty} u_j(t) = 0, \tag{15}$$

$$u_j(t + N/c) = u_{j-N}(t) \quad (c \neq 0). \tag{16}$$

By a spectrum analysis of the linearized operator around a steady state, and the construction of suitable super/subsolutions, we are able to derive the uniqueness and asymptotic stability of traveling wave.

The most difficult part is to derive the existence of traveling wave. For this, we introduce (for $c \neq 0$)

$$w_j(x) := u_j(t)\Big|_{ct=j-x} = u_j\left(\frac{j-x}{c}\right) \forall\, x \in \mathbf{R}, j \in \mathbf{Z}.$$

Then we have $w_{j+N}(x) = w_j(x)\ \forall\, x \in \mathbf{R}, j \in \mathbf{Z}$. Also, by (15) and (16),

$$\lim_{x\to\infty} w_j(x) = \lim_{ct\to-\infty} u_j(t) = 1, \quad \lim_{x\to-\infty} w_j(x) = \lim_{ct\to\infty} u_j(t) = 0.$$

Finally, (14) becomes

$$-c\, w_j'(x) = \sum_k a_{j,k} w_{j+k}(x+k) + f_j(w_j(x)). \tag{17}$$

Note that the constant (speed) $c$ in (17) can be zero. Working on the corresponding integral formulation, we are able to derive the existence of traveling wave solution. We refer the reader to [5] for more details.

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