Weak Solutions to a Model for Interface Motion by Interface Diffusion

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1 Introduction

This talk is based on a recent work by Prof. H.-D. Alber and myself, and concerned with a phase field model for the evolution of an interface in an elastically deformable solid, which moves by diffusion of atoms along this interface. This study was started in [1] and continued in [2, 3].

The interface separates the body in two regions consisting of atoms of different types and having different elastic properties. No exchange of atoms across the interface occurs, the volumes of the different regions are therefore conserved in time. We call these regions phases. The diffusion of the atoms along the interface is only driven by bulk terms of the free energy, surface terms are neglected.

These properties of the model carry over from those of a related sharp interface model. The phases in the phase field model are characterized by an order parameter, whose evolution is governed by a non-uniformly parabolic equation of fourth order. This equation is formulated in [2] following ideas explained in [1, 2, 3], which suggest that when a certain regularizing parameter ν in this equation tends to zero, then solutions of the model equations converge to solutions of a sharp interface model for interface motion by interface diffusion. In this sharp interface model the normal speed is proportional to the value obtained by application of the surface Laplacian to the jump of the Eshelby tensor across the interface.

The notations are as follows: Ω is an open subset in $\mathbb{R}^3$, it represents a set of material points of a solid body. At the point $x \in \Omega$ at time t the material is in phase 1 or 2 if the value $S(t,x) \in \mathbb{R}$ of the order parameter S is near to zero or one. The other unknowns are the displacement $u(t,x) \in \mathbb{R}^3$ and the Cauchy stress tensor $T(t,x) \in \mathcal{S}^3$ of the point x at t. Here $\mathcal{S}^3$ is the set of symmetric 3×3–matrices. For $(t,x) \in (0,\infty) \times \Omega$, the unknowns satisfy the quasi-static equations

$$-\mathrm{div}_x\, T(t,x) = b(t,x), \tag{1.1}$$

$$T(t,x) = D\big(\varepsilon(\nabla_x u) - \bar\varepsilon S\big)(t,x), \tag{1.2}$$

$$S_t(t,x) = c\,\mathrm{div}_x \Big(\nabla_x\big(\psi_S(\varepsilon(\nabla_x u), S) - \nu\Delta_x S\big)|\nabla_x S|\Big)(t,x) \tag{1.3}$$

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and we prescribe the boundary and initial conditions

$$u(t,x) = \gamma(t,x), \qquad (t,x) \in [0,\infty) \times \partial\Omega, \tag{1.4}$$

$$\frac{\partial}{\partial n} S(t,x) = 0, \qquad (t,x) \in [0,\infty) \times \partial\Omega, \tag{1.5}$$

$$\frac{\partial}{\partial n} \big(\psi_S(\varepsilon, S) - \nu \Delta_x S\big) \, |\nabla_x S|(t,x) = 0, \qquad (t,x) \in [0,\infty) \times \partial\Omega, \tag{1.6}$$

$$S(0,x) = S_0(x), \qquad x \in \bar{\Omega}. \tag{1.7}$$

Here n is the unit outward normal vector, $\nabla_x u$ denotes the 3×3-matrix of first order derivatives of u which is the deformation gradient. And

$$\varepsilon\,(\nabla_x u) = \frac{1}{2}\big(\nabla_x u + (\nabla_x u)^T\big)$$

is the strain tensor, where $(\nabla_x u)^T$, $\bar{\varepsilon} \in \mathcal{S}^3$ are the transposed matrix, a given matrix (the transformation strain), respectively. The elasticity tensor $D : \mathcal{S}^3 \to \mathcal{S}^3$ is a linear, symmetric, positive definite mapping. $\psi_S = \frac{\partial}{\partial S}\psi$ is the partial derivative of the free energy

$$\psi(\varepsilon, S) = \frac{1}{2}\big(D(\varepsilon - \bar{\varepsilon} S)\big) \cdot (\varepsilon - \bar{\varepsilon} S) + \hat{\psi}(S), \tag{1.8}$$

where $\hat{\psi} : \mathbb{R} \to [0,\infty)$ is chosen as a double well potential with minima at points $S = 0$ and $S = 1$, and the scalar product of two matrices is denoted by $A \cdot B = \sum a_{ij} b_{ij}$. Thus,

$$\psi_S(\varepsilon, S) = -T \cdot \bar{\varepsilon} + \hat{\psi}'(S). \tag{1.9}$$

