ON THE EXISTENCE OF BURGERS VORTICES FOR HIGH REYNOLDS NUMBERS

YASUNORI MAEKAWA[1]

DEPARTMENT OF MATHEMATICS, HOKKAIDO UNIVERSITY

SAPPORO, 060-0810, JAPAN.

ABSTRACT. Axisymmetric or non-axisymmetric Burgers vortices have been studied numerically as a model of concentrated vorticity fields. Recently, it is rigorously proved that non-axisymmetric Burgers vortices exist for all values of the vortex Reynolds number if an asymmetric parameter is sufficiently small. On the other hand, several numerical results suggest that Burgers vortices have simpler structures if the vortex Reynolds number is large, even when the asymmetric parameter is not small. In this paper we give a rigorous explanation for this numerical observation and extend the existence results for high vortex Reynolds numbers.

1. INTRODUCTION

In 1948 Burgers [1] found an exact solution to the three dimensional stationary Navier-Stokes equations for viscous incompressible fluids as follows. We consider a two dimensional perturbation of a background straining flow whose velocity is of the form:

$$(1.1) \qquad U(x_1, x_2, x_3) = u_s(x_1, x_2, x_3) + u(x_1, x_2),$$

where u_s represents a given background straining flow with a nonnegative asymmetric parameter λ, and u is an unknown two dimensional perturbation, i.e.,

$$(1.2) \qquad u_s(x_1, x_2, x_3) = (-\frac{1+\lambda}{2}x_1, \ -\frac{1-\lambda}{2}x_2, \ x_3),$$

$$(1.3) \qquad u(x_1, x_2) = (u_1(x_1, x_2), \ u_2(x_1, x_2), \ 0)$$

with $\partial_1 u_1 + \partial_2 u_2 = 0$.

Taking rotation of the velocity U, we find that the vorticity vector has only one component depending only on two spatial variables:

$$(1.4) \qquad \nabla \times U = (0, \ 0, \ \omega(x_1, x_2))$$

where $\omega = \partial_1 u_2 - \partial_2 u_1$. Assuming that U satisfies the three dimensional stationary Navier-Stokes equations for viscous incompressible fluids, we obtain the equations for ω as follows:

$$(B_{\lambda,\alpha}) \qquad \begin{cases} \mathcal{L}\omega = (u, \nabla)\omega - \lambda \mathcal{M}\omega, \ x \in \mathbb{R}^2, \\ u = K * \omega, \\ \displaystyle\int_{\mathbb{R}^2} \omega(x)dx = \alpha. \end{cases}$$

Here the operators $\mathcal{L}$ and $\mathcal{M}$ are given by

$$(1.5) \qquad\qquad \mathcal{L} = \Delta + \frac{x}{2} \cdot \nabla + 1,$$

$$(1.6) \qquad\qquad \mathcal{M} = \frac{1}{2}(x_1 \partial_1 - x_2 \partial_2).$$

The relation between the velocity field u and the vorticity ω is called the Biot-Savart law, and the convolution kernel K is given by

$$(1.7) \qquad\qquad K(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2}, \ \ x^\perp = (-x_2, x_1).$$

The value α represents the total circulation, and $|\alpha|$ is the vortex Reynolds number since the viscosity coefficient is normalized to one in our case. We call a solution to the equation $(B_{\lambda,\alpha})$ the Burgers vortex.

Let G be the two dimensional Gauss kernel:

$$(1.8) \qquad\qquad G(x) = \frac{1}{4\pi} e^{-\frac{|x|^2}{4}}.$$

Then by direct calculations, we see that G satisfies

$$\mathcal{L}G = 0, \ (K * G, \nabla)G = 0.$$

Thus αG solves the equation $(B_{\lambda,\alpha})$ for $\lambda = 0$. This is the exact solution Burgers found, and it is called the axisymmetric Burgers vortex. The stability of the axisymmetric Burgers vortices was firstly discussed by Y. Giga and T. Kambe [7] for small $|\alpha|$ (see also the related work by A. Carpio [2]), and this smallness assumption was removed by Th. Gallay and C. E. Wayne [4].

The case $\lambda \ne 0$ is called non-axisymmetric. In this case, the equation $(B_{\lambda,\alpha})$ has not yet been studied much. As far as the author knows, the only mathematical results are the results by Th. Gallay and C. E. Wayne [5], [6]. In [5] they constructed solutions to the equations $(B_{\lambda,\alpha})$ in the Gaussian weighted L^2 space for any Reynolds numbers $|\alpha|$ when the asymmetric parameter λ is sufficiently small ($\lambda << \frac{1}{2}$). They also recovered the large-Reynolds-number asymptotics of H. K. Moffatt, S. Kida, and K. Okhitani [10] in this case. For not sufficiently small λ, the existence and uniqueness of the Burgers vortex are obtained only when the Reynolds number $|\alpha|$ is sufficiently small (the smallness of $|\alpha|$ depends on $\lambda \in [0, 1)$); see [6]. Roughly speaking, the term $\lambda \mathcal{M}\omega$ leads to the slow spatial decay in x_2 direction, which causes difficulties in controlling the nonlinear term.

The Burgers vortex, or the equation $(B_{\lambda,\alpha})$, has been used as a model which describes local structures of intensed vorticity fields in turbulence. Although there are only a few mathematical results, the Burgers vortices have been well studied numerically; see A. C. Robinson and P. G. Saffman [14], S. Kida and K. Ohkitani [8], H. K. Moffat, S. Kida, and K. Ohkitani [10], A. Prochazka and D. I. Pullin [12], and A. Prochazka and D. I. Pullin [13]. From physical motivations, the case of high Reynolds numbers is mainly investigated. The interesting feature of their results is that the Burgers vortex has simpler structures and better stability when the Reynolds number $|\alpha|$ is large. Especially, it is numerically shown that the shape of the isovorticity contour becomes more circular as the Reynolds number is increasing. In the previous work [11] the author studied a linearized operator for the equation $(B_{\lambda,\alpha})$ and obtained some estimates and spectrum behavior for this operator which are compatible with the numerical results.

In the present paper we consider the equation $(B_{\lambda,\alpha})$ when the Reynolds number $|\alpha|$ is large, and the asymmetric parameter λ is less than $\frac{1}{2}$.

To state our results precisely, let us introduce function spaces.

Let X, Y be the real Hilbert spaces defined as follows.

$$(1.9) \qquad X = \{w \in L^2(\mathbb{R}^2) \mid G^{-\frac{1}{2}}w \in L^2(\mathbb{R}^2), \int_{\mathbb{R}^2} wdx = 0,$$

$$< w_1, w_2 >_X = \int_{\mathbb{R}^2} G^{-1}(x)w_1(x)w_2(x)dx\},$$

$$(1.10) \qquad Y = \{w \in X \mid \partial_i w \in X, \ i = 1, 2,$$

$$< w_1, w_2 >_Y = \int_{\mathbb{R}^2} G^{-1}(x)\big(w_1(x)w_2(x) + \nabla w_1(x) \cdot \nabla w_2(x)\big)dx.\}$$

We also define the subspace of X

$$(1.11) \ W \ = \ \{w \in X \mid , \ G^{-\frac{1}{2}}x_i w \in L^2(\mathbb{R}^2) \ i = 1, 2,$$

$$< w_1, w_2 >_W = \int_{\mathbb{R}^2} G^{-1}(x)\big(w_1(x)w_2(x) + |x|^2 w_1(x)w_2(x)\big)dx.\}$$

Clearly the closed subspace $Y \cap W$ (equipped with the natural scalar product) is compactly embedded in X. Motivated by [11], we set $\mathbb{P}_S X$ as the space of all radially symmetric functions in X, i.e.,

$$(1.12) \quad \mathbb{P}_S X = \{f \in X \mid f(Rx) = f(x) \text{ a.e. } x \in \mathbb{R}^2 \text{ for all orthogonal matrix } R\}.$$

Let $\mathbb{P}_{S^\perp} X$ be the orthogonal complement of $\mathbb{P}_S X$ in X and let $w_\infty \in Y \cap W \cap \mathbb{P}_{S^\perp} X$ be the function which satisfies the equation

$$(1.13) \qquad\qquad\qquad \mathcal{M}G = \Lambda w_\infty.$$

The existence of w_∞ is obtained in [5] (see also [10]). In fact, w_∞ is uniquely determined in $\mathbb{P}_{S^\perp} X$; see Section 2. We are now in position to state our main result.

Theorem 1.1. *Let $\lambda \in [0, \frac{1}{2})$. Then there is a number $R(\lambda) \geq 0$ such that for any $\alpha \in \mathbb{R}$ with $|\alpha| \geq R(\lambda)$, there exists a solution $\omega_{\alpha,\lambda}$ of the equation $(B_{\lambda,\alpha})$ such that $\omega_{\alpha,\lambda} - \alpha G \in Y \cap W$, $\int_{\mathbb{R}^2} x_i \omega_{\alpha,\lambda} dx = 0$, $i = 1, 2$, and*

$$(1.14) \qquad\qquad ||\omega_{\alpha,\lambda} - \alpha G - \lambda w_\infty||_{Y \cap W} \leq \frac{C\lambda}{(1 - 2\lambda)(1 + |\alpha|)}$$

where the constant C is independent of α and λ. The constant $R(\lambda)$ is taken as

$$(1.15) \qquad\qquad\qquad \lim_{\lambda \to \frac{1}{2}} R(\lambda) = \infty.$$

This solution is unique in the following closed ball in X:

$$\{f \in X \mid \int_{\mathbb{R}^2} x_i f(x)dx = 0, \ i = 1, 2, \ ||f - \alpha G - \lambda w_\infty||_{Y \cap W} \leq \frac{C\lambda}{(1 - 2\lambda)(1 + R(\lambda))}\}.$$

Remark 1.1. *It is not difficult to see that the constant $R(\lambda)$ can be taken as zero for sufficiently small λ. So the above theorem improves the existence result obtained by Th. Gallay and C. E. Wayne [5]. In [6] the solution of $(B_{\lambda,\alpha})$ is obtained in the polynomial weighted L^2 space for $\lambda \in [0, 1)$ when the Reynolds number $|\alpha|$ is sufficiently small. In particular, the solution is constructed near αG_λ where*

$$(1.16) \qquad\qquad\qquad G_\lambda(x) = \frac{\sqrt{1 - \lambda^2}}{4\pi} e^{-\frac{1+\lambda}{4}x_1^2 - \frac{1-\lambda}{4}x_2^2}.$$

Note that G_λ is a solution of $(\mathcal{L} + \lambda \mathcal{M})G_\lambda = 0$. The above result shows that the dynamics of the Burgers vortex depends on the Reynolds number and has simpler structures as $|\alpha|$ is increasing, which gives the rigorous explanation for the numerical observation for $\lambda \in [0, \frac{1}{2})$.