Given are the positive constant c, the small positive constant ν, the volume force $b : [0,\infty) \times \Omega \to \mathbb{R}^3$ and the boundary and initial data $\gamma : [0,\infty) \times \partial\Omega \to \mathbb{R}^3$, $S_0 : \Omega \to \mathbb{R}$.

This completes the formulation of an initial-boundary value problem. Equations (1.1) and (1.2), which differ from the system of linear elasticity only by the term $\bar{\varepsilon} S$, determine the elastic properties of the two phases characterized by the values $S \approx 0$ or $S \approx 1$: in the first phase the material is stress free at the strain state $\varepsilon(\nabla_x u) = 0$, in the other phase at $\varepsilon(\nabla_x u) = \bar{\varepsilon}$. We assume that D has the same value at both phases, but it would be important for applications to study the case where D is a function of S with $D[0] \neq D[1]$. Equation (1.3) for S is non-uniformly parabolic because of the term $\operatorname{div}_x\big(\nabla_x(\nu \Delta S)\,|\nabla_x S|\big)$. Up to now, we can only prove existence of weak solutions for the initial-boundary value problem in $1\frac{1}{2}$–space dimensions.

Statement of the main result. In the sharp interface model (2.1) – (2.5) the normal speed of the interface determined by equation (2.3) is proportional to the value obtained by application of the surface Laplacian to the jump in the Eshelby tensor. Therefore, this model and also the regularized model (1.1) – (1.7) is not of interest in a strictly one-dimensional situation, where all unknowns only depend on the first component x_1 of $x \in \mathbb{R}^3$ and of t, since in this case the normal speed of a planar interface $\tilde{\Gamma}(t) = \{(h(t), x_2, x_3) \mid (x_2, x_3) \in \mathbb{R}^2\}$ would be equal to zero, hence $h(t) = \text{const}$. In this talk we thus consider a $1\frac{1}{2}$–dimensional problem. To formulate such a problem, we choose $\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid a < x_1 < d\}$, and assume that

$$b(t,x) = \tilde{b}(t, x_1) + b_1(t, x_2, x_3), \tag{1.10}$$

$$\gamma(t,x) = \tilde{\gamma}(t, x_1) + \gamma_1(t, x_1, x_2, x_3), \tag{1.11}$$

where b_1, γ_1 satisfy the elliptic boundary value problem

$$\begin{aligned} -\operatorname{div}_x \sigma(t,x) &= b_1(t,x_2,x_3), && (1.12)\\ \sigma(t,x) &= D(\varepsilon(\nabla_x v(t,x))), && (1.13)\\ v(t,x) &= \gamma_1(t,x), \quad x\in\partial\Omega && (1.14)\end{aligned}$$

of linear elasticity, which has a solution $x \mapsto (v(t,x), \sigma(t,x)) : \Omega \to \mathbb{R}^3 \times \mathcal{S}^3$ such that

$$\partial_{x_1} \sigma(t,x)\cdot\bar\varepsilon = 0, \quad \nabla_x(\partial^2_{x_2} + \partial^2_{x_3})\,(\sigma(t,x)\cdot\bar\varepsilon) = 0, \quad x\in\Omega. \tag{1.15}$$

It follows that $(\partial^2_{x_2} + \partial^2_{x_3})\,(\sigma\cdot\bar\varepsilon)$ is independent of x. We thus define

$$r(t) = (\partial^2_{x_2} + \partial^2_{x_3})\,(\sigma(t,x)\cdot\bar\varepsilon) \in \mathbb{R}. \tag{1.16}$$