Remark 1.2. *In [5] the asymptotic estimate of $\omega_{\alpha,\lambda}$ at large Reynolds numbers (1.14) is obtained by establishing the uniform estimates for the operator $(\mathcal{L} - \alpha\Lambda_G)^{-1}$ (see (1.17) for the definition of Λ_G) and using the smallness of $\lambda(<< \frac{1}{2})$. In our proof we use the advantage of the equation at large Reynolds numbers instead of the smallness of λ. In particular, it is revealed that how the radially or non-radially symmetric parts of the Burgers vortex are influenced by the value of the Reynolds numbers. We note that the estimate (1.14) shows the validity of the formal asymptotic expansion by H. K. Moffat, S. Kida, and K. Ohkitani [10].*

Remark 1.3. *The restriction $\lambda < \frac{1}{2}$ seems to be essential if we try to find the Burgers vortex in the Gaussian weighted L^2 space X; see (1.14) and the estimates in Lemma 3.1. We also note that the function G_λ belongs to X if and only if $\lambda < \frac{1}{2}$.*

In order to prove the main theorem, we expand the equation $(B_{\lambda,\alpha})$ around $\alpha G + \lambda w_\infty$. Then we get the equation for $w = \omega - \alpha G - \lambda w_\infty$:

$$(1.17) \qquad (\mathcal{L} - \alpha\Lambda_G + \lambda\mathcal{M})w = B(w,w) + \lambda\Lambda_{w_\infty}w + \lambda f_\lambda,$$

where

$$(1.18) \qquad\qquad B(f,h) \;=\; (K * f, \nabla)h$$

$$(1.19) \qquad\qquad \Lambda_h f \;=\; B(h,f) + B(f,h),$$

for $h, f \in Y$.

The function f_λ is defined as

$$(1.20) \qquad\qquad f_\lambda = -\mathcal{L}w_\infty + \lambda\big(B(w_\infty, w_\infty) - \mathcal{M}w_\infty\big).$$

To derive the function f_λ from the equation $(B_{\lambda,\alpha})$, we used the definition of w_∞, and the facts that $\mathcal{L}G = 0$, $B(f,g) = 0$ for f, $g \in Y \cap \mathbb{P}_S X$. By direct calculations, we can check that the functions w_∞, $B(w_\infty, w_\infty)$, and $\mathcal{M}w_\infty$ belongs to $\mathbb{P}_{S^\perp} X$; see Section 2. Hence the function f_λ also belongs to $\mathbb{P}_{S^\perp}X$.

In general, the integro-differential operator Λ_h is not a closed operator in X. To avoid this inconvenience, as in [11], we consider the closure of Λ_G instead of Λ_G itself. For simplicity, we write

$$(1.21) \qquad\qquad \Lambda = \overline{\Lambda_G};\ \text{the closure of}\ \Lambda_G\ \text{in}\ X, \qquad \Lambda_1 = \Lambda_{w_\infty}.$$

Let X_1 be the closed subspace of X defined by

$$(1.22) \qquad\qquad X_1 = \{f \in X \mid \int_{\mathbb{R}^2} x_i f(x)dx = 0,\ i = 1,2\}.$$

Then we see that the equation for $w = \omega - \alpha G - \lambda w_\infty$ is also invariant in X_1. Thus we consider the equation

$$(\tilde{B}_{\lambda,\alpha}) \qquad
\begin{cases}
(\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M})w = B(w,w) + \lambda\Lambda_1 w + f_\lambda,\ x \in \mathbb{R}^2, \\
\displaystyle\int_{\mathbb{R}^2} w(x)dx = \int_{\mathbb{R}^2} x_i w(x)dx = 0,\ i = 1,2.
\end{cases}$$

The reason why we consider the equation in X_1 instead of X is that the kernel of Λ coincides with $\mathbb{P}_S X_1 (= \mathbb{P}_S X)$ in this space by [11]; see Section 2.

We shall construct a solution of $(\tilde{B}_{\lambda,\alpha})$ by the Schauder fixed point theorem. And then, we shall show the uniqueness of solutions under the assumptions of the theorem.

Let us state what is the difficulty and how we overcome it. The main difficulty appears when we deal with the term $\lambda\mathcal{M}w$. In [5] this term is treated as the perturbation. However, since $\mathcal{M}w$ is not a lower order term, we cannot regard the term $\lambda\mathcal{M}w$ as the perturbation if λ is not sufficiently small.

In [11] it is shown that the operator norm of the inverse of $\mathcal{L}_\alpha := (\mathcal{L} - \alpha\Lambda)|_{\mathbb{P}_{S^\perp} X}$ in X is small when $|\alpha|$ is large. But this is still not enough to control the term $\lambda\mathcal{M}w$.

We note that if $||\mathcal{L}_\alpha^{-1}\nabla||_{X\to Y}$ is small when $|\alpha|$ is large, then we could regard $\lambda\mathcal{M}w$ as the perturbation. But it seems that the smallness of $||\mathcal{L}_\alpha^{-1}\nabla||_{X\to Y}$ is not true. So far, we only know $||\mathcal{L}_\alpha^{-1}\nabla||_{X\to Y}$ is uniformly bounded with respect to $|\alpha|$ by [5].

The above observation implies that we should treat the term $\lambda\mathcal{M}w$ as the main part of the equation when λ is not small. That is, we regard the term $(\mathcal{L}-\alpha\Lambda+\lambda\mathcal{M})w$ as the principal term. Thus the most important step is to establish the estimates for the operator $(\mathcal{L}-\alpha\Lambda+\lambda\mathcal{M})^{-1}$ in X_1. We note that even the existence of $(\mathcal{L}-\alpha\Lambda+\lambda\mathcal{M})^{-1}$ is not trivial. The estimates for $\mathcal{L}_\alpha^{-1}$ suggest that $(\mathcal{L}-\alpha\Lambda+\lambda\mathcal{M})^{-1}$ have better estimates if it acts on $\mathbb{P}_{S^\perp}X_1$ (the orthogonal complement of $\mathbb{P}_SX_1$ in X_1) and $|\alpha|$ is large. This is shown to be true, but we need more complicated steps to prove this, since the term $\lambda\mathcal{M}w$ leads to the slow spatial decay in x_2 direction and also gives rise to the interaction between different Fourier modes with respect to the angular variable in polar coordinates. For example, we can easily see that the space $\mathbb{P}_SX_1$ or $\mathbb{P}_{S^\perp}X_1$ is not invariant under the action of the operator $\mathcal{L}-\alpha\Lambda+\lambda\mathcal{M}$. With careful analyses of the interaction between the radially symmetric part and the non-radially symmetric part, we establish the required estimates for $(\mathcal{L}-\alpha\Lambda+\lambda\mathcal{M})^{-1}$; see Section 3 for details.

We construct solutions of $(\tilde{B}_{\lambda,\alpha})$ based on the estimates for the operator $(\mathcal{L}-\alpha\Lambda+\lambda\mathcal{M})^{-1}$. To make use of the advantage at large Reynolds numbers, we decompose a solution of $(\tilde{B}_{\lambda,\alpha})$ into the radially symmetric part and non-radially symmetric part. For the non-radially symmetric part, we obtain better estimates when $|\alpha|$ is large. On the other hand, we do not have any advantage in the estimates of $(\mathcal{L}-\alpha\Lambda+\lambda\mathcal{M})^{-1}$ for the radially symmetric part. However, from the structure of the equation, we see that the radially symmetric part of solutions is essentially expressed by the non-radially symmetric part of them. This enables us to obtain the desired estimates also for the radially symmetric part of solutions. The asymptotic estimates of solutions at large Reynolds numbers directly follow from the estimates of the function $(\mathcal{L}-\alpha\Lambda+\lambda\mathcal{M})^{-1}f_\lambda$.

This paper is organized as follows. In Section 2, we summarize the known results for some linear operators obtained in [4], [5], and [11]. We also prove some properties of the bilinear form $B(f,h)$ and the function w_∞. In Section 3, we establish the estimates for the operator $(\mathcal{L}-\alpha\Lambda+\lambda\mathcal{M})^{-1}$, which is the core of this paper. In Section 4, we construct a solution of the equation $(\tilde{B}_{\lambda,\alpha})$ which gives the proof of the former part of Theorem 1.1. In Section 5, we give the asymptotic estimates (1.14) by deriving the estimates of the function $(\mathcal{L}-\alpha\Lambda+\lambda\mathcal{M})^{-1}f_\lambda$.

2. Preliminaries

2.1. **Known results for some linear operators.** In this section we recall the several known properties for some linear operators we consider in this paper.

First of all, it is well known that the operator $\mathcal{L}$ is self-adjoint in X and its spectrum consists of eigenvalues $\{-\frac{n}{2}|\ n=1,2,\cdots\}$. The associated eigenfunctions for $-\frac{n}{2}$ are the Hermite functions $\{\partial_1^{\beta_1}\partial_2^{\beta_2}G\}$ with $\beta_1+\beta_2=n$. So the subspace X_1 is nothing but the orthogonal complement of $\{\beta_1\partial_1G+\beta_2\partial_2G\mid\beta_i\in\mathbb{R}\}$ in X.

In [5] and [6] Th. Gallay and C. E. Wayne proved the following lemma for the operators Λ_G and $\mathcal{L}-\alpha\Lambda_G$.

Lemma 2.1 ([5], [6])**.**
   (1) $(-\mathcal{L})^{-\frac{1}{2}}$ *is bounded from* X *into* $Y\cap W$.
   (2) Λ_G *is bounded from* Y *into* X.
   (3) Λ_G *is skew-symmetric: for any* $w_1,\ w_2\in Y$, *we have* $<\Lambda_Gw_1,w_2>_X\ +\ <w_1,\Lambda_Gw_2>_X=0$.

(4) $(\mathcal{L} - \alpha\Lambda_G)^{-1}$ *is compact in* X *and bounded from* X *into* Y. *Moreover, its operator norm is bounded uniformly in* α.

In [11] the operator Λ (the closure of Λ_G in X) and $\mathcal{L} - \alpha\Lambda$ are studied. As stated in [11], the operator Λ is expressed in terms of polar coordinates and the Fourier series expansion with respect to the angular variable. We omit the details here. We only state the results in [11] without proofs.

Lemma 2.2 ([11]). *The kernel of* Λ *in* X *is given by*

$$(2.1) \qquad \mathrm{Ker}\,\Lambda = \mathbb{P}_S X \oplus \{\beta_1\partial_1 G + \beta_2\partial_2 G \mid \beta_i \in \mathbb{R}\}.$$

Moreover, let $\mathcal{L}_\alpha := (\mathcal{L} - \alpha\Lambda)|_{\overline{\mathrm{Ran}\,\Lambda}} : D(\mathcal{L}) \cap \overline{\mathrm{Ran}\,\Lambda} \to \overline{\mathrm{Ran}\,\Lambda}$. *Then we have*

$$(2.2) \qquad \lim_{|\alpha|\to\infty} \sup_{\mu\in\sigma(\mathcal{L}_\alpha)} \mathrm{Re}\,(\mu) = -\infty.$$

Here, $\sigma(\mathcal{L}_\alpha)$ *is the spectrum of* $\mathcal{L}_\alpha$ *and* $\mathrm{Re}\,(\mu)$ *is the real part of* μ.

The above characterization of $\mathrm{Ker}\,\Lambda$ shows that $\mathrm{Ker}\,\Lambda = \mathbb{P}_S X_1$ if Λ is restricted on X_1. This fact is essentially used in this paper.

2.2. **The properties of the bilinear form and the function** w_∞. The bilinear form $B(f,h) = (K * f, \nabla)h$ plays important roles in the study of Burgers vortices. We start from the following proposition.

Proposition 2.1. *Let* $2 < r < 3$ *and* $p = \frac{r}{2r-3}$. *Let* $f \in L^r(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$ *and* $h \in Y$. *Then we have*

$$(2.3) \qquad ||B(f,h)||_X \;\;\leq\;\; C||f||_{L^p}^{\frac{1}{4}}||f||_{L^r}^{\frac{3}{4}}||h||_Y,$$

$$(2.4) \qquad ||(-\mathcal{L})^{-\frac{1}{2}}B(f,h)||_X \;\;\leq\;\; C||f||_{L^p}^{\frac{1}{4}}||f||_{L^r}^{\frac{3}{4}}||h||_X.$$

Proof. We first note that by the Gagliardo-Nirenberg inequality, we have

$$||K * f||_{L^\infty} \leq C||K * f||_{L^q}^{\frac{1}{4}}||\nabla K * f||_{L^r}^{\frac{3}{4}},$$

where $\frac{1}{3q} = \frac{1}{2} - \frac{1}{r}$. We note that $2 < q < \infty$ from the condition $2 < r < 3$. Then by the Hardy-Littlewood-Sobolev inequality and the Calderón-Zygmund inequality, we see

$$(2.5) \qquad ||K * f||_{L^\infty} \leq C||f||_{L^p}^{\frac{1}{4}}||f||_{L^r}^{\frac{3}{4}},$$

since $\frac{1}{p} = \frac{1}{q} + \frac{1}{2}$. Thus

$$
\begin{aligned}
||B(f,h)||_X &= ||G^{-\frac{1}{2}}(K * f, \nabla)h||_{L^2} \\
&\leq ||K * f||_{L^\infty}||G^{-\frac{1}{2}}\nabla h||_{L^2} \\
&\leq C||f||_{L^p}^{\frac{1}{4}}||f||_{L^r}^{\frac{3}{4}}||h||_Y.
\end{aligned}
$$

This proves the estimate (2.3).

To show the estimate (2.4), we prove the estimate

$$(2.6) \qquad ||(-\mathcal{L})^{-\frac{1}{2}}\partial_i w||_X \leq C||w||_X, \; i = 1, 2,$$

for $w \in X$. This estimate is obtained by the duality argument. Indeed, we have for any $h \in X$,

$$
\begin{aligned}
<(-\mathcal{L})^{-\frac{1}{2}}\partial_i w, h>_X &= <\partial_i w, (-\mathcal{L})^{-\frac{1}{2}}h>_X \\
&= -\frac{1}{2}<w, x_i(-\mathcal{L})^{-\frac{1}{2}}h>_X - <w, \partial_i(-\mathcal{L})^{-\frac{1}{2}}h>_X.
\end{aligned}
$$

Since $(-\mathcal{L})^{-\frac{1}{2}}$ is bounded from X into $Y \cap W$, we have

$$|<(-\mathcal{L})^{-\frac{1}{2}}\partial_i w, h>_X| \leq C||w||_X||h||_X,$$

this proves (2.6). Now the estimate (2.4) immediately follows from $B(f,h) = \nabla \cdot (h\,K*f)$ and the estimate (2.5). The proof of the proposition is completed.

From the above proposition, we can obtain the estimates for the integro-differential operator Λ_h. We set $A = (-\mathcal{L})^{\frac{1}{2}}$. Then A is sectorial; for example, see [3, Section II, Corollary 4.7]. Since A has a bounded inverse, we set the norm on $D(A^\gamma)$ for $\gamma \in [0,1]$ as

$$(2.7) \qquad\qquad ||f||_{D(A^\gamma)} = ||A^\gamma f||_X,$$

instead of the usual graph norm. By the interpolation arguments, we have the following corollary.

Corollary 2.1. *Let $\gamma_1,\ \gamma_2 \in (0,1]$. Let $f \in D(A^{\gamma_1})$ and $h \in D(A^{\gamma_2})$. Then we have*

$$(2.8) \qquad ||(-\mathcal{L})^{-\frac{1}{2}}\Lambda_h f||_X \le C(||h||_{D(A^{\gamma_2})}||f||_X + ||f||_{D(A^{\gamma_1})}||h||_X),$$

where C depends only on γ_1 and γ_2.

Proof. Let $2 < r < 3$. Then by the Gagliardo-Nirenberg inequality, we have

$$||f||_{L^r} \le C||f||_{L^2}^{1-\sigma}||\nabla f||_{L^2}^{\sigma},$$

for $\sigma = 1 - \frac{2}{r}$. Thus $||f||_{L^r} \le C||f||_X^{1-\sigma}||Af||_X^{\sigma}$ and this shows that

$$(2.9) \qquad\qquad ||f||_{L^r} \le C||f||_{(X,D(A))_{\sigma,1}} \le C||f||_{D(A^{\sigma'})},$$

for $\sigma < \sigma'$; see [9, Section 2.2] for details. We note that if $2 < r < 3$, then $p = \frac{r}{2r-3} \in (1,2)$. Hence

$$||f||_{L^p} = ||G^{\frac{1}{2}}G^{-\frac{1}{2}}f||_{L^p} \le C||f||_X,$$

by the Hölder inequality. Combining these, by choosing suitable r in the estimate (2.4), we obtain the estimate (2.8).

To see the qualitative properties of the bilinear form $B(f,h)$, we consider the representation of $B(f,h)$ in terms of polar coordinates.

Let $n \in \mathbb{Z}$ and let $\mathbb{P}_n$ be the orthogonal projection defined by

$$\mathbb{P}_n w = w_n(r)e^{in\theta},$$
$$w_n(r) = \frac{1}{2\pi}\int_0^{2\pi} w(r\cos\theta, r\sin\theta)e^{-in\theta}\,d\theta.$$

We set

$$(2.10) \qquad\qquad \mathbb{P}_n X = \{\mathbb{P}_n w \mid w \in X\}.$$

Then we have the following proposition for $B(f,h)$.

Proposition 2.2. *Let $f \in Y \cap \mathbb{P}_n X$ and $h \in Y \cap \mathbb{P}_m X$. Then $B(f,h) \in \mathbb{P}_{n+m}X$.*

Proof. We recall the argument of [4, Lemma 4.4]. Let $f = f_n(r)e^{in\theta}$ and $h = h_m(r)e^{im\theta}$ in polar coordinates. We set $v_f = (v_f^{(1)}, v_f^{(2)}) = K * f$. We write $v_f^{(1)} = v_r\cos\theta - v_\theta\sin\theta$ and $v_f^{(2)} = v_r\sin\theta + v_\theta\cos\theta$ where $v_r = \overline{v}_r(r)e^{in\theta}$ and $v_\theta = \overline{v}_\theta e^{in\theta}$.

Then from div $v_f = 0$ and rot $v_f = f$, we obtain the linear ordinary differential equations for $\overline{v}_r(r)$ and $\overline{v}_\theta(r)$

$$(2.11) \qquad \overline{v}'_r + \frac{\overline{v}_r}{r} + in\frac{\overline{v}_\theta}{r} = 0,$$

$$(2.12) \qquad \overline{v}'_\theta + \frac{\overline{v}_\theta}{r} - in\frac{\overline{v}_r}{r} = f_n.$$

When $n \neq 0$, by eliminating $\overline{v}_\theta$, we obtain the equation for $\Omega_n = \frac{1}{2in} r \overline{v}_r$

$$(2.13) \qquad -\frac{1}{r}(r\Omega'_n)' + \frac{n^2}{r^2}\Omega_n - \frac{1}{2}f_n = 0.$$

By the decay at infinity and the local integrability conditions, solution of the above equation is written by

$$(2.14) \qquad \Omega_n(f_n)(r) = \frac{1}{4|n|}\Big(\int_0^r (\frac{s}{r})^{|n|} s f_n(s)ds + \int_r^\infty (\frac{r}{s})^{|n|} s f_n(s)ds \Big).$$

The function $\overline{v}_\theta$ is obtained by $\overline{v}_r$. From the uniqueness of the equation

$$\Delta v_f = \nabla^\perp f = (-\partial_2 f, \partial_1 f),$$

we see that v_f is indeed expressed by the above $\overline{v}_r$ and $\overline{v}_\theta$.

Now by using the relation $\partial_1 = \cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta$ and $\partial_2 = \sin\theta\partial_r + \frac{\cos\theta}{r}\partial_\theta$, we obtain

$$(2.15) \qquad B(f,h) = \Big(\overline{v}_r h'_m + \frac{im}{r}\overline{v}_\theta h_m\Big)e^{i(n+m)\theta}.$$

When $n = 0$, again by the decay at infinity and the local integrability conditions, we see that $\overline{v}_r = 0$ and $\overline{v}_\theta(r) = \int_r^\infty \frac{r}{s} f_n(s)ds$ from (2.11), (2.12). Thus

$$(2.16) \qquad B(f,h) = \frac{im}{r}\overline{v}_\theta h_m e^{im\theta}.$$

This completes the proof.

Corollary 2.2. *If $h \in Y \cap \mathbb{P}_S X$, then $\Lambda_h f \in \mathbb{P}_{S^\perp} X$ for any $f \in Y \cap \mathbb{P}_{S^\perp} X$.*

Proof. Since $\mathbb{P}_S X = \mathbb{P}_0 X$, the assertion immediately follows from the above proposition. ∎

Corollary 2.3. *The function f_λ belongs to $\mathbb{P}_{S^\perp} X$.*

Proof. We recall that

$$f_\lambda = -\mathcal{L}w_\infty + \lambda(B(w_\infty, w_\infty) - \mathcal{M}w_\infty).$$

In [5, Proposition 3.1] w_∞ is obtained as $w_\infty = \overline{w}(r)\sin 2\theta$ for some function $\overline{w}(r)$. Note that from the characterization of Ker Λ, this is uniquely determined in $\mathbb{P}_{S^\perp} X$.

Since $\mathbb{P}_{S^\perp} X$ is invariant under the action of $\mathcal{L}$, we have $\mathcal{L}w_\infty \in \mathbb{P}_{S^\perp} X$. By direct calculations, we also have $\mathcal{M}w_\infty \in \mathbb{P}_{S^\perp} X$. Moreover, from the above proposition and $w_\infty = \overline{w}(r)\sin 2\theta = \overline{w}(r)\frac{e^{i2\theta} - e^{-i2\theta}}{2i}$, it is not difficult to see $B(w_\infty, w_\infty) \in \mathbb{P}_{S^\perp} X$. This completes the proof. ∎

Finally, we remark the following simple proposition, which guarantees that the space X_1 is invariant under the equation $(\tilde{B}_{\lambda,\alpha})$.