Examples for $b_1, \gamma_1, v, \sigma, \bar\varepsilon$ with theses properties can be readily constructed. In particular, examples can be given with $b_1 = 0$. Of course, if $r \neq 0$, then (v, σ) will be unbounded for $|(x_2, x_3)| \to \infty$. For the solution of problem (1.1) – (1.7), (1.10) – (1.11), we make the ansatz

$$(u,T,S)(t,x) = (\tilde u, \tilde T, \tilde S)(t,x_1) + (v,\sigma,0)(x).$$

Noting that (1.9) and (1.15) imply

$$\psi_S(\varepsilon(\nabla_x u), S)_{x_1} = (-\tilde T\cdot\bar\varepsilon - \sigma\cdot\bar\varepsilon + \hat\psi'(S))_{x_1} = (-\tilde T\cdot\bar\varepsilon + \hat\psi'(S))_{x_1} = \psi_S(\varepsilon(\nabla_x \tilde u), S)_{x_1}\,,$$

we obtain by insertion of this ansatz into (1.1) – (1.7) an initial-boundary value problem for $(\tilde u, \tilde T, \tilde S)$ in one space dimension. We simplify the notation and denote $(\tilde u, \tilde T, \tilde S)$ and $\tilde b$ again by (u,T,S) and by b, respectively. We write x for x_1, let $\Omega = (a,d)$, set $Q_{T_e} = (0,T_e)\times\Omega$, where a, d, T_e (time of existence) are positive constants, and denote

$$\varepsilon(u_x) = \frac{1}{2}\left((u_x,0,0) + (u_x,0,0)^T\right) \in \mathcal{S}^3.$$

With these notations $(u,T,S) : Q_{T_e} \to \mathbb{R}^3\times\mathcal{S}^3\times\mathbb{R}$ must satisfy the equations

$$\begin{aligned} -T_{1x} &= b, && (1.17)\\ T &= D(\varepsilon(u_x) - \bar\varepsilon S), && (1.18)\\ S_t &= c\big((\psi_S(\varepsilon(u_x),S) - \nu S_{xx})_x |S_x|\big)_x + cr(t)|S_x|\,, && (1.19)\end{aligned}$$

and the boundary and initial conditions

$$\begin{aligned} u|_{[0,T_e]\times\partial\Omega} &= 0, && (1.20)\\ S_x|_{[0,T_e]\times\partial\Omega} &= 0, && (1.21)\\ \big((\psi_S - \nu S_{xx})_x|S_x|\big)|_{[0,T_e]\times\partial\Omega} &= 0, && (1.22)\\ S|_{\{0\}\times\Omega} &= S_0. && (1.23)\end{aligned}$$

Here $T_1(t,x)$ denotes the first column of the matrix $T(t,x)$. In (1.20) we assumed, without restriction of generality, that the function $\tilde\gamma$ from (1.11) vanishes since equations (1.17), (1.18) are linear. Given are $b(t,x)\in\mathbb{R}^3$ and $S_0(x)\in\mathbb{R}$. Since $r(t)\in\mathbb{R}$ defined in (1.16) can be computed by solving the boundary value problem (1.12) – (1.14), also $r : [0,T_e) \to \mathbb{R}$ can be considered to be given.

Equations (1.17) – (1.19) constitute the initial-boundary value problem in one space dimension, which models a planar interface propagating with speed $cr(t)$. To state the main result for this problem we need some notations and definitions.

For a subset $\mathcal{A}$ of Q_{T_e}, for a function $g : \mathcal{A} \to V$ with values in some set V and for $t \in [0, T_e]$ let

$$\mathcal{A}(t) = \{x \mid (t,x) \in \mathcal{A}\} \subseteq \mathbb{R} \quad \text{and} \quad g(t) : \mathcal{A}(t) \to V,\ g(t)(x) = g(t,x).$$

We show that the component S in a solution (u, T, S) of the initial-boundary value problem has the weak derivative S_{xxx}, however on a set $\mathcal{A} \subseteq Q_{T_e}$, for which $\mathcal{A}(t)$ is open for almost all t, but which itself is not open in $\mathbb{R}^2$. We now define these general derivatives.