Proposition 2.3. *Let $f, h \in Y$. Then $\Lambda_h f \in X_1$.*

Proof. We set $v_f = (v_f^{(1)}, v_f^{(2)}) = K * f$ and $v_h = (v_h^{(1)}, v_h^{(2)}) = K * h$. The proof is given by the integration by parts. Indeed, by the definition of Λ_h, we have

$$
\begin{aligned}
\int_{\mathbb{R}^2} x_1 \Lambda_h f dx &= \int_{\mathbb{R}^2} x_1 \nabla \cdot (v_h f + v_f h) dx \\
&= -\int_{\mathbb{R}^2} (v_h^{(1)} f + v_f^{(1)} h) dx \\
&= \int_{\mathbb{R}^2} -v_h^{(1)}(-\partial_2 v_f^{(1)} + \partial_1 v_f^{(2)}) - v_f^{(1)}(-\partial_2 v_h^{(1)} + \partial_1 v_h^{(2)}) dx \\
&= \int_{\mathbb{R}^2} (-\partial_2 v_h^{(1)} v_f^{(1)} + \partial_1 v_h^{(1)} v_f^{(2)} + v_f^{(1)} \partial_2 v_h^{(1)} + \partial_1 v_f^{(1)} v_h^{(2)}) dx \\
&= -\int_{\mathbb{R}^2} (\partial_2 v_h^{(2)} v_f^{(2)} + \partial_2 v_f^{(2)} v_h^{(2)}) dx \\
&= 0.
\end{aligned}
$$

Similarly, we have $\int_{\mathbb{R}^2} x_2 \Lambda_h f dx = 0$. It is obvious that $\int_{\mathbb{R}^2} \Lambda_h f dx = 0$. Now the proof is completed.

3. THE ESTIMATES FOR THE LINEARIZED OPERATOR

In this section, we establish the estimates for the linearized operator $\mathcal{L} - \alpha \Lambda + \lambda \mathcal{M}$. The following lemma is the core of this paper. We recall that $A = (-\mathcal{L})^{\frac{1}{2}}$.

Lemma 3.1. *Let $\lambda \in [0, \frac{1}{2})$ and $\gamma \in [0, 1)$. Then there is some $R_1(\lambda) \geq 0$ independent of γ such that for any α with $|\alpha| \geq R_1(\lambda)$ and $f \in X_1$, we have*

(3.1)
$$
||(\mathcal{L} - \alpha \Lambda + \lambda \mathcal{M})^{-1} f||_{Y \cap W} \leq \frac{K_1}{1 - 2\lambda} ||(-\mathcal{L})^{-\frac{1}{2}} f||_X
$$

(3.2)
$$
||\mathbb{P}_{S^\perp}(\mathcal{L} - \alpha \Lambda + \lambda \mathcal{M})^{-1} f||_{D(A^\gamma)} \leq \delta_1(|\alpha|, \gamma) \left(||\mathbb{P}_{S^\perp}(-\mathcal{L})^{-\frac{1}{2}} f||_X + \frac{K_1 \lambda}{1 - 2\lambda} ||(-\mathcal{L})^{-\frac{1}{2}} f||_X \right)
$$

(3.3)
$$
\begin{aligned}
||\mathbb{P}_S(\mathcal{L} - \alpha \Lambda + \lambda \mathcal{M})^{-1} f||_{D(A^\gamma)} &\leq (1 + \lambda) ||\mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}} f||_X \\
&+ \lambda(1 + \lambda) \delta_2(|\alpha|, \gamma) \left(||\mathbb{P}_{S^\perp}(-\mathcal{L})^{-\frac{1}{2}} f||_X + \frac{K_2 K_1 \lambda}{1 - 2\lambda} ||(-\mathcal{L})^{-\frac{1}{2}} f||_X \right).
\end{aligned}
$$

Here, the constants K_1 and K_2 are independent of λ, γ, and α with $|\alpha| \geq R_1(\lambda)$. The constants $\delta_1(|\alpha|, \gamma)$ and $\delta_2(|\alpha|, \gamma)$ are bounded with respect to $|\alpha| \in [R_1(\lambda), \infty)$ and $\gamma \in [0, 1)$, and satisfy that

(3.4)
$$
\lim_{|\alpha| \to \infty} \delta_1(|\alpha|, \gamma) = \lim_{|\alpha| \to \infty} \delta_2(|\alpha|, \gamma) = 0.
$$

Remark 3.1. *It is not difficult to see that the norm of $Y \cap W$ is equivalent with $|| \cdot ||_{D(A)}$. So the estimate (3.1) is a special case of (3.2) and (3.3). However, we do not have the property (3.4) in this case.*

To prove the above lemma, we first consider the operator $\mathcal{L} - \alpha \Lambda + \lambda \mathcal{M} - \lambda I$. Since $\mathcal{L}$ is self-adjoint and $-\mathcal{L} \geq 1$ in X_1, we can write

$$
\mathcal{L} - \alpha \Lambda + \lambda \mathcal{M} - \lambda I = -(-\mathcal{L})^{\frac{1}{2}} (I + \alpha \Sigma - \lambda \Pi + \lambda(-\mathcal{L})^{-1})(-\mathcal{L})^{\frac{1}{2}},
$$

where

(3.5)
$$
\Sigma = (-\mathcal{L})^{-\frac{1}{2}} \Lambda (-\mathcal{L})^{-\frac{1}{2}},
$$

(3.6)
$$
\Pi = (-\mathcal{L})^{-\frac{1}{2}} \mathcal{M} (-\mathcal{L})^{-\frac{1}{2}}.
$$

By the results in [5], we already know that Σ is compact and skew-symmetric in X and Π is bounded in X. We shall show the following proposition.

Proposition 3.1. *Let $\lambda \in [0, \frac{1}{2})$. Then the operator $(I + \alpha\Sigma - \lambda\Pi + \lambda(-\mathcal{L})^{-1})$ has a bounded inverse in X satisfying the estimate*

$$(3.7) \qquad \|(I + \alpha\Sigma - \lambda\Pi + \lambda(-\mathcal{L})^{-1})^{-1}f\|_X \leq \frac{1}{1-2\lambda}\|f\|_X.$$

Proof of Proposition 3.1. Let $Q_{\alpha,\lambda}$ be a bilinear form on X defined by

$$(3.8) \qquad Q_{\alpha,\lambda}(f,h) = < (I + \alpha\Sigma - \lambda\Pi + \lambda(-\mathcal{L})^{-1})f, h >_X .$$

Clearly, $Q_{\alpha,\lambda}$ is bounded, i.e., there is some constant K such that $|Q_{\alpha,\lambda}(f,h)| \leq K\|f\|_X\|h\|_X$ for all f, $h \in X$. Since Σ is skew-symmetric, we have $< \Sigma f, f >_X = 0$. We also recall the equality

$$
\begin{aligned}
\|(-\mathcal{L})^{\frac{1}{2}}h\|_X^2 &= < h, (-\mathcal{L})h >_X \\
&= \int_{\mathbb{R}^2} |\nabla(G^{-\frac{1}{2}}(x)h(x))|^2 dx + \frac{1}{16}\||x|h\|_X^2 - \frac{1}{2}\|h\|_X^2,
\end{aligned}
$$

which leads to the inequality

$$(3.9) \qquad \|(-\mathcal{L})^{-\frac{1}{2}}f\|_X^2 \geq \frac{1}{8}\||x|(-\mathcal{L})^{-\frac{1}{2}}f\|_X^2 - 2\|f\|_X^2.$$

Combining these, we have

$$
\begin{aligned}
Q_{\alpha,\lambda}(f,f) &= < (I + \alpha\Sigma - \lambda\Pi + \lambda(-\mathcal{L})^{-1})f, f >_X \\
&\geq \|f\|_X^2 - \lambda < (-\mathcal{L})^{-\frac{1}{2}}\mathcal{M}(-\mathcal{L})^{-\frac{1}{2}}f, f >_X + \frac{\lambda}{8}\||x|(-\mathcal{L})^{-\frac{1}{2}}f\|_X^2 - 2\lambda\|f\|_X^2 \\
&= \|f\|_X^2 - \lambda < \mathcal{M}(-\mathcal{L})^{-\frac{1}{2}}f, (-\mathcal{L})^{-\frac{1}{2}}f >_X + \frac{\lambda}{8}\||x|(-\mathcal{L})^{-\frac{1}{2}}f\|_X^2 - 2\lambda\|f\|_X^2 \\
&= \|f\|_X^2 - \frac{\lambda}{8}\||x|(-\mathcal{L})^{-\frac{1}{2}}f\|_X^2 + \frac{\lambda}{8}\||x|(-\mathcal{L})^{-\frac{1}{2}}f\|_X^2 - 2\lambda\|f\|_X^2 \\
&= (1-2\lambda)\|f\|_X^2,
\end{aligned}
$$

thus $Q_{\alpha,\lambda}$ is coercive. By the Lax-Milgram theorem, $(I + \alpha\Sigma - \lambda\Pi + \lambda(-\mathcal{L})^{-1})$ is invertible in X, and the estimate (3.7) follows from the above inequality. This completes the proof of the proposition.

From Proposition 3.1, we see that $\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M} - \lambda I$ is invertible and its inverse $(\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M} - \lambda I)^{-1} = -(-\mathcal{L})^{-\frac{1}{2}}(I + \alpha\Sigma - \lambda\Pi + \lambda(-\mathcal{L})^{-1})^{-1}(-\mathcal{L})^{-\frac{1}{2}}$ has the estimates

$$(3.10) \qquad \|(\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M} - \lambda I)^{-1}f\|_X \leq \frac{1}{1-2\lambda}\|f\|_X$$

$$(3.11) \qquad \|(\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M} - \lambda I)^{-1}f\|_{Y \cap W} \leq \frac{C}{1-2\lambda}\|f\|_X.$$

Next we improve the above estimates for large $|\alpha|$. We set $h = (\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M} - \lambda I)^{-1}f$. Then $h \in Y \cap W$ and we have

$$(I + \alpha\Sigma - \lambda\Pi + \lambda(-\mathcal{L})^{-1})(-\mathcal{L})^{\frac{1}{2}}h = -(-\mathcal{L})^{-\frac{1}{2}}f,$$

so

$$
\begin{aligned}
(I + \alpha\Sigma + \lambda(-\mathcal{L})^{-1})(-\mathcal{L})^{\frac{1}{2}}h &= -(-\mathcal{L})^{-\frac{1}{2}}f + \lambda\Pi(-\mathcal{L})^{\frac{1}{2}}h \\
&= -(-\mathcal{L})^{-\frac{1}{2}}f + \lambda(-\mathcal{L})^{-\frac{1}{2}}\mathcal{M}h.
\end{aligned}
$$

Thus we have the relation

$$
(3.12) \quad
\begin{aligned}
h &= -(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{\alpha,\lambda}^{-1}(-\mathcal{L})^{-\frac{1}{2}}f \\
&\quad + \lambda(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{\alpha,\lambda}^{-1}(-\mathcal{L})^{-\frac{1}{2}}\mathcal{M}h,
\end{aligned}
$$

where

$$
(3.13) \quad \Gamma_{\alpha,\lambda} = I + \alpha\Sigma + \lambda(-\mathcal{L})^{-1}.
$$

Let $\mathbb{P}_S$ be the projection from X_1 onto $\mathbb{P}_S X_1$; the closed subspace of all radially symmetric functions in X_1. Let $\mathbb{P}_{S^\perp} = I - \mathbb{P}_S$. We note that the projection $\mathbb{P}_S$ commutes with the operators $(-\mathcal{L})^{-\frac{1}{2}}$, Σ. In fact, we have $\mathbb{P}_S\Sigma = 0$. Hence we can verify

$$
(3.14) \quad
\begin{aligned}
\mathbb{P}_{S^\perp}h &= -(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{\alpha,\lambda}^{-1}\mathbb{P}_{S^\perp}(-\mathcal{L})^{-\frac{1}{2}}f \\
&\quad + \lambda(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{\alpha,\lambda}^{-1}\mathbb{P}_{S^\perp}(-\mathcal{L})^{-\frac{1}{2}}\mathcal{M}h,
\end{aligned}
$$

$$
(3.15) \quad
\begin{aligned}
\mathbb{P}_S h &= -(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{0,\lambda}^{-1}\mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}}f \\
&\quad + \lambda(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{0,\lambda}^{-1}\mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}}\mathcal{M}\mathbb{P}_{S^\perp}h.
\end{aligned}
$$

In the last line, we used the fact that

$$
\Gamma_{\alpha,\lambda}^{-1}\mathbb{P}_S = (I + \alpha\Sigma + \lambda(-\mathcal{L})^{-1})^{-1}\mathbb{P}_S = (I + \lambda(-\mathcal{L})^{-1})^{-1}\mathbb{P}_S,
$$

and $\mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}}\mathcal{M}h = \mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}}\mathcal{M}\mathbb{P}_{S^\perp}h$ for h with $\mathbb{P}_{S^\perp}h \in Y \cap W \cap \mathbb{P}_{S^\perp}X$. Note that $\mathbb{P}_{S^\perp}h \in Y \cap W \cap \mathbb{P}_{S^\perp}X$ follows from the representation (3.14), since $(-\mathcal{L})^{-\frac{1}{2}}\mathcal{M}$ is bounded from $Y \cap W$ to X.

The following lemma is crucial.

Lemma 3.2. *Let $\lambda \geq 0$ and $\gamma \in [0,1)$. Then we have for any $f \in X_1$,*

$$
(3.16) \quad \|(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{0,\lambda}^{-1}\mathbb{P}_S f\|_{D(A^\gamma)} \leq \|f\|_X,
$$

$$
(3.17) \quad \|(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{\alpha,\lambda}^{-1}\mathbb{P}_{S^\perp} f\|_{D(A^\gamma)} \leq \epsilon_1(|\alpha|,\gamma)\|f\|_X,
$$

$$
(3.18) \quad \|(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{0,\lambda}^{-1}\mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}}\mathcal{M}(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{\alpha,\lambda}^{-1}\mathbb{P}_{S^\perp} f\|_{D(A^\gamma)} \leq \epsilon_2(|\alpha|,\gamma)\|f\|_X.
$$

Here, $\epsilon_1(|\alpha|,\gamma)$ and $\epsilon_2(|\alpha|,\gamma)$ are uniformly bounded with respect to $|\alpha| \geq 0$, $\gamma \in [0,1)$, and $\lambda \geq 0$. Moreover, they satisfy that

$$
(3.19) \quad \lim_{|\alpha|\to\infty} \epsilon_i(|\alpha|,\gamma) = 0, \quad i = 1,2.
$$

Proof of Lemma 3.2. First we note that

$$
(3.20) \quad \|\Gamma_{\alpha,\lambda}^{-1}f\|_X \leq \|f\|_X.
$$

Indeed, we have

$$
\|\Gamma_{\alpha,\lambda}f\|_X \leq (1 + \alpha + \lambda)C\|f\|_X,
$$

and

$$
\begin{aligned}
< \Gamma_{\alpha,\lambda}f, f >_X &= < f + \alpha\Sigma f + \lambda(-\mathcal{L})^{-1}f, f >_X \\
&= \|f\|_X^2 + \lambda\|(-\mathcal{L})^{-\frac{1}{2}}f\|_X^2 \\
&\geq \|f\|_X^2.
\end{aligned}
$$

These estimates give (3.20) by the Lax-Milgram theorem. From the estimate (3.20) we obtain

$$||(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{0,\lambda}^{-1}\mathbb{P}_S f||_{D(A^\gamma)} \le ||f||_X,$$

$$||(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{\alpha,\lambda}^{-1}\mathbb{P}_{S^\perp} f||_{D(A^\gamma)} \le ||f||_X,$$

$$||(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{0,\lambda}^{-1}\mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}}\mathcal{M}(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{\alpha,\lambda}^{-1}\mathbb{P}_{S^\perp} f||_{D(A^\gamma)} \le C'||f||_X,$$

where $C' = ||(-\mathcal{L})^{-\frac{1}{2}}\mathcal{M}(-\mathcal{L})^{-\frac{1}{2}}||_{X \to X}$. Here we used the estimate $||(-\mathcal{L})^{-\frac{1}{2}}f||_{D(A^\gamma)} \le ||f||_X$ for $f \in X_1$.

We prove the estimates (3.17) and (3.18) by deriving a contradiction. Without loss of generality, we may assume that $\alpha > 0$. Set $\epsilon_1(\alpha, \gamma) := ||(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{\alpha,\lambda}^{-1}\mathbb{P}_{S^\perp}||_{X \to D(A^\gamma)}$.

We assume that $\limsup_{\alpha \to \infty} \epsilon_1(\alpha, \gamma) > 0$.

Then, there exists a sequence $\{\alpha_i\}_{i \in \mathbb{N}}$, $\alpha_i \to \infty$ as $i \to \infty$, such that $\epsilon_1 = \inf_{i \in \mathbb{N}} \epsilon_1(\alpha_i) > 0$.

Then we have a sequence of functions $\{f_i\}_{i \in \mathbb{N}}$ with $||f_i||_X = 1$ such that

$$||(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{\alpha,\lambda}^{-1}\mathbb{P}_{S^\perp} f_i||_{D(A^\gamma)} = ||(-\mathcal{L})^{\frac{-1+\gamma}{2}}\Gamma_{\alpha,\lambda}^{-1}\mathbb{P}_{S^\perp} f_i||_X \ge \frac{\epsilon_1(\alpha_i)}{2}||f_i||_X \ge \frac{\epsilon_1}{2} > 0.$$

We set $h_i = (-\mathcal{L})^{\frac{-1+\gamma}{2}}\Gamma_{\alpha,\lambda}^{-1}\mathbb{P}_{S^\perp} f_i \in \mathbb{P}_{S^\perp} X_1$.

Since $(-\mathcal{L})^{\frac{-1+\gamma}{2}}$ is compact (because $(\mathcal{L})^{-\frac{1}{2}}$ is compact) and $\{\Gamma_{\alpha,\lambda}^{-1}\mathbb{P}_{S^\perp} f_i\}$ is bounded in X, we have a subsequence $\{h_j\}$ of $\{h_i\}$ such that h_j converges to a function $h_\infty \in \mathbb{P}_{S^\perp} X_1$ strongly in X_1. Then h_∞ satisfies $(-\mathcal{L})^{\frac{1-\gamma}{2}}h_\infty \in X_1$ and $||h_\infty||_X \ge \frac{\epsilon_1}{2} > 0$.

On the other hand, for any $f \in X_1$, we see

$$< (-\mathcal{L})^{-\frac{1}{2}}\Lambda(-\mathcal{L})^{-\frac{\gamma}{2}}h_\infty, f >_X$$
$$= -< (-\mathcal{L})^{-\frac{\gamma}{2}}h_\infty, \Lambda(-\mathcal{L})^{-\frac{1}{2}}f >_X$$
$$= -\lim_{j \to \infty} < (-\mathcal{L})^{-\frac{\gamma}{2}}h_j, \Lambda(-\mathcal{L})^{-\frac{1}{2}}f >_X$$
$$= \lim_{j \to \infty} < (-\mathcal{L})^{-\frac{1}{2}}\Lambda(-\mathcal{L})^{-\frac{\gamma}{2}}h_j, f >_X$$
$$= \lim_{j \to \infty} \frac{1}{\alpha_j}(< \Gamma_{\alpha_j,\lambda}(-\mathcal{L})^{\frac{1-\gamma}{2}}h_j, f >_X - < (-\mathcal{L})^{\frac{1-\gamma}{2}}h_j, f >_X - \lambda < (-\mathcal{L})^{-\frac{1+\gamma}{2}}h_j, f >_X)$$
$$= \lim_{j \to \infty} \frac{1}{\alpha_j}(< \mathbb{P}_{S^\perp} f_j, f >_X - < (-\mathcal{L})^{\frac{1-\gamma}{2}}h_j, f >_X - \lambda < (-\mathcal{L})^{-\frac{1+\gamma}{2}}h_j, f >_X)$$
$$= 0.$$

Thus $(-\mathcal{L})^{-\frac{1}{2}}\Lambda(-\mathcal{L})^{-\frac{\gamma}{2}}h_\infty = 0$, that is, $\Lambda(-\mathcal{L})^{-\frac{\gamma}{2}}h_\infty = 0$. However, since $\mathrm{Ker}\,\Lambda = \mathbb{P}_S X_1$ and $h_\infty \in \mathbb{P}_{S^\perp} X_1$ (and thus $(-\mathcal{L})^{-\frac{\gamma}{2}}h_\infty \in \mathbb{P}_{S^\perp} X_1$), we must have $(-\mathcal{L})^{-\frac{\gamma}{2}}h_\infty = 0$. Hence $h_\infty = 0$. This contradicts with $||h_\infty||_X > 0$. Now the estimate (3.17) has been proved.

From the estimate (3.17), we have the following claim:

Let $\{f_i\}_{i \in \mathbb{N}}$ be any bounded sequence in X_1. Then for any sequence $\{\alpha_i\}_{i \in \mathbb{N}}$ in $\mathbb{R}$ such that $\alpha_i \to \infty$ as $i \to \infty$, the sequence $h_i = \Gamma_{\alpha_i,\lambda}^{-1}\mathbb{P}_{S^\perp} f_i$ weakly converges to 0 in X_1.

Indeed, for any $f \in X_1 \cap D(\mathcal{L})$, we have

$$\lim_{i \to \infty} < h_i, f >_X = \lim_{i \to \infty} < h_i, (-\mathcal{L})^{\frac{-1+\gamma}{2}}(-\mathcal{L})^{\frac{1-\gamma}{2}} f >_X$$
$$= \lim_{i \to \infty} < (-\mathcal{L})^{\frac{-1+\gamma}{2}}h_i, (-\mathcal{L})^{\frac{1-\gamma}{2}} f >_X$$
$$= 0.$$

Since $D(\mathcal{L})$ is dense in X_1 and h_i is bounded in X by the estimate (3.17), we have the claim.

The estimate (3.18) is shown by the above claim. We set

$$
\begin{aligned}
\epsilon_2(\alpha,\gamma) &= ||(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{0,\lambda}^{-1}\mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}}\mathcal{M}(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{\alpha,\lambda}^{-1}\mathbb{P}_{S^\perp}||_{X\to D(A^\gamma)} \\
&= ||(-\mathcal{L})^{\frac{-1+\gamma}{2}}\Gamma_{0,\lambda}^{-1}\mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}}\mathcal{M}(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{\alpha,\lambda}^{-1}\mathbb{P}_{S^\perp}||_{X\to X}.
\end{aligned}
$$

Again we assume that there exists a sequence $\{\alpha_i\}_{i\in\mathbb{N}}$, $\alpha_i \to \infty$ as $i \to \infty$, satisfying $\epsilon_2 = \inf_{i\in\mathbb{N}} \alpha_i > 0$. Then we have $\{f_i\}_{i\in\mathbb{N}}$ with $||f_i||_X = 1$ such that

$$
h_i = (-\mathcal{L})^{\frac{-1+\gamma}{2}}\Gamma_{0,\lambda}^{-1}\mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}}\mathcal{M}(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{\alpha_i,\lambda}^{-1}\mathbb{P}_{S^\perp}f_i
$$

satisfies $\inf_{i\in\mathbb{N}} ||h_i||_X \geq \frac{\epsilon_2}{2} > 0$.