Definition 1.1 *Let $\mathcal{A} \subset Q_{T_e}$ such that $\mathcal{A}(t)$ is open for almost all $t \in [0, T_e]$, and let $\alpha \in \mathbb{N}_0$. We call $g : \mathcal{A} \to \mathbb{R}$ the α-th local weak L^2–derivative of $S \in L^2(Q_{T_e})$ with respect to x in $\mathcal{A}$, if for almost all $t \in [0, T_e]$ the function $g(t)$ belongs to $L^{2,\mathrm{loc}}(\mathcal{A}(t))$ and is the local weak derivative of S in the usual sense:*

$$g(t) = \partial_x^\alpha S(t)|_{\mathcal{A}(t)}, \tag{1.24}$$

and if moreover there exists a sequence $\{\mathcal{A}_n\}_n$ of measurable sets $\mathcal{A}_n \subset \mathcal{A}$ with $g|_{\mathcal{A}_n} \in L^2(\mathcal{A}_n)$ for all $n \in \mathbb{N}$, such that

$$\text{meas}\ \big(\mathcal{A} \setminus \bigcup_{n=1}^{\infty} \mathcal{A}_n\big) = 0.$$

Local weak derivatives in the sense of this definition are unique because of (1.24), and it is immediately seen that if $\mathcal{A}$ is open then the local weak derivative in the sense of this definition coincides with the ordinary weak derivative. For $S \in L^2(0, T_e; H^2(\Omega))$ let

$$\mathcal{A}^S = \{(t,x) \in Q_{T_e} \mid |S_x(t,x)| > 0\}.$$

Since by the Sobolev embedding theorem $S_x(t)$ is continuous for almost all t, it follows that $\mathcal{A}^S(t)$ is open for almost all t.

Definition 1.2 *Let $b \in L^\infty(0, T_e; L^2(\Omega))$, $r \in L^\infty(0, T_e)$ and $S_0 \in L^2(\Omega)$. A function (u, T, S) with*

$$u \in L^2(0, T_e; H^2(\Omega)), \qquad u(t) \in H_0^1(\Omega)\ \text{a.e. in}\ (0, T_e), \tag{1.25}$$

$$T \in L^2(0, T_e; H^1(\Omega)), \tag{1.26}$$

$$S \in L^2(0, T_e; H^2(\Omega)) \cap L^\infty(Q_{T_e}), \qquad S_x(t) \in H_0^1(\Omega)\ \text{a.e. in}\ (0, T_e), \tag{1.27}$$

is a weak solution of the problem (1.17) – (1.23), if (u, T, S) solves (1.17), (1.18) *weakly, if S has the local weak derivative S_{xxx} in $\mathcal{A}^S$ with $|S_x| S_{xxx} \in L^1(\mathcal{A}^S)$ and if*

$$\begin{aligned}(S, \varphi_t)_{Q_{T_e}} + c(\nu S_{xxx}|S_x|, \varphi_x)_{\mathcal{A}^S} + c\Big(\Big(T \cdot \bar{\varepsilon} - \hat{\psi}'(S)\Big)_x |S_x|, \varphi_x\Big)_{Q_{T_e}} \\ + (cr|S_x|, \varphi)_{Q_{T_e}} = -(S_0, \varphi(0))_\Omega\end{aligned} \tag{1.28}$$

holds for all $\varphi \in C_0^\infty((-\infty, T_e) \times \mathbb{R})$.