Since $(-\mathcal{L})^{-\frac{1}{2}}\mathcal{M}(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{\alpha_i,\lambda}^{-1}\mathbb{P}_{S^\perp}$ is bounded in X (note that $(-\mathcal{L})^{-\frac{1}{2}}x_i$ is bounded in X which is obtained in [5]), we have a subsequence $\{h_j\}_{j\in\mathbb{N}}$ of $\{h_i\}_{i\in\mathbb{N}}$ such that h_j strongly converges to a nontrivial function h_∞ in X_1. Now for any $f \in X_1$,

$$
\begin{aligned}
&< h_\infty, f >_X \\
={}& \lim_{j\to\infty} < h_j, f >_X \\
={}& \lim_{j\to\infty} < (-\mathcal{L})^{\frac{-1+\gamma}{2}}\Gamma_{0,\lambda}^{-1}\mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}}\mathcal{M}(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{\alpha_j,\lambda}^{-1}\mathbb{P}_{S^\perp}f_j, f >_X \\
={}& \lim_{j\to\infty} < \mathcal{M}(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{\alpha_j,\lambda}^{-1}\mathbb{P}_{S^\perp}f_j, (-\mathcal{L})^{-\frac{1}{2}}\mathbb{P}_S\Gamma_{0,\lambda}^{-1}(-\mathcal{L})^{\frac{-1+\gamma}{2}}f >_X \\
={}& -\frac{1}{2}\lim_{j\to\infty} < (-\mathcal{L})^{-\frac{1}{2}}\Gamma_{\alpha_j,\lambda}^{-1}\mathbb{P}_{S^\perp}f_j, (x_1^2-x_2^2)(-\mathcal{L})^{-\frac{1}{2}}\mathbb{P}_S\Gamma_{0,\lambda}^{-1}(-\mathcal{L})^{\frac{-1+\gamma}{2}}f >_X \\
&- \lim_{j\to\infty} < (-\mathcal{L})^{-\frac{1}{2}}\Gamma_{\alpha_j,\lambda}^{-1}\mathbb{P}_{S^\perp}f_j, \mathcal{M}(-\mathcal{L})^{-\frac{1}{2}}\mathbb{P}_S\Gamma_{0,\lambda}^{-1}(-\mathcal{L})^{\frac{-1+\gamma}{2}}f >_X \\
={}& -\frac{1}{2}\lim_{j\to\infty} < \Gamma_{\alpha_j,\lambda}^{-1}\mathbb{P}_{S^\perp}f_j, (-\mathcal{L})^{-\frac{1}{2}}(x_1^2-x_2^2)(-\mathcal{L})^{-\frac{1}{2}}\mathbb{P}_S\Gamma_{0,\lambda}^{-1}(-\mathcal{L})^{\frac{-1+\gamma}{2}}f >_X \\
&- \lim_{j\to\infty} < \Gamma_{\alpha_j,\lambda}^{-1}\mathbb{P}_{S^\perp}f_j, (-\mathcal{L})^{-\frac{1}{2}}\mathcal{M}(-\mathcal{L})^{-\frac{1}{2}}\mathbb{P}_S\Gamma_{0,\lambda}^{-1}(-\mathcal{L})^{\frac{-1+\gamma}{2}}f >_X \\
={}& 0,
\end{aligned}
$$

by the above claim. This implies $h_\infty = 0$, which leads to a contradiction. Now the proof of the lemma is completed.

Proof of Lemma 3.1. Let $\tilde{f} \in X_1$. We consider a solution h of the equation

$$
(\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M})h = \tilde{f}.
$$

Then, h satisfies the equation

(3.21)
$$
(\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M} - \lambda I)h = \tilde{f} - \lambda h.
$$

Thus, from the estimate (3.11) for $f = \tilde{f} - \lambda h$, we have the estimate

$$
\begin{aligned}
||h||_{Y\cap W} &\leq \frac{C}{1-2\lambda}||(-\mathcal{L})^{-\frac{1}{2}}(\tilde{f}-\lambda h)||_X \\
(3.22) &\leq \frac{C}{1-2\lambda}||(-\mathcal{L})^{-\frac{1}{2}}\tilde{f}||_X + \frac{C\lambda}{1-2\lambda}(||\mathbb{P}_{S^\perp}h||_X + ||\mathbb{P}_S h||_X).
\end{aligned}
$$

For simplicity, we write $\epsilon_1 = \epsilon_1(|\alpha|,\gamma)$, $\epsilon_2 = \epsilon_2(|\alpha|,\gamma)$ in Lemma 3.2. We apply Lemma 3.2 to the expression (3.14). Then we have

$$
\begin{aligned}
||\mathbb{P}_{S^\perp}h||_{D(A^\gamma)} &\leq \epsilon_1(||\mathbb{P}_{S^\perp}(-\mathcal{L})^{-\frac{1}{2}}\tilde{f}||_X + \lambda||\mathbb{P}_{S^\perp}(-\mathcal{L})^{-\frac{1}{2}}h||_X) \\
&\quad + \lambda\epsilon_1||\mathbb{P}_{S^\perp}(-\mathcal{L})^{-\frac{1}{2}}\mathcal{M}h||_X \\
&\leq \epsilon_1(||\mathbb{P}_{S^\perp}(-\mathcal{L})^{-\frac{1}{2}}\tilde{f}||_X + \lambda||\mathbb{P}_{S^\perp}h||_X) + C\lambda\epsilon_1||h||_Y,
\end{aligned}
$$

so

$$(3.23) \qquad ||\mathbb{P}_{S^\perp}h||_{D(A^\gamma)} \le 2\epsilon_1||\mathbb{P}_{S^\perp}(-\mathcal{L})^{-\frac{1}{2}}\tilde{f}||_X + 2C\lambda\epsilon_1||h||_Y,$$

if $|\alpha|$ is sufficiently large or λ is sufficiently small.

We also have from (3.15) and Lemma 3.2,

$$
\begin{aligned}
||\mathbb{P}_S h||_{D(A^\gamma)} \;\le\; & ||\mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}}\tilde{f}||_X + \lambda||(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{0,\lambda}^{-1}\mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}}h||_X \\
& + \lambda||(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{0,\lambda}^{-1}\mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}}\mathcal{M}(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{\alpha,\lambda}^{-1}\mathbb{P}_{S^\perp}(-\mathcal{L})^{-\frac{1}{2}}(-\tilde{f}+\lambda\mathcal{M}h)||_X \\
\le\; & ||\mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}}\tilde{f}||_X + \lambda||(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{0,\lambda}^{-1}\mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}}h||_X \\
& + \lambda\epsilon_2||\mathbb{P}_{S^\perp}(-\mathcal{L})^{-\frac{1}{2}}\tilde{f}||_X + \lambda^2\epsilon_2||\mathbb{P}_{S^\perp}(-\mathcal{L})^{-\frac{1}{2}}\mathcal{M}h||_X \\
\le\; & ||\mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}}\tilde{f}||_X + \lambda||(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{0,\lambda}^{-1}\mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}}h||_X \\
& + \lambda\epsilon_2||\mathbb{P}_{S^\perp}(-\mathcal{L})^{-\frac{1}{2}}\tilde{f}||_X + C\lambda^2\epsilon_2||h||_Y.
\end{aligned}
$$

By the relation

$$< (-\mathcal{L}+\lambda I)h, h >_X \ge (1+\lambda)||h||_X^2,$$

we see that

$$
\begin{aligned}
(3.24) \qquad \lambda||(-\mathcal{L})^{-\frac{1}{2}}\Gamma_{0,\lambda}^{-1}\mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}}h||_X \;=\; & \lambda||(-\mathcal{L}+\lambda I)^{-1}\mathbb{P}_S h||_X \\
\le\; & \frac{\lambda}{1+\lambda}||\mathbb{P}_S h||_X \\
=\; & \frac{\lambda}{1+\lambda}||(-\mathcal{L})^{-\frac{\gamma}{2}}(-\mathcal{L})^{\frac{\gamma}{2}}\mathbb{P}_S h||_X \\
\le\; & \frac{\lambda}{1+\lambda}||\mathbb{P}_S h||_{D(A^\gamma)}.
\end{aligned}
$$

Hence we obtain

$$(3.25)$$
$$||\mathbb{P}_S h||_{D(A^\gamma)} \le (1+\lambda)\left(||\mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}}\tilde{f}||_X + \lambda\epsilon_2||\mathbb{P}_{S^\perp}(-\mathcal{L})^{-\frac{1}{2}}\tilde{f}||_X + C\lambda^2\epsilon_2||h||_Y \right).$$

Combining the estimates (3.23) and (3.25) for $\gamma = 0$, we have

$$
\begin{aligned}
& ||\mathbb{P}_{S^\perp}h||_X + ||\mathbb{P}_{S^\perp}h||_X \\
\le\; & (1+\lambda)||\mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}}\tilde{f}||_X + \big(2\epsilon_1 + \lambda(1+\lambda)\epsilon_2\big)||\mathbb{P}_{S^\perp}(-\mathcal{L})^{-\frac{1}{2}}\tilde{f}||_X \\
& + \big(2\epsilon_1\lambda + C(1+\lambda)\lambda^2\epsilon_2\big)||h||_Y,
\end{aligned}
$$

thus substituting this into (3.22), we get

$$
\begin{aligned}
||h||_{Y\cap W} \;\le\; & 2\big(\frac{C}{1-2\lambda} + (1+\lambda)\big)||\mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}}\tilde{f}||_X \\
& + 2\big(\frac{C}{1-2\lambda} + 2\epsilon_1 + \lambda(1+\lambda)\epsilon_2\big)||\mathbb{P}_{S^\perp}(-\mathcal{L})^{-\frac{1}{2}}\tilde{f}||_X, \\
(3.26) \qquad \le\; & \frac{K_1}{1-2\lambda}||(-\mathcal{L})^{-\frac{1}{2}}\tilde{f}||_X,
\end{aligned}
$$

if $|\alpha|$ is sufficiently large or λ is sufficiently small. This estimate proves the existence and boundedness of $(\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M})^{-1}$ by the Fredholm alternative.

By substituting (3.26) into (3.23) and (3.25), we finally obtain

$$(3.27) \qquad ||\mathbb{P}_{S^\perp} h||_{D(A^\gamma)} \leq 2\epsilon_1 \left(||\mathbb{P}_{S^\perp}(-\mathcal{L})^{-\frac{1}{2}} \tilde{f}||_X + \frac{K_1 \lambda}{1-2\lambda} ||(-\mathcal{L})^{-\frac{1}{2}} \tilde{f}||_X \right)$$

$$(3.28) \qquad ||\mathbb{P}_S h||_{D(A^\gamma)} \leq (1+\lambda) ||\mathbb{P}_S(-\mathcal{L})^{-\frac{1}{2}} \tilde{f}||_X$$
$$+ \lambda(1+\lambda)\epsilon_2 \left(||\mathbb{P}_{S^\perp}(-\mathcal{L})^{-\frac{1}{2}} \tilde{f}||_X + \frac{K_1 K_2 \lambda}{1-2\lambda} ||(-\mathcal{L})^{-\frac{1}{2}} \tilde{f}||_X \right).$$

The proof of Lemma 3.1 is now completed.

4. Construction of solutions

In this section we construct a solution of the equation

$$(4.1) \qquad w = (\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M})^{-1} \big(B(w,w) + \lambda\Lambda_1 w + \lambda f_\lambda \big),$$

where Λ_1 and f_λ are given by (1.21) and (1.20).

In order to use the estimates in Lemma 3.1 effectively, we decompose the equation into the radially symmetric part and the non-radially symmetric part. That is, we construct a solution of the form

$$w = w_S + w_{S^\perp}, \ w_S \in Y \cap W \cap \mathbb{P}_S X_1, \ w_{S^\perp} \in Y \cap W \cap \mathbb{P}_{S^\perp} X_1.$$

Then we see

$$\begin{aligned} B(w,w) &= B(w_S, w_{S^\perp}) + B(w_{S^\perp}, w_S) + B(w_{S^\perp}, w_{S^\perp}) \\ &= \Lambda_{w_S} w_{S^\perp} + \frac{1}{2}\Lambda_{w_{S^\perp}} w_{S^\perp}, \\ \Lambda_1 w &= \Lambda_1 w_S + \Lambda_1 w_{S^\perp}. \end{aligned}$$

Note that the functions $\Lambda_{w_S} w_{S^\perp}$, $\Lambda_1 w_S$, and f_λ belong to $\mathbb{P}_{S^\perp} X_1$.

We identify $D(A^\gamma)$ in X_1 with $\mathbb{P}_S D(A^\gamma) \times \mathbb{P}_{S^\perp} D(A^\gamma)$. Here $\mathbb{P}_S D(A^\gamma) = D(A^\gamma) \cap \mathbb{P}_S X_1$ and $\mathbb{P}_{S^\perp} D(A^\gamma) = D(A^\gamma) \cap \mathbb{P}_{S^\perp} X_1$.

For $(f,h) \in \mathbb{P}_S D(A^\gamma) \times \mathbb{P}_{S^\perp} D(A^\gamma)$, we set

$$(4.2) \qquad H_1(f,h) = (\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M})^{-1} \Lambda_f h,$$

$$(4.3) \qquad H_2(f,h) = \frac{1}{2}(\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M})^{-1} \Lambda_h h,$$

$$(4.4) \qquad H_3(f,h) = \lambda(\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M})^{-1} \Lambda_1 f,$$

$$(4.5) \qquad H_4(f,h) = \lambda(\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M})^{-1} \Lambda_1 h,$$

$$(4.6) \qquad F_{\alpha,\lambda} = \lambda(\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M})^{-1} f_\lambda,$$

and

$$(4.7) \qquad H_{\alpha,\lambda}(f,h) = \sum_{i=1}^{4} H_i(f,h) + F_{\alpha,\lambda}.$$

The term $(\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M})^{-1} \Lambda_f h$ makes sense for any $f, \ h \in D(A^\gamma)$ with $\gamma \in (0,1]$. Indeed, by Lemma 3.1 and (2.8), we have

$$\begin{aligned} ||(\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M})^{-1} \Lambda_f h||_{Y \cap W} &\leq \frac{K_1}{1-2\lambda} ||(-\mathcal{L})^{-\frac{1}{2}} \Lambda_f h||_X \\ &\leq \frac{C K_1}{1-2\lambda} (||h||_{D(A^\gamma)} ||f||_X + ||f||_{D(A^\gamma)} ||h||_X). \end{aligned}$$

Thus the above $H_{\alpha,\lambda}$ maps $\mathbb{P}_S D(A^\gamma) \times \mathbb{P}_{S^\perp} D(A^\gamma)$ into $Y \cap W \cap X_1$ for any $\gamma \in (0,1]$. We fix $\gamma \in (0,1)$ and write $D_S = \mathbb{P}_S D(A^\gamma)$, $D_{S^\perp} = \mathbb{P}_{S^\perp} D(A^\gamma)$ for simplicity. Now we define the map $\Phi_{\alpha,\lambda}$ on $D_S \times D_{S^\perp}$ by

(4.8) $$\Phi_{\alpha,\lambda}(f,h) = (\mathbb{P}_S H_{\alpha,\lambda}(f,h), \mathbb{P}_{S^\perp} H_{\alpha,\lambda}(f,h)).$$

By Lemma 3.1, this map $\Phi_{\alpha,\lambda}$ is well-defined. Let κ_1, $\kappa_2 > 0$ and let X_{κ_1,κ_2} be a closed convex subset in $D_S \times D_{S^\perp}$ such that

(4.9) $$X_{\kappa_1,\kappa_2} = \{(f,h) \in D_S \times D_{S^\perp} \mid ||f||_{D(A^\gamma)} \leq \kappa_1, \ ||h||_{D(A^\gamma)} \leq \kappa_2\}.$$

The following proposition leads to Theorem 1.1.

Proposition 4.1. *Let $\lambda \in [0, \frac{1}{2})$. Then there exist $\kappa_1(\lambda)$, $\kappa_2(\lambda)$, and $R_2(\lambda) \geq 0$ such that for any α with $|\alpha| \geq R_2(\lambda)$, the above $\Phi_{\alpha,\lambda}$ has a unique fixed point in $X_{\kappa_1(\lambda),\kappa_2(\lambda)}$.*

Proof. First we show that $\Phi_{\alpha,\lambda}$ is a compact mapping on $X_{\kappa_1(\lambda),\kappa_2(\lambda)}$ into itself for suitable $\kappa_1(\lambda)$, $\kappa_2(\lambda)$, and $R_1(\lambda) \geq 0$. By Lemma 3.1, we have

$$
\begin{aligned}
||\mathbb{P}_S H_1(f,h)||_{D(A^\gamma)} &\leq \lambda(1+\lambda)\delta_2\Big(||\mathbb{P}_{S^\perp}(-\mathcal{L})^{-\frac{1}{2}}\Lambda_f h||_X \\
&\quad + \frac{K_2 K_1 \lambda}{1-2\lambda}||(-\mathcal{L})^{-\frac{1}{2}}\Lambda_f h||_X\Big) \\
&\leq C\lambda(1+\lambda)(1+\frac{\lambda}{1-2\lambda})\delta_2(||f||_X||h||_{D(A^\gamma)} + ||f||_{D(A^\gamma)}||h||_X).
\end{aligned}
$$

Here, we used the estimates (2.8). Similarly, we obtain from Lemma 3.1,

$$||\mathbb{P}_{S^\perp} H_1(f,h)||_{D(A^\gamma)} \leq C(1+\frac{\lambda}{1-2\lambda})\delta_1(||f||_X||h||_{D(A^\gamma)} + ||f||_{D(A^\gamma)}||h||_X),$$

$$||\mathbb{P}_S H_2(f,h)||_{D(A^\gamma)} \leq C(1+\lambda)\big(1+\lambda(1+\frac{\lambda}{1-2\lambda})\delta_2\big)||h||_X||h||_{D(A^\gamma)},$$

$$||\mathbb{P}_{S^\perp} H_2(f,h)||_{D(A^\gamma)} \leq C(1+\frac{\lambda}{1-2\lambda})\delta_1||h||_X||h||_{D(A^\gamma)},$$

$$||\mathbb{P}_S H_3(f,h)||_{D(A^\gamma)} \leq C\lambda^2(1+\lambda)(1+\frac{\lambda}{1-2\lambda})\delta_2||f||_{D(A^\gamma)},$$

$$||\mathbb{P}_{S^\perp} H_3(f,h)||_{D(A^\gamma)} \leq C\lambda(1+\frac{\lambda}{1-2\lambda})\delta_1||f||_{D(A^\gamma)},$$

$$||\mathbb{P}_S H_4(f,h)||_{D(A^\gamma)} \leq C\lambda(1+\lambda)\big(1+\lambda(1+\frac{\lambda}{1-2\lambda})\delta_2\big)||h||_{D(A^\gamma)},$$

$$||\mathbb{P}_{S^\perp} H_4(f,h)||_{D(A^\gamma)} \leq C\lambda(1+\frac{\lambda}{1-2\lambda})\delta_1||h||_{D(A^\gamma)}.$$

In the estimates for H_3, we used the fact that $\Lambda_1 f \in \mathbb{P}_{S^\perp} X_1$ by Corollary 2.2. We also remark that the estimates for $||\mathbb{P}_S H_2(f,h)||_{D(A^\gamma)}$ and $||\mathbb{P}_S H_4(f,h)||_{D(A^\gamma)}$ imply that we potentially require the smallness of $||h||_{D(A^\gamma)}$ itself. Especially, the fact that the term $\mathbb{P}_S H_4(f,h)$ does not depend on f is crucial, since the prefactor constant is not sufficiently small when λ is not small enough.

The estimate for $F_{\alpha,\lambda}$ is

$$||\mathbb{P}_S F_{\alpha,\lambda}||_{D(A^\gamma)} \leq C\lambda^2(1+\lambda)(1+\frac{\lambda}{1-2\lambda})\delta_2||(-\mathcal{L})^{-\frac{1}{2}}f_\lambda||_X,$$

$$||\mathbb{P}_{S^\perp} F_{\alpha,\lambda}||_{D(A^\gamma)} \leq C\lambda(1+\frac{\lambda}{1-2\lambda})\delta_1||(-\mathcal{L})^{-\frac{1}{2}}f_\lambda||_X,$$

hence especially we have

(4.10) $$\lim_{|\alpha|\to\infty} ||F_{\alpha,\lambda}||_{D(A^\gamma)} = 0.$$

We also note that $||F_{\alpha,\lambda}||_{D(A^\gamma)}$ is sufficiently small uniformly in $|\alpha|$ if λ is sufficiently small. Combining these above, we obtain

(4.11)

$$
\begin{aligned}
||\mathbb{P}_S H(f,h)||_{D(A^\gamma)} \leq\; & C_0\lambda(1+\lambda)(1+\frac{\lambda}{1-2\lambda})\delta_2\Big\{||f||_X||h||_{D(A^\gamma)} + ||f||_{D(A^\gamma)}||h||_X \\
& +||h||_X||h||_{D(A^\gamma)} + \lambda(||f||_{D(A^\gamma)} + ||h||_{D(A^\gamma)})\Big\} \\
& +C_0(1+\lambda)\big\{||h||_X||h||_{D(A^\gamma)} + \lambda||h||_{D(A^\gamma)}\big\} + ||\mathbb{P}_S F_{\alpha,\lambda}||_{D(A^\gamma)},
\end{aligned}
$$

(4.12)

$$
\begin{aligned}
||\mathbb{P}_{S^\perp} H(f,h)||_{D(A^\gamma)} \leq\; & C_0(1+\frac{\lambda}{1-2\lambda})\delta_1\Big\{||f||_X||h||_{D(A^\gamma)} + ||f||_{D(A^\gamma)}||h||_X \\
& +||h||_X||h||_{D(A^\gamma)} + \lambda(||f||_{D(A^\gamma)} + ||h||_{D(A^\gamma)})\Big\} + ||\mathbb{P}_{S^\perp} F_{\alpha,\lambda}||_{D(A^\gamma)}.
\end{aligned}
$$

Here C_0 is a numerical constant.