The main result of this talk is

Theorem 1.3 *Assume that there exists a constant* $M > 0$ *such that the double well potential* $\hat{\psi} \in C^3(\mathbb{R}, [0,\infty))$ *satisfies*

$$\max\left\{\hat{\psi}'(S)^2, S^2\right\} \leq M(\hat{\psi}(S) + 1). \tag{1.29}$$

Then to all $S_0 \in H^1(\Omega)$, $r \in L^\infty(0,T_e)$ *and* $b \in L^2(Q_{T_e})$ *with* $b_t \in L^2(Q_{T_e})$ *there exists a weak solution* (u,T,S) *of* (1.17) *–* (1.23)*, which in addition to (1.25) – (1.28) satisfies*

$$u \in L^\infty(0,T_e;H^2(\Omega)), \qquad T \in L^\infty(0,T_e;H^1(\Omega)) \tag{1.30}$$

$$S \in L^\infty(0,T_e;H^1(\Omega)), \qquad S_t \in L^{\frac{4}{3}}(0,T_e;W^{-1,\frac{4}{3}}(\Omega)), \tag{1.31}$$

$$|S_x|S_{xxx} \in L^{\frac{4}{3}}(Q_{T_e}), \tag{1.32}$$

where we defined $|S_x|S_{xxx} = 0$ *on* $Q_{T_e} \setminus \mathcal{A}^S$ *.*

The ideas of the proof of this theorem is as follows: to replace the degenerate parabolic equation (1.19) by the non-degenerate equation

$$S_t = c\Big(\,(\psi_S - \nu S_{xx})_x\,(|S_x|_\kappa + \kappa)\Big)_x + cr|S_x|_\kappa\,, \tag{1.33}$$

where $|y|_\kappa = |y|^2/\sqrt{|y|^2+\kappa}$, with a constant $\kappa > 0$, and to use some famous lemmas, i.e. the Aubin–Lions lemma and the Egolov lemma to investigate the limits of the approximate solution of (1.17) – (1.23) as $\kappa \to 0$.

2 The sharp interface problem

To state the sharp interface model, we let the interface be given by a sufficiently smooth three-dimensional manifold $\tilde{\Gamma}$ in $[t_1,t_2] \times \Omega \subset \mathbb{R}^4$ such that for all $t \in [t_1,t_2]$

$$\tilde{\Gamma}(t) = \left\{x \in \Omega \mid (t,x) \in \tilde{\Gamma}\right\}$$

is a two-dimensional manifold. The two different phases are characterized by the values of a discontinuous order parameter S, which has the constant values 0 and 1 in the regions separated by the phase interface, and which jumps along the interface. The sharp interface problem, which determines the unknown position of the interface and the unknown functions u, T, consists of the equations

$$-\mathrm{div}_x T = b, \tag{2.1}$$

$$T = D(\varepsilon(\nabla_x u) - \bar{\varepsilon}S), \tag{2.2}$$

$$s[S] = -c\,\Delta_{\tilde{\Gamma}(t)}(n \cdot [C]n), \tag{2.3}$$

$$[u] = 0, \tag{2.4}$$

$$[T]n = 0, \tag{2.5}$$

and of suitable initial and boundary conditions. (2.1) and (2.2) must hold on $([t_1,t_2] \times \Omega)\backslash\tilde{\Gamma}$, the jump conditions (2.3) – (2.5) are given on $\tilde{\Gamma}$. Here $n(t,x) \in \mathbb{R}^3$ is the unit normal vector to $\tilde{\Gamma}(t)$ at $x \in \tilde{\Gamma}(t)$ pointing into the region where $S = 1$, and $s(t,x) \in \mathbb{R}$

is the normal speed of $\tilde{\Gamma}(t)$ at $x \in \tilde{\Gamma}(t)$ in direction $n(t,x)$. Also, $\Delta_{\tilde{\Gamma}(t)}$ is the surface Laplacian on $\tilde{\Gamma}(t)$, and $[u],[T],[S],[C]$ denote the jumps of u,T,S and of the Eshelby tensor

$$C(\nabla_x u, S) = \psi(\varepsilon(\nabla_x u), S)I - (\nabla_x u)^T T$$

across $\tilde{\Gamma}$, where I is the 3×3–unit matrix and ψ is the free energy given in (1.8). We use the notation $(\nabla_x u)^T T$ to denote the matrix product.

The evolution law (2.3) describes motion of the interface $\tilde{\Gamma}(t)$ due to diffusion of atoms along the interface. The flux is given by $-c\nabla_{\tilde{\Gamma}(t)}(n\cdot[C]n)$ with the surface gradient $\nabla_{\tilde{\Gamma}(t)}$. There is no exchange of atoms between the phases, hence the volume $\int_\Omega S(x,t)dx$ of one of the phases is conserved in time. The evolution law is derived in the standard way by application of the second law of thermodynamics under the assumption that the free energy is given by $\Psi(t) = \int_\Omega \psi(\varepsilon, S)dx$ and thus contains only bulk terms: the Clausius-Duhem inequality must be satisfied, which for this free energy leads to the flux term given above. For this derivation we refer to [1], where the application of the second law of thermodynamics to an interface problem is discussed with mathematical rigour.

If one assumes more generally that the free energy is a sum of bulk and surface terms

$$\Psi(t) = \alpha_1 \int_\Omega \psi(\varepsilon(\nabla_x u(t,x)), S(t,x))dx + \alpha_2 \int_{\tilde{\Gamma}(t)} d\sigma$$

with $\alpha_1, \alpha_2 \geq 0$, then the evolution law obtained is

$$s[S] = -c\,\Delta_{\tilde{\Gamma}(t)} \left(\alpha_1(n\cdot[C]n) + \alpha_2\kappa_{\tilde{\Gamma}(t)}\right), \tag{2.6}$$

where $\kappa_{\tilde{\Gamma}(t)}$ is the mean curvature of $\tilde{\Gamma}(t)$. Then we obtain several variations of our model, however we omit the details here due to the limitation of length.

Second law of thermodynamics. The second law requires that there exist a free energy ψ^* and a flux q such that the Clausius-Duhem inequality $\frac{\partial}{\partial t}\psi^* + \operatorname{div}_x q \leq b\cdot u_t$ holds. With ψ given in (1.8) we choose

$$\psi^*(\varepsilon, S, \nabla_x S) = \psi(\varepsilon, S) + \frac{\nu}{2}|\nabla_x S|^2, \tag{2.7}$$

$$\begin{aligned} &q(u_t, S_t, \varepsilon, \nabla_x\varepsilon, S, \ldots, \nabla_x^3 S) \\ &\qquad = -T\cdot u_t - \nu S_t\cdot\nabla_x S - c(\psi_S - \nu\Delta_x S)\nabla_x(\psi_S - \nu\Delta_x S)\,|\nabla_x S|, \end{aligned}$$

and apply (1.1) and the relation $\nabla_\varepsilon\psi\cdot\varepsilon_t = T\cdot u_t$, which holds by (1.2) and the symmetry of T, to obtain after a short computation

$$\begin{aligned} &\frac{\partial}{\partial t}\psi^* + \operatorname{div}_x q - b\cdot u_t \\ &\qquad = (\psi_S - \nu\Delta_x S)S_t - c\,\operatorname{div}_x\big((\psi_S - \nu\Delta_x S)\,\nabla_x(\psi_S - \nu\Delta_x S)|\,\nabla_x S|\big). \end{aligned}$$

Insertion of (1.3) into this equation results in

$$\frac{\partial}{\partial t}\psi^* + \operatorname{div}_x q - b\cdot u_t = -c\,|\nabla_x(\psi_S - \nu\Delta_x S)|^2|\nabla_x S| \leq 0,$$

which shows that the second law holds for the system (1.1) – (1.3).

3 Main Steps of the Proof of Theorem 1.3

The proof of Theorem 1.3 consists of the following two steps.
Step 1. To construct approximate solutions to (1.17) – (1.23) by solving the following quasilinear, uniformly parabolic initial-boundary value problem