Let $\kappa_2 \leq \kappa_1 \leq 1$. We take $|\alpha|$ sufficiently large (or λ sufficiently small) enough to satisfy

(4.13)
$$
C_0\lambda(1+\lambda)(1+\frac{\lambda}{1-2\lambda})\delta_2 \leq \frac{1}{N},
$$

where $N \geq 100$ is determined later. Then for $(f,h) \in X_{\kappa_1,\kappa_2}$, we have

$$
||\mathbb{P}_S H(f,h)||_{D(A^\gamma)} \leq \frac{1}{4}\kappa_1 + C_0(1+\lambda)(\kappa_2^2 + \lambda\kappa_2) + ||\mathbb{P}_S F_{\alpha,\lambda}||_{D(A^\gamma)}.
$$

Next we consider the estimate of (4.12). If λ is not small, then we take $|\alpha|$ sufficiently large enough to satisfy

(4.14)
$$
C_0(1+\frac{\lambda}{1-2\lambda})\delta_1 \leq \frac{1}{N}.
$$

If λ is sufficiently small enough to satisfy

(4.15)
$$
C_0\lambda(1+\frac{\lambda}{1-2\lambda})\sup_{|\alpha|}\delta_1(|\alpha|,\lambda) \leq \frac{1}{N},
$$

then we take κ_1 and κ_2 sufficiently small enough such as

(4.16)
$$
2C_0(1+\frac{\lambda}{1-2\lambda})(\kappa_1+\kappa_2)\sup_{|\alpha|}\delta_1(|\alpha|,\lambda) \leq \frac{1}{N}.
$$

In each case, we have

$$
||\mathbb{P}_{S^\perp} H(f,h)||_{D(A^\gamma)} \leq \frac{1}{4}\kappa_2 + \frac{1}{N}(\kappa_1+\kappa_2) + ||\mathbb{P}_{S^\perp} F_{\alpha,\lambda}||_{D(A^\gamma)}
$$

We take $8C_0(1+\lambda)\kappa_2 = \kappa_1$. Then we have

(4.17)
$$
||\mathbb{P}_S H(f,h)||_{D(A^\gamma)} \leq \frac{1}{2}\kappa_1 + ||\mathbb{P}_S F_{\alpha,\lambda}||_{D(A^\gamma)},
$$

(4.18)
$$
||\mathbb{P}_{S^\perp} H(f,h)||_{D(A^\gamma)} \leq \frac{1}{4}\kappa_2 + \frac{8C_0(1+\lambda)+1}{N}\kappa_2 + ||\mathbb{P}_{S^\perp} F_{\alpha,\lambda}||_{D(A^\gamma)},
$$

thus if we take N as $\frac{8C_0(1+\lambda)+1}{N} \leq \frac{1}{4}$, then $\Phi_{\alpha,\lambda}$ maps X_{κ_1,κ_2} into itself, because $||\mathbb{P}_S F_{\alpha,\lambda}||_{D(A^\gamma)}$ and $||\mathbb{P}_{S^\perp} F_{\alpha,\lambda}||_{D(A^\gamma)}$ are sufficiently small if we take $|\alpha|$ large or λ small enough. We omit the details. Since H is a mapping from $D_S \times D_{S^\perp}$ into $D(A)$, it is easy to see that $\Phi_{\alpha,\lambda}$ is completely continuous.

Hence by the Schauder fixed point theorem, $\Phi_{\alpha,\lambda}$ has at least one fixed point on X_{κ_1,κ_2}.

Now we shall prove that a fixed point of $\Phi_{\alpha,\lambda}$ on X_{κ_1,κ_2} is unique. Let (f_1,h_1), $(f_2,h_2) \in X_{\kappa_1,\kappa_2}$ be fixed points of $\Phi_{\alpha,\lambda}$. Then, arguing as same as above, we obtain

$$
\begin{aligned}
||f_1 - f_2||_{D(A^\gamma)} &= ||\mathbb{P}_S H(f_1,h_1) - \mathbb{P}_S H(f_2,h_2)||_{D(A^\gamma)} \\
&\leq \frac{1}{4}||f_1 - f_2||_{D(A^\gamma)} + C||h_1 - h_2||_{D(A^\gamma)}, \\
||h_1 - h_2||_{D(A^\gamma)} &= ||\mathbb{P}_{S^\perp} H(f_1,h_1) - \mathbb{P}_{S^\perp} H(f_2,h_2)||_{D(A^\gamma)} \\
&\leq \frac{1}{4}||h_1 - h_2||_{D(A^\gamma)} + \frac{C}{N}||f_1 - f_2||_{D(A^\gamma)},
\end{aligned}
$$

for a numerical constant C independent of N.

Hence $||f_1 - f_2||_{D(A^\gamma)} \leq \frac{4C}{3}||h_1 - h_2||_{D(A^\gamma)}$ and

$$
||h_1 - h_2||_{D(A^\gamma)} \leq \frac{1}{4}||h_1 - h_2||_{D(A^\gamma)} + \frac{4C^2}{3N}||h_1 - h_2||_{D(A^\gamma)} \leq \frac{1}{2}||h_1 - h_2||_{D(A^\gamma)},
$$

since N is large. This gives $h_1 = h_2$, and also $f_1 = f_2$. This completes the proof of the proposition.

5. Large Reynolds number asymptotics

In this section, we prove the asymptotic estimate (1.14). Let $w_{\alpha,\lambda}$ be the solution obtained in Proposition 4.1. Let $0 < \gamma < 1$. Then it is not difficult to see

$$(5.1) \qquad\qquad\qquad ||w_{\alpha,\lambda}||_{D(A^\gamma)} \leq C||F_{\alpha,\lambda}||_{D(A^\gamma)},$$

for a numerical constant $C > 0$ by the estimates (4.11) and (4.12). We give a proof only to the case of large $|\alpha|$. In this case, we may assume that the constant $C_0(1 + \frac{\lambda}{1-2\lambda})(\delta_1 + \delta_2)$ in (4.11), (4.12) is sufficiently small. Note that we already have

$$||\mathbb{P}_S w_{\alpha,\lambda}||_{D(A^\gamma)}, \ ||\mathbb{P}_{S^\perp} w_{\alpha,\lambda}||_{D(A^\gamma)} \leq 1$$

by the proof of Proposition 4.1. Thus we obtain

$$
\begin{aligned}
||\mathbb{P}_S w_{\alpha,\lambda}||_{D(A^\gamma)} &\leq C||\mathbb{P}_{S^\perp} w_{\alpha,\lambda}||_{D(A^\gamma)} + 2||\mathbb{P}_S F_{\alpha,\lambda}||_{D(A^\gamma)}, \\
||\mathbb{P}_{S^\perp} w_{\alpha,\lambda}||_{D(A^\gamma)} &\leq C\lambda(1 + \frac{\lambda}{1-2\lambda})\delta_1||\mathbb{P}_S w_{\alpha,\lambda}||_{D(A^\gamma)} + 2||\mathbb{P}_{S^\perp} F_{\alpha,\lambda}||_{D(A^\gamma)}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&||\mathbb{P}_{S^\perp} w_{\alpha,\lambda}||_{D(A^\gamma)} \\
&\leq C\lambda(1 + \frac{\lambda}{1-2\lambda})\delta_1(||\mathbb{P}_{S^\perp} w_{\alpha,\lambda}||_{D(A^\gamma)} + 2||\mathbb{P}_S F_{\alpha,\lambda}||_{D(A^\gamma)}) + ||\mathbb{P}_{S^\perp} F_{\alpha,\lambda}||_{D(A^\gamma)},
\end{aligned}
$$

that is,

$$||\mathbb{P}_{S^\perp} w_{\alpha,\lambda}||_{D(A^\gamma)} \leq C||F_{\alpha,\lambda}||_{D(A^\gamma)},$$

The estimate (5.1) is now easily obtained.

Since $w_{\alpha,\lambda}$ is a solution of the equation (4.1), by the estimate (3.1), we have the estimate of $||w_{\alpha,\lambda}||_{Y \cap W}$ such that

$$
\begin{aligned}
||w_{\alpha,\lambda}||_{Y \cap W} &\leq \frac{C}{1-2\lambda}(||w_{\alpha,\lambda}||_{D(A^\gamma)} + \lambda)||w_{\alpha,\lambda}||_{D(A^\gamma)} + ||F_{\alpha,\lambda}||_{Y \cap W} \\
(5.2) \qquad &\leq \frac{C}{1-2\lambda}(||F_{\alpha,\lambda}||_{D(A^\gamma)} + \lambda)||F_{\alpha,\lambda}||_{D(A^\gamma)} + ||F_{\alpha,\lambda}||_{Y \cap W}.
\end{aligned}
$$

Hence the large Reynolds number asymptotics of solutions is controlled by the behavior of $F_{\alpha,\lambda} = \lambda(\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M})^{-1}f_\lambda$. By the arguments in [5], we obtain the desired estimate as follows.

Proposition 5.1. *Let $\lambda \in [0, \frac{1}{2})$ and let $R_2(\lambda)$ be the number obtained in Proposition 4.1. Then for any α with $|\alpha| \geq R_2(\lambda)$, the function $F_{\alpha,\lambda}$ satisfies*

$$(5.3) \qquad ||F_{\alpha,\lambda}||_{Y \cap W} \leq \frac{C\lambda}{(1-2\lambda)(1+|\alpha|)}.$$

Proof. We only give the proof for the case $|\alpha| >> 1$. By Corollary 2.3, we already know $f_\lambda \in \mathbb{P}_{S^\perp} X_1 = \overline{\text{Ran } \Lambda}$. Moreover, by investigating the equation $\mathcal{M}G = \Lambda w_\infty$, we see that $f_\lambda \in \text{Ran } \Lambda$ and the function h_λ satisfying $f_\lambda = \Lambda h_\lambda$ also belongs to $D(\mathcal{L})$; see [5, Section 3]. We omit the details here. Now we use the argument in [5, Proposition 3.4].

$$
\begin{aligned}
&\quad -(\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M})^{-1} f_\lambda \\
&= -(\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M})^{-1} \Lambda h_\lambda \\
&= \frac{1}{\alpha}(\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M})^{-1}(\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M}) h_\lambda - \frac{1}{\alpha}(\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M})^{-1}(\mathcal{L} + \lambda\mathcal{M}) h_\lambda \\
&= \frac{1}{\alpha}\left\{ h_\lambda + (\mathcal{L} - \alpha\Lambda + \lambda\mathcal{M})^{-1}(\mathcal{L} + \lambda\mathcal{M}) h_\lambda \right\}.
\end{aligned}
$$

Thus we have from (3.1),

$$
\begin{aligned}
||F_{\alpha,\lambda}||_{Y \cap W} &\leq \frac{C\lambda}{|\alpha|}\left\{ ||h_\lambda||_{Y \cap W} + \frac{1}{1-2\lambda}||(-\mathcal{L})^{-\frac{1}{2}}(\mathcal{L} + \lambda\mathcal{M}) h_\lambda)||_X \right\} \\
&\leq \frac{C\lambda}{(1-2\lambda)|\alpha|}||h_\lambda||_{Y \cap W}.
\end{aligned}
$$

This gives the desired estimate for $|\alpha| >> 1$.

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