$$
\begin{aligned}
-T_{1x} &= b, && (3.1)\\
T &= D(\varepsilon(u_x) - \bar{\varepsilon}S), && (3.2)\\
S_t &= c\Big(\big((\psi_S(\varepsilon(u_x), S) - \nu S_{xx}\big)_x(|S_x|_\kappa + \kappa)\Big)_x + cr|S_x|_\kappa\,, && (3.3)\\
u &= 0, \qquad \text{on } [0, T_e] \times \partial\Omega, && (3.4)\\
S_x &= 0, \qquad \text{on } [0, T_e] \times \partial\Omega, && (3.5)\\
(\psi_S - \nu S_{xx})_x &= 0, \qquad \text{on } [0, T_e] \times \partial\Omega, && (3.6)\\
S(0, x) &= S_0(x), \quad x \in \Omega\,, && (3.7)
\end{aligned}
$$

with a fixed positive parameter κ. By definition, $(u, T, S) \in L^2(0, T_e; H^1(\Omega)^3)$ with $S_{xxx} \in L^2(Q_{T_e})$ is a weak solution of (3.1) – (3.7) if (3.1), (3.2), (3.4) (3.5) are satisfied weakly and if for all $\varphi \in C_0^\infty((-\infty, T_e) \times \mathbb{R})$

$$
\begin{aligned}
-(S, \varphi_t)_{Q_{T_e}} &= (S_0, \varphi(0))_\Omega \\
&+ c\big((|S_x|_\kappa + \kappa)(\nu S_{xxx} - \psi_{S,x}), \varphi_x\big)_{Q_{T_e}} + c(r|S_x|_\kappa\,, \varphi)_{Q_{T_e}}\,. \qquad (3.8)
\end{aligned}
$$

We have

Theorem 3.1 *Assume that $S_0 \in H^1(\Omega)$, $r \in L^\infty(0, T_e)$ and $b \in L^2(Q_{T_e})$ with $b_t \in L^2(Q_{T_e})$. Then there is a weak solution (u, T, S) of (3.1) – (3.7) with $S \in L^2(0, T_e; H^3(\Omega)) \subseteq L^2(0, T_e; C^{2+\alpha}(\bar{\Omega}))$, $\alpha < \frac{1}{2}$, and with $S_t \in L^{\frac{4}{3}}(0, T_e; W^{-1,\frac{4}{3}}(\Omega))$.*

The proof is divided into two steps, i.e. Lemmas 3.2 and 3.3: firstly, to construct a sequence of approximate solutions, secondly, to derive uniform a-priori bounds for the solutions, then to pass to limit. The approximate problem consists of the equations and initial boundary conditions

$$
\begin{aligned}
-T_{1x} &= b, && (3.9)\\
T &= D(\varepsilon(u_x) - \bar{\varepsilon}S), && (3.10)\\
S_t &= c\Big((\,|\widetilde{\hat{S}_x}|_\kappa + \kappa)(\psi_S(\varepsilon(u_x), S) - \nu S_{xx})_x\Big)_x + cr|S_x|_\kappa\,, && (3.11)
\end{aligned}
$$

and of the boundary and initial conditions (3.4) – (3.7). Here $\hat{S} \in L^2(0, T_e; H^2(\Omega))$ is a given function and

$$
\widetilde{\hat{S}_x}(t, x) = (\chi_\eta * S_x)(t, x) = \int_{Q_{T_e}} \chi_\eta(t - \tau, x - y)\, \hat{S}_x(\tau, y)\, d(\tau, y)\,, \qquad (3.12)
$$

with the standard mollifier $\chi_\eta \in C_0^\infty(\{x \in \mathbb{R}^2 \mid |x| \le \eta\})$.

Lemma 3.2 *(Existence) Let $0 < \alpha < \frac{1}{2}$. To every $\hat{S} \in L^2(Q_{T_e})$, $b \in C^{\frac{\alpha}{4},\alpha}(\bar{Q}_{T_e})$, $r \in C^{\frac{\alpha}{4}}([0, T_e])$ and $S_0 \in C^{4+\alpha}(\bar{\Omega})$ there is a unique solution (u, T, S) of the initial-boundary value problem* (3.9) – (3.11), (3.4) – (3.7)*. This solution belongs to the space*

$$
L^\infty(0, T_e; C^{2+\alpha}(\bar{\Omega})) \times L^\infty(0, T_e; C^{1+\alpha}(\bar{\Omega})) \times C^{1+\frac{\alpha}{4},4+\alpha}(\bar{Q}_{T_e})
$$

and satisfies $S_{xxt} \in L^2(Q_{T_e})$.

Lemma 3.3 *(A-priori estimates) There is a constant $\bar{C}$ independent of η and κ but depending on K, and another constant C independent of η, such that for every $\hat{S} \in L^2(0,T_e;H^2(\Omega))$, all (b,r,S_0) satisfying suitable conditions and for any $t \in [0,T_e]$*

$$\begin{aligned} \|S(t)\|_{H^1(\Omega)} + \|S\|_{L^\infty(Q_{T_e})} &\leq \bar{C}, \quad (3.13) \\ \|u(t)\|_{H^2(\Omega)} + \|T(t)\|_{H^1(\Omega)} + \|u\|_{L^\infty(Q_{T_e})} + \|T\|_{L^\infty(Q_{T_e})} &\leq \bar{C}, \quad (3.14) \\ \|(\widetilde{|\hat{S}_x|_\kappa} + \kappa)^{\frac{1}{2}} S_{xxx}\|_{L^2(Q_{T_e})} + \|S_{xx}\|_{L^2(Q_t)} + \|S_t\|_{L^{\frac{4}{3}}(0,T_e;W^{-1,\frac{4}{3}}(\Omega))} &\leq C. \quad (3.15) \end{aligned}$$

Step 2. We need the following a-priori estimates independent of κ and a lemma concerning the limit of the third order derivative. In this step, we make use of Egorov's theorem, the Aubin-Lions Lemma etc.

Lemma 3.4 *For any $t \in [0,T_e]$ there hold*

$$\|S^\kappa_{xx}\|_{L^2(Q_{T_e})} + \|(|S^\kappa_x|_\kappa + \kappa)^{\frac{1}{2}} S^\kappa_{xxx}\|_{L^2(Q_{T_e})} \leq \bar{C}, \tag{3.16}$$

$$\|\, |S^\kappa_x|_\kappa S^\kappa_{xxx}\|_{L^{\frac{4}{3}}(Q_t)} + \|\partial_t S^\kappa\|_{L^{\frac{4}{3}}(0,T_e;W^{-1,\frac{4}{3}}(\Omega))} \leq \bar{C}. \tag{3.17}$$

Lemma 3.5 *The limit function S has the local weak L^2–derivative S_{xxx} on $\mathcal{A}^S$ in the sense of Definition 1.1. Moreover, there exists a subsequence S^κ such that*

$$|S^\kappa_x|_\kappa S^\kappa_{xxx} \rightharpoonup \chi, \quad \text{weakly in } L^{\frac{4}{3}}(Q_{T_e}), \tag{3.18}$$

where the function $\chi \in L^{\frac{4}{3}}(Q_{T_e})$ is given by

$$\chi(t,x) = \begin{cases} 0, & \text{if } S_x(t,x) = 0 \\ |S_x| S_{xxx}, & \text{if } S_x(t,x) \neq 0. \end{cases} \tag{3.19}$$

References

[1] Alber, H.-D. (2000) *Evolving microstructure and homogenization,* Continuum. Mech. Thermodyn. **12**, 235–286

[2] Alber, H.-D. and Zhu, P. (2004) *Evolution of phase boundaries by configurational forces,* Preprint Depart. Math., Technische Universität Darmstadt **2329**, to appear in Archive Rat. Mech. Anal. (2007)

[3] Alber, H.-D., and Zhu, P. (2006) *Solutions to a model with nonuniformly parabolic terms for phase evolution driven by configurational forces,* SIAM J. Appl. Math. **66**, No. 2, 680–699

[4] Alber, H.-D. and Zhu, P. (2007) *Solutions to a Model for Interface Motion by Interface Diffusion,* Preprint Depart. Math., Technische Universität Darmstadt **2491**. Submitted

[5] Cahn, J. and Taylor, J. (1994) *Surface motion by surface diffusion*, Acta Metall. Mater. **42** No. 4, 1045–1063